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CONTENTS

No. 1. JANUARY

On Boundary Conditions in the Non-Symmetric Unified Field Theory ..	Mineo Ikeda	1
Meson Reactions in Two-Nucleon System	Shigeo Minami	12
On the Theories of Higher Derivatives and Non-Local Couplings, II	Tosiya Taniuti	19
He-He Repulsive Potential, I	Michiko Sakamoto and Eiichi Ishiguro	37
Extensions of Variational Methods, II— <i>Two Parameter Eigenvalue Problem for the Deuteron State</i> —	Takashi Kikuta	50
On the Exchange Magnetic Moment of the Two Nucleon System	Shigeaki Hatano and Tadashi Kaneno	63

Letters to the Editor :

The Photodisintegration of Deuteron at High Energy ..	D. Ito, M. Ōno, T. Kato and Y. Takahashi	74
Relation between the Quadrupole Moments and the Widths of the Giant Resonance of Photonuclear Reaction	K. Okamoto	75
A Method of Calculation of Electrical Conductivity	H. Nakano	77
Nature of Nuclear Force Indicated by the Photodisintegration of the Deuteron, I	S. H. Hsieh and M. Nakagawa	79
Mass Reversal and Space-Time Inversions	S. Watanabe	81
Remarks on Heisenberg's Non-Linear Field Theory	H. Kita	83
Determination of the Pion Coupling Constant in Nuclear Forces	J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari	86

No. 2. FEBRUARY

Populations and Evolution of Stars	Mitsuo Taketani, Takeo Hatanaka and Shinya Obi	89
Theory of Electron Multiplication in Silicon	Jiro Yamashita	95
Supernova Origin of Cosmic Rays	Satio Hayakawa	111
Pion Theory of Nuclear Forces	Kazuhiko Inoue, Shigeru Machida, Mitsuo Taketani and Toshiyuki Toyoda	122
Intermediate Coupling Meson Theory of Nuclear Forces, II		

.....	Yukihisa Nogami and Hiroichi Hasegawa	137
Quantum Statistical Mechanics of Electron-Phonon System	
.....	Hiroshi Ichimura	151
On a Regular Formulation of Quantum Field Theory, I		
— <i>Non-Relativistic Theory</i> —	Ken-iti Goto 167
Letters to the Editor :		
On Solutions of He^3 and He^4	S. K. Trikha and V. S. Nanda 178
Note on the Decay Interactions of Hyperons and Heavy Mesons	M. Kawaguchi and K. Nishijima 180
Note on the Decays of Σ Particles	M. Kawaguchi and K. Nishijima 182
On the Energy Dissipation of Conduction Electrons undergoing Elastic Scattering by Impurities	T. Yamamoto, K. Tani and K. Okada 184
Interaction of Antinucleons in Matter, I	J. Iwadare and S. Hatano 185
Errata	187

No. 3. MARCH

On the One Pion Exchange Potential	
.....	Kichiro Hiida, Junji Iwadare and Shigeru Machida	189
Electromagnetic Radiation from Electron Plasma	
.....	Satio Hayakawa and Noboru Hokkyo	193
Deuteron Stripping Reactions and Nuclear Shell Structure	
.....	Sueji Okai and Mitsuo Sano	203
Effect of Hard Core on the Binding Energies of H^3 and He^3, I	
.....	Takashi Kikuta, Masato Morita and Masami Yamada	222
On the Use of Feynman Amplitudes in the Quantum Field Theory	
.....	Ziro Maki	237
A Collective Description of the Surface Oscillation of Atomic Nuclei		
— <i>Extension of Tomonaga's Method to Three Dimensional Nucleus</i> —	
.....	Tatsuoki Miyazima and Taro Tamura	255
Coulomb Interactions and the Diamagnetism of Free Electrons	
.....	Hideo Kanazawa	273

Letters to the Editor:

Interaction of π -Mesons with π -Mesons.....	Z. Koba	294
Surface Tension of Liquid He ⁴ and Liquid He ³	S. K. Trikha and O. P. Rustgi	296
On the Renormalization Theory of Quantum Electrodynamics		
..... S. Kamefuchi and H. Umezawa		298
On the Elementarity of the Weak Boson-Fermion Interaction, I		
..... S. Ôneda, S. Hori and A. Wakasa		300
On the Elementarity of the Weak Boson-Fermion Interaction, II		
..... S. Ôneda, S. Hori and A. Wakasa		302
On K-Meson Decays and the Universal Interactions.....	S. Ôneda, S. Hori and A. Wakasa	304
Theory of Antiferromagnetic Resonance in CuCl ₂ ·2H ₂ O (Errata).....	T. Nagamiya	306

No. 4. APRIL

Absorption Spectra of Unsymmetrical Cyanines		
..... Gentaro Araki and Shigeru Huzinaga		307
On the Wave Theory of Light in General Relativity, III— <i>Electromagnetic</i> <i>Four Potential</i> —	Hyôitirô Takeno and Yoshio Ueno	322
A Relativistic Field Theory of an Extended Particle, I....	Tadao Nakano	333
A Statistical Theory of Linear Dissipative Systems, II		
..... Natsuki Hashitsume		369

Letters to the Editor:

Some Relations between the Bound State Problem and Scattering Theory	N. Fukuda	414
Note on the Nuclear Level of Spin 2 ⁺	M. Sakai	416
On the Renormalization Cut-off....	H. Umezawa, S. Kamefuchi, Y. Tomozawa and M. Konuma	417

Errata		419
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No. 5. MAY

The Atmospheric Effects on the Intensity of High Energy μ -Mesons		
.. Takashi Murayama, Kazuaki Murakami, Ryokichi Tanaka and Shuzo Ogawa		421
Theory of Mass Reversal in the Quantized Field Theory		
..... Tadashi Ouchi, Kei Senba and Minoru Yonezawa		431
Proposal for Experiments for Determination of Beta-Decay Interaction and Theory of Triple Cascade Transition.....	Masato Morita	445

Remarks on the Pion-Pion Interaction.....	Ziro Koba	461
On the Relativistic Thomas-Fermi Atom.....	G. S. Murty	473
The Statistical Mechanical Aspect of the H-Theorem, II.....	Ei Teramoto	480
On the Universality of the Weak Interaction	Shuzo Ogawa	487
Interaction between Electrons in Two-Dimensional Free-Electron Model for Conjugated Systems.....	Shigeru Huzinaga	495
One-Center Expansion of Molecular Wave Function	Shigeru Huzinaga	501
Letters to the Editor :		
On the Feynman's Theory of Polarons.....	K. Yamazaki	508
Hydrodynamical Theory of the Multiple Production of Particles in High Energy Nucleon-Nucleus Collision	C. Iso and M. Sato	510

No. 6. JUNE

Orbits of an Electron in Static Electromagnetic Fields, I.....	Giiti Iwata	513
Symmetry in Time and Tanikawa's Method of Superquantization in Regard to Negative Energy Fields	Satosi Watanabe	523
Deuteron Photodisintegration at High Energies	Ryozi Suzuki	536
Elastic Scattering of Alpha-Particle by Heavy Elements.....		
..... Nobuo Oda and Kichinosuke Harada		545
The Theory of Radiative K Capture, I.....	Jiro Yukawa	561
Interpretation on Nitrogen- and Proton-Induced Nuclear Reactions.....		
..... Ryuzo Nakasima, Yoshihisa Tanaka and Ken Kikuchi		574
Letters to the Editor :		
Surface Rigidity and Shell Structure of Nuclei....	T. Marumori, S. Suekane and A. Yamamoto	582
Nuclear Quadrupole Moment and Rigidity of Nuclear Core		
..... T. Marumori, S. Suekane and A. Yamamoto		584
Rotational Excitation with Neutron Inelastic Scattering	M. Yasuno	586
On the Hydrodynamical Description of the Behaviours of Elementary Particles ..	M. Hamaguchi	588
Contents and Authors' Index to Volume 15	i~vii	
Subject Index (1946~1955)	1~62	

On Boundary Conditions in the Non-Symmetric Unified Field Theory

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We study boundary conditions in the non-symmetric unified field theory from the covariant point of view. The covariant expression of spatial infinity is obtained by generalizing the one ordinarily used in physics. We propose a new boundary condition, which is covariant for arbitrary coordinate transformations and reduces in a special case to the ordinary condition in general relativity. We compare the newly-proposed condition with the non-covariant one which has been employed by many authors. For the static solutions of the field equations in the electric or magnetic case, the new condition is shown to be equivalent to those which we proposed previously by using the scalar potentials.

§ 1. Introduction

There are many extended theories of general relativity, in which the equations are covariant for arbitrary transformations of coordinates. This covariant feature is based on the so-called principle of general covariance and is essential to the formulation of the theories. Among these extended theories, the non-symmetric unified field theory proposed by Einstein¹⁾ is the most interesting. There, the space-time is *not* Riemannian, and the non-symmetric connection $\Gamma_{\lambda\mu}^{\nu}$ is generally expressible in terms of the *non-symmetric* tensor $g_{\lambda\mu}$ and its first derivatives. The symmetric part of this $g_{\lambda\mu}$, i.e., $g_{(\lambda\mu)}$, refers to the gravitational, while the skew-symmetric part $g_{[\lambda\mu]}$ does to the electromagnetic field. On eliminating $\Gamma_{\lambda\mu}^{\nu}$, the field equations take the forms which involve $g_{\lambda\mu}$ and its derivatives alone. Since they have the covariant character, the boundary conditions for their solutions must be chosen so as to be covariant for arbitrary transformations of coordinates. Unfortunately, it is not the case with the conditions hitherto used by many authors, as will be discussed in the following.

Since the field vanishes at a large distance from matter and charge, it is generally assumed that :

$$g_{\lambda\mu} \rightarrow \gamma_{\lambda\mu} \quad \text{at spatial infinity.} \quad (1.1)$$

Here, $\gamma_{\lambda\mu}$, usually called the Galilean tensor, is the metric tensor of a Minkowski (flat Riemannian) space-time S_4 to which the space-time under consideration should reduce in the absence of the field. As stated above, this condition must satisfy the requirement of general covariance; in more detail, if we denote the transforms of $g_{\lambda\mu}$ and $\gamma_{\lambda\mu}$ by $'g_{\lambda\mu}$ and $'\gamma_{\lambda\mu}$ respectively, $'g_{\lambda\mu} \rightarrow '\gamma_{\lambda\mu}$ must follow from (1.1). It was, however, pointed out by Wyman that (1.1) has no such covariant character and has different meanings according to the coordinate systems chosen.²⁾ For this reason Bonnor chose a certain advantageous

coordinate system in putting boundary conditions.³⁾ This difficulty was partially overcome by the present author for the static $g_{\lambda\mu}$ in the electric or magnetic case.⁴⁾ He introduced "scalar potentials" for the static electromagnetic field, and proposed certain boundary conditions by using these potentials. These conditions are covariant at least for the coordinate transformations which leave the expression of spatial infinity, $r \rightarrow \infty$ for $t = \text{const.}$, invariant. In this case, the discussion on covariance for arbitrary coordinate transformations is meaningless, because the expression of spatial infinity is covariant only for a restricted class of transformations.

Thus it comes into question to express the spatial infinity in a covariant manner, and to seek for the boundary conditions satisfying the requirement of general covariance. The chief aim of the present paper is to solve these problems.

For this purpose we shall first take up the boundary condition in general relativity:

$$\overset{\circ}{g}_{\lambda\mu} \rightarrow \eta_{\lambda\mu} \text{ at spatial infinity,} \quad (1.2)$$

where $\overset{\circ}{g}_{\lambda\mu}$ is the metric tensor of the space-time. Throughout this paper the metric tensor of a (Riemannian) space-time is denoted by this notation, in order to distinguish it from the non-metric $g_{\lambda\mu}$ in the unified field theory. In § 2, the spatial infinity will be expressed in a covariant manner. Then, in § 3, we shall write (1.2) in a new form by using only scalar quantities. The resultant "scalar conditions" will be shown to be covariant for arbitrary transformations of coordinates. Thus the condition (1.2) is satisfactory from the standpoint of covariance.

The remaining sections are concerned with the non-symmetric unified field theory. In § 4, we shall generalize the above scalar conditions to this case. The conditions thus obtained also satisfy the requirement of general covariance. They are clearly different from the non-covariant condition (1.1), and the difference is explained in detail in the case of the spherically symmetric $g_{\lambda\mu}$. In the last section, we shall clarify the relation of the new-found conditions to those which were proposed in our previous work.

§ 2. Covariant expression of spatial infinity

The boundary conditions to be discussed in the following involve the concept of spatial infinity. This concept has usually been considered as self-evident, and has been treated without ambiguity. Its ordinary expression is, however, obviously dependent on the choice of space-time coordinates. So we shall show a method of expressing it in a covariant manner.

Since we first intend to deal with the condition in general relativity, the space-time V_4 is assumed to be *Riemannian* throughout this section. The metric tensor $\overset{\circ}{g}_{\lambda\mu}$ of the V_4 , representing the gravitational potentials, is of signature -2 .

It should be remembered that the boundary conditions are generally used to single out from a great variety of $\overset{\circ}{g}_{\lambda\mu}$'s (e.g., solutions of the field equations) a particular one which takes the prescribed values on the boundary. Hence, in solving the boundary value problems, one must treat a great variety of space-times with the metric tensors $\overset{\circ}{g}_{\lambda\mu}$'s at the same time. This feature is characteristic of the case of general relativity (and of its ex-

tended theories), and needs a particular treatment in the formulation of boundary conditions. So somewhat detailed explanations will be presented on this point in Appendix. As will be seen there, all the $\hat{g}_{\lambda\mu}$'s usually have some *common form* in concrete problems.

For the covariant expression of spatial infinity, it is necessary to consider a family of space-like hypersurfaces defined by

$$F(x^1, \dots, x^4) = \text{const.} (\equiv k), \quad \hat{g}^{\lambda\mu} F_{,\lambda} F_{,\mu} > 0, \quad (2.1)$$

where $\hat{g}^{\lambda\mu}$ is given by $\hat{g}_{\lambda\lambda} \hat{g}^{\mu\mu} = \partial_{\lambda}^{\mu}$ and the comma followed by a Greek index denotes ordinary differentiation with respect to the x 's. (2.1) can also be written as

$$x^{\lambda} = x^{\lambda}(u^1, u^2, u^3; k),$$

where the u 's are parameters on the hypersurfaces. Here, the rank of the matrix $\|x^{\lambda}_{,i}\|$ is 3, the comma followed by a Latin index denoting differentiation with respect to the u 's.* Let $\bar{g}_{ij} = -\hat{g}_{\lambda\mu} x^{\lambda}_{,i} x^{\mu}_{,j}$, then each of the hypersurfaces can be considered as a 3-dimensional Riemannian space \bar{V}_3 with the positive definite metric tensor \bar{g}_{ij} . It should be noticed that on account of the diversity of $\hat{g}_{\lambda\mu}$'s a great variety of \bar{g}_{ij} 's are obtained for a *single* hypersurface. All these \bar{g}_{ij} 's cannot be regarded as the metric tensors of the same \bar{V}_3 . So, to a single hypersurface there correspond a great variety of \bar{V}_3 's. Thus the spatial infinity in the hypersurface must be specified independently of all these \bar{V}_3 's, and its expression must be formulated without the metric tensor \bar{g}_{ij} .

Now suppose a mapping of \bar{V}_3 on a 3-dimensional Euclidean space S_3 , where the same values are assigned to the coordinates of the corresponding points. This means the introduction of a Euclidean metric tensor γ_{ij} at points of \bar{V}_3 . Of course there are obtained also a great number of γ_{ij} 's, but this time they can be considered as the metric tensors of the same S_3 referred to different parameter systems. The spatial infinity in the S_3 can be specified in a unique way in spite of the variety of γ_{ij} 's. This may be carried out as follows.

Let P_0 and P be arbitrary points in the S_3 , such that the parameter system u^i is valid on the half-line extending from P_0 in the direction of P . Then the distance between P_0 and P is given by

$$\rho \equiv \rho(P_0, P) = \int_{P_0}^P \sqrt{\gamma_{ij} du^i du^j}. \quad (2.2)$$

As $\rho \rightarrow \infty$, P approaches to some point which belongs to the (3-dimensional) "domain of spatial infinity". $\rho \rightarrow \infty$ is independent of all the \bar{V}_3 's corresponding to the hypersurface under consideration. It has also an invariant meaning for arbitrary transformations of coordinates x^{λ} and parameters u^i , as can easily be seen from (2.2). However, its *concrete* expression depends on the choice of parameter system. For instance, if we choose $x^4 =$

* Latin indices take the values 1, 2, 3, while Greek ones 1, 2, 3, 4.

const. as (2.1) and identify u^i with x^i , the expression is given by $r \rightarrow \infty$ for (A.2), whereas $r \rightarrow 0$ for (A.3). (See Appendix.) The latter seems to be strange at a glance, but is, in fact, quite reasonable, because (A.3) is obtained from (A.2) by the transformation $r \rightarrow r^{-1}$.

The above expression of spatial infinity is a generalization of the ordinary one hitherto used in physics. In the following we shall adopt it as the mathematical expression of "spatial infinity" in the boundary conditions. It is to be noted that this definition is given with respect to the S_3 . Strictly speaking, for the determination of spatial infinity in \overline{V}_3 , we must consider the metric property of \overline{V}_3 . For, the expression $\rho \rightarrow \infty$ is not always equivalent to $\int_{p^0}^p \sqrt{g_{ij}} du^i du^j \rightarrow \infty$. This equivalence is to be examined for each $\check{g}_{\lambda\mu}$ satisfying boundary conditions. At the present stage, however, such an examination is impossible, because we are studying the way of choosing boundary conditions. This is the reason why we consider the spatial infinity in \overline{V}_3 as corresponding to that in the S_3 .

In the above formulation we have used a particular family of hypersurfaces given by (2.1). If we take another family, the domain of spatial infinity may not change as a whole, excluding a few exceptional points. Hence the results obtained do not depend on the choice of the family. This is also true for the following sections, because the hypersurfaces will not be used except for the expression of spatial infinity.

§ 3. Covariance of the boundary condition in general relativity

Now by making use of the covariant expression of spatial infinity, the condition (1.2) in general relativity is written as

$$\check{g}_{\lambda\mu} \rightarrow \eta_{\lambda\mu} \quad \text{as} \quad \rho \rightarrow \infty. \quad (3.1)$$

For the $\check{g}_{\lambda\mu}$'s under consideration there exist a great variety of $\eta_{\lambda\mu}$'s, and they have the same common form as the $\check{g}_{\lambda\mu}$'s. For instance, when dealing with the spherically symmetric $\check{g}_{\lambda\mu}$'s of the form indicated by (A.1), the corresponding $\eta_{\lambda\mu}$'s are given by (A.2) (A.3), (A.4), and so on. (Those such as (4.9) are excluded.) This diversity of $\eta_{\lambda\mu}$'s may cause some ambiguities in using the condition (3.1), and so we shall write it in a form which does not involve $\eta_{\lambda\mu}$ explicitly. By using this result we shall verify the covariance of (3.1) under arbitrary transformations of coordinates.

Let us now introduce an orthogonal ennuple $\hat{\xi}_{\sigma i}^\lambda$ in the S_4 with the metric tensor $\eta_{\lambda\mu}$,* such that $\hat{\xi}_{4i}^\lambda$ is normal to a hypersurface defined by (2.1).** The vectors $\hat{\xi}_{\sigma i}^\lambda$ satisfy the conditions

$$\eta_{\lambda\mu} \hat{\xi}_{\sigma i}^\lambda \hat{\xi}_{\tau j}^\mu = e_{\sigma\tau}, \quad (e_1 = e_2 = e_3 = -1, e_4 = 1). \quad (3.2)$$

Although $\hat{\xi}_{4i}^\lambda$ is determined uniquely, $\hat{\xi}_{\sigma i}^\lambda$ can be chosen arbitrarily to some extent. If

* All the $\eta_{\lambda\mu}$'s can be regarded as the metric tensors of the same S_4 referred to different coordinate systems.

** The index σ of $\hat{\xi}_{\sigma i}^\lambda$, indicating the vector, must not be confused with those indicating components of a tensor. For such an index the summation convention is not used, and will be replaced by the symbol ξ if necessary.

we take another ennuple η_{σ}^{λ} such as $\eta_{i1}^{\lambda} = \hat{\xi}_{i1}^{\lambda}$, the vectors $\eta_{\sigma1}^{\lambda}$ are expressible linearly in terms of $\hat{\xi}_{\sigma1}^{\lambda}$, i.e.,

$$\eta_{\sigma1}^{\lambda} = \sum_{\tau} t_{\sigma\tau} \hat{\xi}_{\tau1}^{\lambda}, \quad (3.3)$$

where $t_{\sigma\tau}$ satisfy the conditions

$$\sum_{\sigma} t_{\sigma\tau} t_{bc} = \delta_{ab}, \quad t_{a1} = t_{1a} = 0, \quad t_{11} = 1, \quad (3.4)$$

by means of (3.2) and a similar equation for η_{σ}^{λ} . (3.3) with (3.4) is the general relation between orthogonal ennuples under consideration.

If we put $\hat{\xi}_{\sigma1\lambda} = \eta_{\lambda\mu} \hat{\xi}_{\sigma1}^{\mu}$, $\eta_{\lambda\mu}$ can be written

$$\eta_{\lambda\mu} = \sum_{\sigma} e_{\sigma} \hat{\xi}_{\sigma1\lambda} \hat{\xi}_{\sigma1\mu}. \quad (3.5)$$

Similarly, $\hat{g}_{\lambda\mu}$ can be expressed as

$$\hat{g}_{\lambda\mu} = \sum_{\sigma, \tau} e_{\sigma} e_{\tau} \hat{a}_{\sigma\tau} \hat{\xi}_{\sigma1\lambda} \hat{\xi}_{\tau1\mu}, \quad (3.6)$$

where $\hat{a}_{\sigma\tau}$ are scalars given by

$$\hat{a}_{\sigma\tau} = \hat{g}_{\lambda\mu} \hat{\xi}_{\sigma1}^{\lambda} \hat{\xi}_{\tau1}^{\mu}. \quad (3.7)$$

On account of (3.2) and (3.7), the condition (3.1) becomes

$$\hat{a}_{\sigma\tau} \rightarrow e_{\sigma} \delta_{\sigma\tau} \quad \text{as} \quad \rho \rightarrow \infty, \quad (3.8)$$

and conversely (3.1) is obtained from (3.8) using (3.5) and (3.6). This is the required expression of (3.1) which does not involve $\eta_{\lambda\mu}$ explicitly. As easily seen from its scalar character, (3.8) is independent of the coordinate system. However, we must show that it is also independent of the particular choice of ennuple $\hat{\xi}_{\sigma1}^{\lambda}$. For this purpose we introduce scalars $\hat{b}_{\sigma\tau}$ for any other ennuple $\eta_{\sigma1}^{\lambda}$ in like manner as in (3.7). Then under the condition (3.8), we have from (3.3) and (3.4)

$$\hat{b}_{\sigma\tau} = \sum_{\alpha, \beta} \hat{g}_{\lambda\mu} t_{\sigma\alpha} t_{\tau\beta} \hat{\xi}_{\alpha1}^{\lambda} \hat{\xi}_{\beta1}^{\mu} = \sum_{\alpha, \beta} \hat{a}_{\alpha\beta} t_{\sigma\alpha} t_{\tau\beta} \rightarrow \sum_{\alpha, \beta} e_{\alpha} \delta_{\alpha\beta} t_{\sigma\alpha} t_{\tau\beta} = e_{\sigma} \delta_{\sigma\tau}. \quad (3.9)$$

This shows the required invariance of (3.8) under changes of the ennuple.

Thus the boundary condition (1.2) hitherto used in general relativity is consistent with the principle of general covariance, in the sense that it is covariant for arbitrary transformations of coordinates. This is the reason why the problem under discussion has not come into question in general relativity. Whereas, in the non-symmetric unified field theory, we do not know any covariant way of choosing boundary conditions as yet, hence we shall make some investigations on this subject.

§ 4. Boundary conditions in the non-symmetric unified field theory

In the non-symmetric unified field theory of Einstein¹⁾, many authors have used the boundary condition (1.1), which is a formal generalization of (1.2). At first sight this

condition seems to be reasonable. However, it is not covariant for coordinate transformations, and is significant only when the coordinate system is fixed.²⁾ For example, when the condition is imposed on Wyman's static spherically symmetric solution in the electric case using "polar coordinates", $g_{\lambda\mu}$ vanishes identically and the solution reduces to that for the pure gravitational field. Whereas, in "cartesian coordinates", the same condition implies that $g_{\lambda\mu}$ does not always vanish and Wyman's solution contains both the gravitational and electromagnetic fields.

A method of avoiding this difficulty was put forward by the present author by introducing a scalar potential for the electrostatic or magnetostatic field.⁴⁾ But this method is applicable only to the static $g_{\lambda\mu}$ in the electric or magnetic case. Moreover the boundary conditions adopted there are covariant only for the restricted class of transformations, i.e., linear transformations of time coordinate and transformations of spatial coordinates leaving $r \rightarrow \infty$ invariant. Hence it comes into question to propose new boundary conditions which are available for the general $g_{\lambda\mu}$ and are covariant for arbitrary transformations of coordinates. This problem is solved completely in the following manner.

In what follows, ' $\rho \rightarrow \infty$ ', ' S_1 ', ' $\hat{\xi}_\sigma^\lambda$ ', and so on, have the same meanings as those in §§ 2, 3, with a proviso that in their definitions $\hat{g}_{\lambda\mu}$ is replaced by $g_{\lambda\mu}$.

It is obvious that the scalar character of (3.8) contributed much to the success of the last section. Without these scalar conditions we cannot even presume the covariance of (1.2). On the other hand, the unsatisfactory feature of (1.1) seems to be caused by the fact that it was obtained as a formal generalization of (1.2). These suggest that satisfactory conditions may be obtained by generalizing the scalar conditions (3.8). This may really bring about good results, as will be shown in the following.

We now introduce the scalars

$$a_{\sigma\tau} = g_{\lambda\mu} \hat{\xi}_{\sigma 1}^\lambda \hat{\xi}_{\tau 1}^\mu, \quad (4.1)$$

which correspond to $\hat{a}_{\sigma\tau}$ in the last section. Then it seems reasonable to choose

$$a_{\sigma\tau} \rightarrow e_\sigma \hat{\partial}_{\sigma\tau} \quad \text{as} \quad \rho \rightarrow \infty, \quad (4.2)$$

as scalar conditions corresponding to (3.8). It is also written

$$\underline{a}_{\sigma\tau} \rightarrow e_\sigma \hat{\partial}_{\sigma\tau}, \quad \underline{a}_{\sigma\tau} \rightarrow 0 \quad \text{as} \quad \rho \rightarrow \infty,$$

where $a_{\sigma\tau}$ and $\underline{a}_{\sigma\tau}$ are the symmetric and skew-symmetric parts of $a_{\sigma\tau}$ respectively. The symmetric part of (4.2) is independent of the choice of orthogonal ennuple as (3.8) is. For the skew-symmetric part the same result holds also as follows: If $b_{\sigma\tau}$ are scalars formed from $g_{\lambda\mu}$ with respect to $\eta_{\sigma 1}^\lambda$, we have

$$b_{\sigma\tau} = \sum_{\alpha, \beta} t_{\sigma\alpha} t_{\tau\beta} a_{\alpha\beta}.$$

Since $t_{\sigma\tau}$ are finite by means of (3.4), it holds that $b_{\sigma\tau} \rightarrow 0$ as $\underline{a}_{\sigma\tau} \rightarrow 0$. Thus the covariance of (4.2) has been proved.

From the standpoint of covariance, it is thus desirable to choose not (1.1) but (4.2)

as boundary conditions. The symmetric part of (4.2) is equivalent to that of (1.1), as can be shown in the same way as in § 3. The difference of (4.2) from (1.1) really lies in the skew-symmetric part. We show this by taking the spherically symmetric $g_{\lambda\mu}$, which is given by

$$g_{\lambda\mu} = \begin{pmatrix} -A(r, t), & 0, & 0, & f(r, t) \\ 0 & -B(r, t), & h(r, t) \sin \theta, & 0 \\ 0, & -h(r, t) \sin \theta, & -B(r, t) \sin^2 \theta, & 0 \\ -f(r, t) & 0, & 0, & C(r, t) \end{pmatrix}, \quad (4.3)$$

in polar coordinates, where A , B and C are positive. We choose (A.2) as the corresponding line element of the S_4 , and

$$\hat{\xi}_{11}^\lambda = \delta_1^\lambda, \quad \hat{\xi}_{21}^\lambda = \delta_2^\lambda / r, \quad \hat{\xi}_{31}^\lambda = \delta_3^\lambda / r \sin \theta, \quad \hat{\xi}_{41}^\lambda = \delta_4^\lambda \quad (4.4)$$

as an orthogonal ennuple in the S_4 . If we calculate $a_{\sigma\tau}$ corresponding to (4.3) and (4.4) and substitute them into (4.2), we have

$$A \rightarrow 1, \quad B \rightarrow r^2, \quad C \rightarrow 1 \quad \text{as} \quad r \rightarrow \infty \quad (4.5)$$

for the symmetric part, and

$$f \rightarrow 0, \quad h/r^2 \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty \quad (4.6)$$

for the skew-symmetric part. The symmetric part of (1.1) has the same expression (4.5) as that of (4.2). On the other hand, its skew-symmetric part reduces to

$$f \rightarrow 0, \quad h \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty, \quad (4.7)$$

which is more stringent than (4.6). Thus the desired result is obtained.

Next, we pass to cartesian coordinate system by the transformation

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta, \quad t = t.$$

Then $g_{\lambda\mu}$ and the line element of the S_4 reduce to

$$g_{\lambda\mu} = \begin{pmatrix} 0, & vz/r, & -vy/r, & fx/r \\ -vz/r, & 0, & vx/r, & fy/r \\ vy/r, & -vx/r, & 0, & fz/r \\ -fx/r, & -fy/r, & -fz/r, & 0 \end{pmatrix}, \quad v = h/r^2, \quad (4.8)$$

and

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 \quad (4.9)$$

respectively. We can choose the orthogonal ennuple

$$\hat{\xi}_{\sigma 1}^\lambda = \delta_{\sigma}^\lambda, \quad (4.10)$$

then the scalars $a_{\sigma\tau}$ take the same form as the components of $g_{\lambda\mu}$ given in (4.8). This time the skew-symmetric part of (1.1) has the expression (4.6), which is different from

that in polar coordinates. This is the non-covariant feature of (1.1) pointed out by Wyman.

In general, the conditions (1.1) and (4.2) have the common expression in cartesian coordinates, because there exists the orthogonal ennuple (4.10) in this case. For example, for the spherically symmetric $g_{\lambda\mu}$ the common expression is given by (4.5) and (4.6). Thus, as far as cartesian coordinates are chosen, (1.1) can be used as a covariant condition, in the sense that it has the same expression as (4.2). In his work on static spherically symmetric solutions, Bonnor used (1.1) in such coordinates.³⁾ This is now confirmed as satisfactory by virtue of the above result.

§ 5. Comparison with the conditions used in our previous work

In the previous work on static solutions of the non-symmetric unified field theory, we used the boundary conditions:

$$g_{\lambda\mu} \rightarrow \eta_{\lambda\mu}, \quad \phi \rightarrow \text{const.} \quad \text{as} \quad r \rightarrow \infty \quad (5.1)_a$$

for the electric case, and

$$g_{\lambda\mu} \rightarrow \eta_{\lambda\mu}, \quad \psi \rightarrow \text{const.} \quad \text{as} \quad r \rightarrow \infty. \quad (5.1)_b$$

for the magnetic case. Here ϕ and ψ are "scalar potentials"*, which we introduced for the electrostatic and magnetostatic fields, and whose derivatives are expressible in terms of $g_{\lambda\mu}$. These conditions are covariant at least for the coordinate transformations leaving the expression of spatial infinity, $r \rightarrow \infty$ for $t = \text{const.}$, invariant. In the following, we shall study the relation between (5.1) and (4.2), and shall show that they put the same restriction on the static solutions of the field equations in the electric or magnetic case.

Previously we treated the static field, for which $g_{\lambda\mu}$ is restricted by

$$g_{i4} = 0, \quad \dot{g}_{\lambda\mu} = 0, \quad (5.2)$$

in the coordinate system where (2.1) takes the form

$$x^4 = \text{const.} \quad (5.3)$$

In this case x^i can be identified with u^i , as will be done throughout this section. By means of (5.3) we can choose $\eta_{\lambda\mu}$ and $\hat{\epsilon}_{a|}{}^\lambda$ such as

$$\eta_{\lambda 4} = \partial_{\lambda 4}, \quad (5.4)$$

$$\hat{\epsilon}_{a|}{}^\lambda = \partial_a{}^\lambda, \quad \hat{\epsilon}_{a|}{}^4 = 0. \quad (5.5)$$

The derivatives of scalar potentials are

$$\phi_{;i} = \epsilon_{ij}{}^k h^{jl} h^{km} g_{lm} g / 2 \sqrt{-g} \quad (5.6)$$

for the electrostatic field, and

* These are scalars for coordinate transformations in \bar{V}_3 , but not for those in V_4 .

$$\phi_{,i} = -h^{44} g_{i4} \underline{g}/g \quad (5.7)$$

for the magnetostatic field. In these equations, $g = \det. g_{\lambda\mu}$, $\underline{g} = \det. g_{\lambda\mu}$, $h^{\lambda\mu}$ is defined by $g_{\lambda\nu} h^{\mu\nu} = \delta_{\lambda}^{\mu}$, and $\epsilon_{\lambda\mu\nu\sigma}$ is the Levi-Civita tensor density.

It follows from (5.6) and (5.7) that*

$$\begin{aligned} \phi_{|i} &= \phi_{,i} \xi_{a|i}^i = \sum_{b,c,d,e} \epsilon_{abcd} \bar{a}_{bd} \bar{a}_{ce} a_{de} |a_{\sigma\tau}| / 2 \sqrt{-|a_{\sigma\tau}|}, \\ \phi_{|i} &= \phi_{,i} \xi_{a|i}^i = -a_{a1} |a_{\sigma\tau}| / g_{44} |a_{\sigma\tau}|, \end{aligned} \quad (5.8)$$

where $\bar{a}_{ab} = h^{\lambda\mu} \xi_{a|\lambda} \xi_{b|\mu}$. In consequence of (4.2), we have as $\rho \rightarrow \infty$,

$$\begin{aligned} (\phi_{a1})_{\infty} &= - (a_{bc})_{\infty}, \quad (a, b, c = \text{cycl.}(1, 2, 3)) \\ (\phi_{a1})_{\infty} &= - (a_{a4})_{\infty}. \end{aligned} \quad (5.9)$$

Hence the skew-symmetric part of (4.2) reduces to

$$\phi_{a1} \rightarrow 0, \quad \phi_{a1} \rightarrow 0 \quad \text{as} \quad \rho \rightarrow \infty. \quad (5.10)$$

Next choose the vector ξ_{a1}^{λ} for some suffix a in the direction of $\vec{P}_0 P$ given in § 2. Along this half-line, ϕ is a function of ρ , and we have $\xi_{a1}^{\lambda} = dx^{\lambda}/d\rho$, hence

$$\phi_{a1} = \phi_{,i} (dx^i/d\rho) = d\phi/d\rho.$$

By means of (5.10), ϕ_{a1} must have the asymptotic form

$$\phi_{a1} \sim \rho^{-n} (a + b\rho^{-1} + \dots), \quad 1 \leq n > 0, \quad (5.11)$$

where a and b are some constants. When a does not vanish, $\phi \rightarrow \infty$ as $\rho \rightarrow \infty$. Therefore, in order for ϕ to approach to some constant, ϕ_a hence a_{bc} must approach to zero faster than ρ^{-1} . From this and a similar result for ϕ , the conditions (5.1) are more stringent than (4.2) for the static $g_{\lambda\mu}$ in the electric or magnetic case. The difference between the two conditions can be interpreted physically as follows.

In Maxwell's electromagnetic theory, the Coulomb potential is given by $r^{-1} + \text{const.}$, and the corresponding electric field is of the order r^{-2} . In the present case, ϕ or ψ represents the potential, and $g_{\lambda\mu}$ is usually interpreted as representing the field strength.**

* (5.8) can be obtained, if one expresses the various tensors in (5.6) and (5.7) in terms of $\xi_{\sigma 1}^{\lambda}$ and then uses the following formulae:

$$g = |e_{\sigma} \xi_{\sigma 1}^{\lambda}|^2 \cdot |a_{\sigma\tau}|, \quad \underline{g} = |e_{\sigma} \xi_{\sigma 1}^{\lambda}|^2 \cdot |a_{\sigma\tau}|, \quad |e_{\sigma} \xi_{\sigma 1}^{\lambda}| \cdot |e_{\tau} \xi_{\tau 1}^{\mu}| = -1,$$

$$|e_{\sigma} \xi_{\sigma 1}^{\lambda}| = \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \epsilon_{\lambda\mu\nu\sigma} \xi_1^{\lambda} \xi_2^{\mu} \xi_3^{\nu} \xi_4^{\sigma}.$$

** In the previous papers, we assumed that the strength of the electromagnetic field is represented not by $g_{\lambda\mu}$ or its conjugate, but by a certain tensor $F_{\lambda\mu}$. This tensor is expressible in terms of $g_{\lambda\mu}$, and reduces for the weak field to

$$F_{\lambda\mu} \simeq \sum_{\alpha, \beta} \epsilon_{\lambda\mu\alpha\beta} e_{\alpha} e_{\beta} g_{\alpha\beta}.$$

Therefore, one can consider that $F_{\lambda\mu}$ has the same meaning as the conjugate of $g_{\lambda\mu}$ for $\rho \gg 1$.

Therefore, on the analogy of Maxwell's theory, it seems to be reasonable that $a_{\sigma\tau}$, scalars formed from $g_{\lambda\mu}$, approach to zero in the higher order by one than ϕ or ζ^h does to some limiting constant as in (5.1). The above difference between the two conditions actually has such a character.

For the static solutions of the field equations in the electric or magnetic case, this difference does not appear at all and the conditions (4.2) are equivalent to (5.1). In fact, there are no solutions for which the constant a does not vanish, as can be shown as follows: Since the field is weak for $\rho \gg 1$, $g_{\lambda\mu}$ corresponding to the leading term of (5.11) must satisfy the field equations in the first approximation⁶⁾:

$$\sum_{\mu} \epsilon_{\mu} g_{\lambda\mu,\mu} = 0, \quad \square (g_{\lambda\mu,\nu} + g_{\mu\nu,\lambda} + g_{\nu\lambda,\mu}) = 0. \quad (5.12)$$

These equations are linear homogeneous with respect to $g_{\lambda\mu}$. Accordingly, if the source of the field is regarded as consisting of many point sources, the field is obtained by superposing the fields caused by respective point sources. The latter fields are spherically symmetric with different centres, and a simple calculation shows that we have no corresponding solutions for which a does not vanish.

Conclusion

Both general relativity and the non-symmetric unified field theory are based on the so-called principle of general covariance. So the boundary conditions in these theories are to be chosen so as to be covariant for arbitrary transformations of coordinates. This problem was taken up in this paper, and the following results were obtained.

(1) The boundary condition (1.2) ordinarily used in general relativity is covariant, with a proviso that the spatial infinity is expressed covariantly as in § 2.

(2) In the non-symmetric unified field theory, one should use the new conditions (4.2), which are independent of the coordinate system employed.

(3) In "cartesian coordinates", the non-covariant condition (1.1) has the same expression as (4.2). Hence (1.1) can be used as a covariant condition, provided that this coordinate system is chosen.

(4) For the static solutions of the field equations in the electric or magnetic case, the conditions (4.2) are equivalent to those propounded in our previous papers by using scalar potentials.

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Appendix

The boundary conditions are generally used to single out from a great variety of $\dot{g}_{\lambda\mu}$'s (or, $g_{\lambda\mu}$'s) a particular one which takes the prescribed values on the boundary. This fact is important in formulating the boundary conditions. So we shall give here some detailed explanations on this point.

In concrete problems, the $\overset{\circ}{g}_{\lambda\mu}$'s are often restricted by some conditions, such as spatial symmetry, or staticness, or orthogonality for coordinate system. As a result the $\overset{\circ}{g}_{\lambda\mu}$'s under consideration have a *common* form. But there still remains some *arbitrariness* in the form of arbitrary constants or functions of the x 's, and we have as many $\overset{\circ}{g}_{\lambda\mu}$'s as the diversity of these functions or constants.

As an example, let us consider the spherically symmetric space-times whose line elements are of the common form

$$ds^2 = -A(r, t) dr^2 - B(r, t) (d\theta^2 + \sin^2\theta d\varphi^2) + C(r, t) dt^2, \quad A, B, C > 0, \quad (\text{A} \cdot 1)$$

in polar coordinates. Each individual $\overset{\circ}{g}_{\lambda\mu}$ is obtained by giving particular functional forms to A, B and C . The diversity of $\overset{\circ}{g}_{\lambda\mu}$'s corresponds to that of the forms taken by A, B and C .

Among the line elements of the form (A.1) there are

$$ds^2 = -dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) + dt^2, \quad (\text{A} \cdot 2)$$

$$ds^2 = -r^{-4} dr^2 - r^{-2} (d\theta^2 + \sin^2\theta d\varphi^2) + dt^2, \quad (\text{A} \cdot 3)$$

$$ds^2 = -dr^2 - r^2 (d\theta^2 + \sin^2\theta d\varphi^2) + C(t) dt^2, \quad (\text{A} \cdot 4)$$

and so on. The last two are obtained from the first by carrying out the transformations $r \rightarrow r^{-1}$ and $t \rightarrow \int^t \sqrt{C(t)} dt$ respectively. This illustrates that some of the tensors $\overset{\circ}{g}_{\lambda\mu}$'s are reducible to each other. Such tensors can be regarded as giving the metric of the same space-time. On the other hand, there are some tensors which are by no means transformable to each other. These are the metric tensors of the *different* space-times. For instance, if the functions A, B and C assume the form $(1-2m/r)^{-1}, r^2, 1-2m/r$ ($m \neq 0$) respectively, we obtain the famous Schwarzschild line element, which is not reducible to any of (A.2), (A.3) and (A.4).

These considerations will be helpful for understanding of §§ 2, 3.

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Meson Reactions in Two-Nucleon System

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With the charge independence hypothesis, the meson-nucleon interaction in two-nucleon system is studied by analyzing the data of meson production in nucleon-nucleon collision and of meson-deuteron reaction.

§ 1. Introduction

Among the various meson reactions in two-nucleon system the meson production in nucleon-nucleon collision is one of the most important. Although several approaches to this problem have been made from the experimental as well as theoretical points of view, there are still many things to be solved in the problems of the qualitative character. In this paper, we intend to examine this meson reaction, concentrating our attention to the charge space.

Since the interaction between a meson and two nucleons is taken into consideration, it may be allowed to adopt the charge independence hypothesis. In order to specify the states of isotopic spin in our process, let us reexamine the various methods¹⁾⁻⁴⁾ by which the composition has been done. It is common to compose the isotopic spins of the two nucleons first and then bring the isotopic spin of the meson to the composition. (Such composition as this will be cited as L - S scheme hereafter, as mentioned in the previous paper⁵⁾.) This method, however, is not suitable for examining the meson-nucleon interaction. Therefore, we try another method of composition of isotopic spins, that is, to compose the isotopic spin of meson and that of one nucleon first, and then bring the isotopic spin of the other nucleon to the composition. (Such composition as this will be cited as j - j scheme.)

There is of course one-to-one correspondence between L - S scheme and j - j scheme. As for meson-nucleon interaction, detailed analyses have been done of meson-nucleon scattering. Tracing the knowledge deduced from these analyses, we set the following assumption which may be considered as a natural one, that is, the state of $j=3/2$ composed by the isotopic spin of a meson and of a nucleon contributes predominantly in the energy region where the resonance will take place in our process corresponding to the resonance phenomena in meson-nucleon scattering. Then we will obtain the branching ratio for our process, such as the cross section for π^0 meson production in proton-proton collision is one-sixth of the total cross section for meson production in this collision. This result is examined by comparison both with the lower energy data in proton-proton collision and

with those in neutron-proton collision, that is, how strongly each of the charge states contributes and how each contribution depends on the energy are examined.

Moreover, paying our attention to the fact that our process has a certain relation with meson-deuteron reaction, our results are examined by the analysis of the data for this reaction.

§ 2. Composition of isotopic spins

i) L - S scheme

Since the isotopic spin of a nucleon is $1/2$, the resultant isotopic spin L of two nucleons is one or zero. The values of the total isotopic spin T composed by $L=1$ and the isotopic spin of meson are 2, 1 or 0. The reaction amplitudes for these final states are denoted by the notations of $T_2(s)$, $T_1(s)$ and $T_0(s)$ respectively. And the reaction amplitude for the final state of $T=1$ which is composed by $L=0$ and the isotopic spin of meson is denoted by the notation of $T_1(a)$. Where s and a in the bracket stand for the symmetric and antisymmetric parts for the two nucleons respectively. Moreover $T_1(a)$ is separated into two parts, that is, $T_{1f}(a)$ and $T_{1D}(a)$ which represent respectively the amplitudes for free state in the two-nucleon system and for deuteron state.

In the process of meson production in nucleon-nucleon collision, however, the state of $T=2$ does not exist. Because of the orthogonality of the symmetric and antisymmetric parts, and for want of interference between $T_1(s)$ and $T_0(s)^*$, the cross sections for our process can be written down as follows:

$$\begin{aligned}\sigma(p+p \rightarrow \pi^0 + p+p) &= (1/2) |T_1(s)|^2, \\ \sigma(p+p \rightarrow \pi^+ + p+n) &= (1/2) |T_1(s)|^2 + |T_{1f}(a)|^2, \\ \sigma(p+p \rightarrow \pi^+ + D) &= |T_{1D}(a)|^2, \\ T_{1f}(a) + T_{1D}(a) &= T_1(a),\end{aligned}\tag{1}$$

and

$$\begin{aligned}\sigma(n+p \rightarrow \pi^+ + n+n) &= (1/4) |T_1(s)|^2 + (1/6) |T_0(s)|^2, \\ \sigma(n+p \rightarrow \pi^- + p+p) &= (1/4) |T_1(s)|^2 + (1/6) |T_0(s)|^2, \\ \sigma(n+p \rightarrow \pi^0 + n+p) &= (1/6) |T_0(s)|^2 + (1/2) |T_{1f}(a)|^2, \\ \sigma(n+p \rightarrow \pi^0 + D) &= (1/2) |T_{1D}(a)|^2.\end{aligned}\tag{2}$$

ii) The relation between L - S scheme and j - j scheme

There are of course two isotopic spin states $j=3/2$ and $j=1/2$ in the one-meson and one-nucleon system. The reaction amplitudes for our process can easily be expressed

* This fact is obvious from the consideration of the initial states of isotopic spins.

in terms of those of j - j scheme. For simplicity we omit these expressions and write down the relation of the reaction amplitudes in L - S scheme and j - j scheme.

$$\begin{aligned} T_1(s) &= \sqrt{1/3} T_1(3/2)_s + \sqrt{2/3} T_1(1/2)_s, \\ T_1(a) &= \sqrt{2/3} T_1(3/2)_a - \sqrt{1/3} T_1(1/2)_a, \\ T_0(s) &= T_0(1/2)_s, \end{aligned} \quad (3)$$

where the suffices 2, 1 and 0 show the states of the total isotopic spin $T=2, 1$ and 0 respectively, and $3/2, 1/2$ in the bracket on the right hand side mean $j=3/2, j=1/2$ in the j - j scheme respectively.

§ 3. Branching ratio in the neighbourhood of 600 Mev

Recalling the data for meson-nucleon scattering which have shown the resonance of $j=3/2$ state in the energy of about 190 Mev ($\gamma \cong 1.6$), we set the following assumption which may be considered as a natural one, that is, in our process the state of $j=3/2$ reacts also predominantly in the energy region of about 600 Mev ($\gamma \sim 1.6^*$) corresponding to the energy for the resonance of meson-nucleon scattering, and put the following relation for simplicity.

$$|T_1(3/2)_s| = |T_1(3/2)_a|. \quad (4)^{**}$$

If we neglect the contribution from the state of $j=1/2$, then from eq. (3)

$$|T_1(a)|^2 = 2|T_1(s)|^2. \quad (5)$$

Let us consider the following ratio concerning the cross sections.

$$\frac{\sigma(p+p \rightarrow \pi^+ + n + p) + \sigma(p+p \rightarrow \pi^+ + D)}{\sigma(p+p \rightarrow \pi^0 + p + p)} = \frac{|T_1(a)|^2 + (1/2)|T_1(s)|^2}{(1/2)|T_1(s)|^2}. \quad (6)$$

On the assumption mentioned above (c.f. (5)),

$$\frac{\sigma(p+p \rightarrow \pi^+ + n + p) + \sigma(p+p \rightarrow \pi^+ + D)}{\sigma(p+p \rightarrow \pi^0 + p + p)} = \frac{5}{1}. \quad (7)$$

Then the cross section for π^0 meson production turns out to be one-sixth of the total cross section for meson production in proton-proton collision.

§ 4. Comparison with the experimental data

i) Meson production in proton-proton collision

When we plot the data which have been cited in the papers of Rosenfeld⁽⁶⁾ or Gell-Mann and Watson⁽⁷⁾, the results will be shown in Fig. 1.

* In this case, γ is the maximum center-of-mass momentum available to the pion, measured in units of μ .

** With regard to this assumption, there is enough ground for controversy.

Since the cross sections for $p+p \rightarrow \pi^0 + p + p$ are available at only two energies, namely 341 Mev and 430 Mev, we are obliged to adopt the formula deduced by Rosenfeld⁽⁶⁾ or Gell-Mann and Watson⁽⁷⁾ from phenomenological point of view.

$$\sigma(p+p \rightarrow \pi^0 + p + p) = 0.2\eta^8. \quad (8)$$

But the usefulness of this formula has to be restricted to the region between the energy of 341 Mev and 430 Mev.

On the other hand, the formula for the cross section of π^+ -meson production, $\sigma(p+p \rightarrow \pi^+ + n + p) + \sigma(p+p \rightarrow \pi^+ + D) = 1.8\eta^3$, has also been given by these authors, but we do not adopt it. The reason is that their formula gives too small value for $\sigma(p+p \rightarrow \pi^+ + n + p) + \sigma(p+p \rightarrow \pi^+ + D)$ to explain the experimental data in high energy region as shown in Fig. 1. This fact is also obvious from their result that the predicted ratio $\sigma(p+p \rightarrow \pi^+ + n + p) / \sigma(p+p \rightarrow \pi^+ + D)$ appears to be about a factor 1/2 smaller than the observed one in spite of the fine estimation of $\sigma(p+p \rightarrow \pi^+ + D)$. Therefore we make use of the experimental data for the cross section of π^+ -meson production.

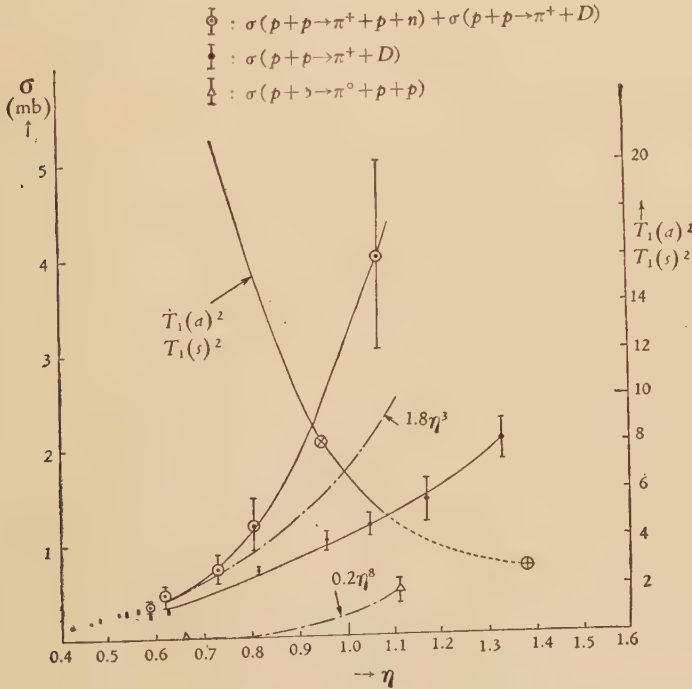


Fig. 1. The cross sections for meson production in p - p collision and the values of $|T_1(a)|^2/|T_1(s)|^2$.

\oplus : The value of $|T_1'(a)|^2/|T_1'(s)|^2$ calculated from the data of π^-D reaction.

\otimes : The value of $|T_1(a)|^2/|T_1(s)|^2$ calculated from the data of n - p collision.

Thus, from eq. (6), the value $|T_1(a)|^2/|T_1(s)|^2$ is calculated and its energy dependence is shown by the solid line in Fig. 1.

A complete analysis should of course be accompanied by the study of angular momentum. By combining, however, the eqs. (1), (3) and the curve of Fig. 1, we may probably infer that firstly, in the lower energy region, $T_1(s)$ is affected not only by the selection rule⁸⁾ which is derived from the conservation of total angular momentum and parity, but also by the interference between $T_1(3/2)_s$ and $T_1(1/2)_s$, causing the value of the cross section for π^0 production to be much smaller than that of π^\pm production, and secondly, in the higher energy region, the state of $j=3/2$ affects more strongly than that of $j=1/2$, causing the values of the two cross sections to approach the same order.

ii) Meson production in neutron-proton collision at 410 Mev

The data of meson production in proton-proton collision contain some large errors, particularly in high energy region, as shown in Fig. 1. In order to examine our result further and to get more minute knowledge of the reaction amplitudes indicated in eqs. (1) and (2), let us analyze the data for meson production in neutron-proton collision. In this case, not only the state of $T=1$ but also the state of $T=0$ takes part in the process, so it is necessary to know more data than those in the case of i). Fortunately we have enough data to analyze the results in 410 Mev.

By Yodh's result⁹⁾, the cross section for π^+ or π^- meson production in neutron-proton collision is of about 0.16 ± 0.04 mb. From eq. (2)

$$\begin{aligned}\sigma(n+p \rightarrow \pi^+ + n + n) &= \sigma(n+p \rightarrow \pi^- + p + p) \\ &= (1/4)|T_1(s)|^2 + (1/6)|T_0(s)|^2 = 0.16 \text{ mb.}\end{aligned}\quad (9)$$

The cross section for $p+p \rightarrow \pi^0 + p + p$ in 410 Mev is estimated by eq. (8), and the result is as follows:

$$\sigma(p+p \rightarrow \pi^0 + p + p) = (1/2)|T_1(s)|^2 \cong 0.133 \text{ mb.} \quad (10)$$

From (9) and (10)

$$(1/6)|T_0(s)|^2 \cong 0.0936 \text{ mb.} \quad (11)$$

Now we make use of the value of the cross section for the process $n+p \rightarrow \pi^0 + D$ which has been measured by Hildebrand¹⁰⁾ to verify the charge independence hypothesis

$$\sigma(n+p \rightarrow \pi^0 + D) = (1/2)\sigma(p+p \rightarrow \pi^+ + D) \cong 0.47 \text{ mb.} \quad (12)$$

and of the following experimental ratio observed by him¹⁰⁾

$$\frac{\sigma(n+p \rightarrow \pi^0 + n + p)}{\sigma(n+p \rightarrow \pi^0 + n + p) + \sigma(n+p \rightarrow \pi^0 + D)} = 0.60 \pm 0.15. \quad (13)$$

From these and eq. (2)

$$\sigma(n+p \rightarrow \pi^0 + n + p) = (1/6)|T_0(s)|^2 + (1/2)|T_{1f}(a)|^2 \cong 0.705 \text{ mb.} \quad (14)$$

Using the result of (11),

$$(1/2) |T_{1f}(a)|^2 \cong 0.611 \text{ mb.} \quad (15)$$

As we have $|T_{1D}(a)|^2 \cong 0.94 \text{ mb}$ from (12),

$$|T_1(a)|^2 = |T_{1f}(a)|^2 + |T_{1D}(a)|^2 \cong 2.16 \text{ mb.} \quad (16)$$

Thus we obtain the following result from eqs. (10) and (16) :

$$|T_1(a)|^2 / |T_1(s)|^2 \cong 8.1. \quad (17)$$

Comparing this with the ratio $|T_1(a)|^2 / |T_1(s)|^2$ in the case of proton-proton collision at 410 Mev, it is found that the agreement is fairly well.

§ 5. On meson-deuteron reaction

For meson-deuteron reaction (for example, π^- -D reaction) there are the following five processes :

$$\pi^- + D \rightarrow \pi^- + D \quad (a')$$

$$\pi^- + D \rightarrow \pi^- + n + p \quad (b')$$

$$\pi^- + D \rightarrow \pi^0 + n + n \quad (c')$$

$$\pi^- + D \rightarrow n + n \quad (d')$$

$$\pi^- + D \rightarrow n + n + \gamma \quad (e')$$

Since the magnitude of the cross section for (e') is obviously small and the process of (d') is the inverse one of $n + n \rightarrow \pi^- + D$, we do not take into consideration of these processes.

Noting that only the state of $T=1$ takes part in this reaction as in proton-proton collision, the cross sections for these processes can easily be written down with the same way as in § 2.

$$\begin{aligned} \sigma(\pi^- + D \rightarrow \pi^0 + n + n) &= (1/2) |T_1'(s)|^2, \\ \sigma(\pi^- + D \rightarrow \pi^- + n + p) &= (1/2) |T_1'(s)|^2 + |T_{1f}'(a)|^2, \\ \sigma(\pi^- + D \rightarrow \pi^- + D) &= |T_{1D}'(a)|^2, \\ T_{1f}'(a) + T_{1D}'(a) &= T_1'(a), \end{aligned} \quad (18)$$

where $T_1'(s)$, $T_1'(a)$, $T_{1f}'(a)$ and $T_{1D}'(a)$ are the reaction amplitudes in L - S scheme. (c.f. (1))

In this case also, we have the similar relations between L - S scheme and j - j scheme with those in (3). If the branching ratio for these processes in the energy of about 150 Mev* are discussed with the assumption mentioned in § 3, the following relation can easily be found.

* The energy of 150 Mev in this process corresponds nearly to the resonance energy.

$$|T_1'(a)|^2 = 2|T_1'(s)|^2 \quad (19)$$

and

$$\frac{\sigma(\pi^- + D \rightarrow \pi^- + p + n) + \sigma(\pi^- + D \rightarrow \pi^- + D)}{\sigma(\pi^- + D \rightarrow \pi^0 + n + n)} = \frac{|T_1'(a)|^2 + (1/2)|T_1'(s)|^2}{(1/2)|T_1'(s)|^2} = \frac{5}{1}. \quad (20)$$

Although there are no data to be compared directly with this, we have some interesting results reported by Nagle¹¹⁾. As his data are at the energy of 120 Mev, the reactions can be expected to be the phenomena in the neighbourhood of resonance.

According to his result,

$$\begin{aligned} \sigma(\pi^- + D \rightarrow \pi^- + p + n) + \sigma(\pi^- + D \rightarrow \pi^- + D) &= 96 \pm 5 \text{ mb}, \\ \sigma(\pi^- + D \rightarrow \pi^0 + n + n) &= 14 \pm 2 \text{ mb}. \end{aligned}$$

Therefore

$$|T_1'(a)|^2 / |T_1'(s)|^2 \cong 2.8. \quad (21)$$

Comparing this with eq. (19), we see our assumption is fairly well. Then it may be expected that the relation of (7) for meson production in proton-proton collision will be satisfied in the energy of about 600 Mev and that the curve of $|T_1(a)|^2 / |T_1(s)|^2$ will almost follow the dotted line in Fig. 1. In order to say the above conclusion with confidence, however, it is necessary to have much experimental data in detail.

In conclusion the author expresses his sincere thanks to Dr. D. Ito for his valuable discussions about the problem of the relation between L - S scheme and j - j scheme.

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On the Theories of Higher Derivative and Non-Local Couplings, II

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It is generally granted that in theories of non-local couplings, the field equations (in mechanics, the equations of motion) are the integro-differential equations containing in their interaction terms an arbitrary function, the so-called form factor, and in a previous paper of this series, discussions have been done on these (*c*-number) equations, assuming the validity of the perturbation theory. In this paper, the convergency of the perturbation method is investigated and then it is shown that among the general solutions of these integro-differential equations, we can construct some special solutions which tend to the solutions of free equations as the coupling constant approaches zero. It is, moreover, found that the motion specified by these special solutions conforms to the equations which are localized in the time-like direction and have as manifold solutions as free equations. The similar arguments are also done on the theories of the higher derivative couplings whose formal solutions have been constructed extensively in the previous paper.

§ 1. Introduction

The first article of this series¹⁾ was concerned with a formal construction of special solutions which can be given by the perturbation method and have the same degrees of freedom as the free equation. There we have, in general, assumed without proof that the solutions obtained by the method involving a sort of infinite process, e.g., the iteration method or the perturbation method, converge to the desired solutions. The contribution of this paper is intended to be twofold: first to show how a rigorous mathematical justification can be given for them, and second to investigate whether those special solutions are equivalent to the general solution of some differential equation which has the same degrees of freedom as the free equation. The next section deals with the non-local couplings in mechanical theories and a proof for the existence of solution is given by the iteration method under some restrictions. Moreover it is shown that the integro-differential equations representing the non-local couplings are entirely equivalent to the corresponding differential equations, whenever a development in the neighborhood of linearized vibrations is discussed.

In § 3, the similar discussions will be extended to the systems with the higher derivative couplings. However we shall face, in this case, the grave difficulty in order to prove the existence of iterative solutions.

In § 4, we shall deal with the field theories with non-localized interactions. The existence proof of the iterative solutions may be given in the similar manner as was done for the mechanical theories. However, the equivalence to the partial differential equation can not be proved contrary to mechanical theories, because of the space-like non-localizability, and then we see that in this theory the finite domain of dependence does not exist and

the causal relations in the classical meaning are lost entirely.

§ 2. Non-local couplings in mechanical theories

(a) The existence proof of iterative solutions

We consider the integro-differential equation

$$\ddot{q} + \omega^2 q = \epsilon \int_{-\infty}^{\infty} f(t, t_1, \dots, t_r) G(q(t), \dot{q}(t), q(t_1), \dot{q}(t_1), \dots, q(t_r), \dot{q}(t_r)) dt_1 \dots dt_r \quad (2.1)$$

for $q(t)$, in which ϵ is a small parameter. G and f are the regular functions of $q(t)$, $\dot{q}(t)$, $q(t_1)$, $\dot{q}(t_1)$, \dots , $q(t_r)$, $\dot{q}(t_r)$ and of t , t_1, \dots, t_r respectively. We assume that these functions satisfy the following conditions:

(i) G is bounded in the domain D in the $\{q(t), \dot{q}(t), q(t_1), \dot{q}(t_1), \dots, q(t_r), \dot{q}(t_r)\}$ space, specified by $|q(t_s)| < C$, $|\dot{q}(t_s)| < C$ ($s=1, 2, \dots, r$), where C is a positive number, and there exists a positive number M satisfying the relation $|G| < M$, in D .

(ii) G satisfies a Lipschitz condition for the values of its arguments,

$$\begin{aligned} & |G(y, y', y_1, y_1', \dots, y_r, y_r') - G(z, z', z_1, z_1', \dots, z_r, z_r')| \\ & \leq K[|y - z| + |y' - z'| + \sum_{i=1}^r \{|y_i - z_i| + |y_i' - z_i'|\}], \end{aligned}$$

where K is a positive number.

(iii) $f(t, t_1, \dots, t_r)$ satisfies the condition:

$$\int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} |f(t', t_1, \dots, t_r)| dt' dt_1 \dots dt_r < \alpha,$$

where α is a positive number.

We are interested in establishing the existence of the iterative solutions:

$$\begin{aligned} q_\lambda(t) = q_0(t) + (\epsilon/\omega) \int_{-\infty}^{+\infty} \sin \omega(t-t') \{ f(t', t_1, \dots, t_r) G(q_{\lambda-1}(t), \dot{q}_{\lambda-1}(t), \dots, q_{\lambda-1}(t_r), \dot{q}_{\lambda-1}(t_r)) \\ \cdot dt' dt_1 \dots dt_r. \end{aligned} \quad (2.2)$$

We shall first prove the following theorem:

Theorem:

The iteration method (2.2) converges uniformly for all $t (-\infty < t < +\infty)$, and a unique solution $q(t) = \lim_{\lambda \rightarrow \infty} q_\lambda(t)$ exists satisfying the initial conditions:

$$q(t_0) = A, \quad \dot{q}(t_0) = \omega B \quad \text{at } t = t_0,$$

if the following conditions (A), (B) for A , B and ϵ , are fulfilled.

$$(A) \quad \text{Max } (|A| \omega, |A|) < C/3.$$

$$\text{Max } (|B| \omega, |B|) < C/3.$$

$$(B) \quad |\epsilon| < \text{Min}(|\omega| / \{K\alpha(r+1)(1+|\omega|)\}, |\omega|C / (3\alpha M), C / (3\alpha M)).$$

Proof:

We get immediately from (A) that $|q_0(t)| < 2C/3$, $|\dot{q}_0(t)| < 2C/3$, in the domain

of t : $(-\infty < t < +\infty)$ and then using (2.2) and (B), we obtain also $|q_1(t)| < C$, $|\dot{q}_1(t)| < C$, in the domain of t : $(-\infty < t < +\infty)$.

Accordingly, we can easily obtain, by the mathematical induction,

$$|q_\lambda(t)| < C, \quad |\dot{q}_\lambda(t)| < C$$

for all values of λ and t : $(-\infty < t < +\infty)$. We can also show from (2.2) that

$$\begin{aligned} |q_1(t) - q_0(t)| &< a, \\ |\dot{q}_1(t) - \dot{q}_0(t)| &< |\omega| a, \end{aligned}$$

where a is defined by $a = |\epsilon| \alpha M / |\omega|$.

Using the mathematical induction, we obtain

$$|q_{\lambda+1}(t) - q_\lambda(t)| < a \rho^\lambda,$$

where ρ is given by

$$\rho = (r+1) (1 + 1/|\omega|) K |\epsilon| \alpha.$$

It can easily be seen from (B) that ρ is smaller than unity, and therefore the sequence $q_\lambda(t)$ converges uniformly for all t $(-\infty < t < +\infty)$. The uniqueness of the solution can be seen as follows:

Let \tilde{q} be another solution satisfying the same initial conditions:

$$\tilde{q} = A, \quad \dot{\tilde{q}} = \omega B, \quad \text{at } t = t_0.$$

\tilde{q} then satisfies the following integral equation:

$$\begin{aligned} \tilde{q}(t) &= A \cos \omega(t - t_0) + B \sin \omega(t - t_0) \\ &+ (\epsilon/\omega) \int_{t_0}^t \sin \omega(t - t') f(t', t_1, \dots, t_r) G(\tilde{q}(t'), \dot{\tilde{q}}(t'), \dots, \tilde{q}_2(t_r), \dot{\tilde{q}}_2(t_r)) dt' dt_1 \dots dt_r. \quad (2.2)' \end{aligned}$$

A similar calculation as was used in the proof of the uniform convergency gives for all λ

$$|q_\lambda - \tilde{q}_\lambda| < a \rho^\lambda, \quad \text{i.e. } |q_\lambda - \tilde{q}_\lambda| \rightarrow 0 \quad (\lambda \rightarrow +\infty).$$

The same relation holds also for $\dot{\tilde{q}}$.*)

(b) The conditions of periodicity

In this subsection, we consider the equation

$$\begin{aligned} \ddot{q} + \omega^2 q &= \epsilon F(q(t), \dot{q}(t)) \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f(t, t_1, \dots, t_r) G(q(t_1), \dot{q}(t_1), \dots, q(t_r), \dot{q}(t_r)) dt_1 \dots dt_r \quad (2.1)' \end{aligned}$$

for $q(t)$, which is obtained by rewriting (2.1).

The function $f(t, t_1, \dots, t_r)$ satisfying the condition (iii) in (a) can not be invariant

*) The above discussion may easily be extended to the system of integro-differential equations such as resulting from the non-local interactions between mass points.

under the time displacement transformation, and $(2.1)'$ (or (2.1)) does not represent a conservative closed system.

However we may assume, without loss of generality, the following form of f :

$$f(t, t_1, \dots, t_r) = \mu(t) \tilde{f}(t - t_1, t - t_2, \dots, t - t_r),$$

in which f depends only on differences of time values $t - t_i$, etc., and $\mu(t)$ is restricted in such a way that the condition (iii) is satisfied. In the subsequent discussions we can construct the periodic solutions of $(2.1)'$ and show that their characteristic features are independent of the explicit behavior of $\mu(t)$ at the remote past and future, if $\mu(t)$ is constant over the time interval $-T \leq t \leq T$ sufficiently large compared with the period of the free equation, $2\pi/\omega$, and \tilde{f} is even with respect to its arguments.

This result shows that under these assumptions, $(2.1)'$ might represent the physical periodic motion. In fact, if we adiabatically switch in and off the interaction at the infinite past and future respectively and use \tilde{f} instead of f for all t , $(2.1)'$ may represent the conservative system corresponding to the autonomous system in the local theory.

We shall really prove the existence of periodic solutions $q(t)$, under the above conditions for $\mu(t)$.

We may assume the initial conditions at $t=0$: $\dot{q}(0)=0$, $q(0)=A$, without loss of generality.

Our purpose is to establish the existence of periodic solutions in a neighborhood of $\epsilon=0$. The period τ (the frequency, ν) will then be a function of ϵ , $\tau=\tau(\epsilon)=2\pi/\nu$, which must approach $2\pi/\omega$ as ϵ approaches zero.

The conditions of periodicity are

$$\left. \begin{aligned} q(\tau, A, \epsilon) &= q(0, A, \epsilon), \\ \dot{q}(\tau, A, \epsilon) &= 0. \end{aligned} \right\} \quad (2.3)$$

To solve the equations (2.3) in a neighborhood of $\epsilon=0$, we apply the same method as was used in the study of non-linear vibrations.²⁾

We introduce the quantities

$$P = \epsilon^{-1} [A - q(\tau, A, \epsilon)],$$

$$Q = -\epsilon^{-1} [\dot{q}(\tau, A, \epsilon)],$$

so that the periodicity conditions are now given by

$$P=Q=0.$$

Putting $\tau = (2\pi/\omega) + \epsilon\eta(\epsilon)$, we obtain from $(2.1)'$

$$P = \epsilon^{-1} A (1 - \cos \omega \epsilon \eta) - (1/\omega) \int_0^{(2\pi/\omega) + \epsilon\eta} \sin \omega(\epsilon \eta - t') \cdot$$

$$\cdot F(t') \int_{-\infty}^{\infty} f(t', t_1, \dots, t_r) G(t_1, \dots, t_r) dt_1 \dots dt_r dt' = 0, \quad (2.4a)$$

$$Q = \epsilon^{-1} A \omega \sin \omega \in \eta$$

$$-\int_0^{(2\pi/\omega) + \epsilon\eta} \cos \omega (\epsilon\eta - t') F(t') \int_{-\infty}^{\infty} f(t', t_1, \dots, t_r) G(t_1, \dots, t_r) dt_1 \dots dt_r dt' = 0, \quad (2.4b)$$

in which

$$F(t') = F(q(t'), \dot{q}(t')),$$

$$G(t_1, \dots, t_r) = G(q(t_1), \dot{q}(t_1), \dots, q(t_r), \dot{q}(t_r)),$$

and it should be noted that the integrals with respect to t' are to be extended over the finite interval $(0, 2\pi/\omega + \epsilon\eta)$, not $(-\infty, +\infty)$, and in this region $\mu(t)$ may be put equal to unity. If it could be shown that (2.4a) and (2.4b) for $\epsilon=0$ possess a solution for (A, η) , it would follow from the implicit function theorem that the periodicity equations could be solved in the neighborhood of $\epsilon=0$ to yield $(A(\epsilon), \eta(\epsilon))$.

The simplest case to be considered would be the case in which F and G are independent of \dot{q} . In this case we have, however

$$P|_{\epsilon=0} = P_0(A)$$

$$= \int_0^{2\pi/\omega} \sin \omega t' F(A \cos \omega t') \int_{-\infty}^{\infty} \tilde{f}(t' - t_1, \dots, t' - t_r) G(A \cos \omega t_1, \dots, A \cos \omega t_r) dt_1 \dots dt_r dt',$$

and one sees that $P_0(A)$ is identically zero. Consequently our procedure fails. In the following investigation we shall see that it can be modified in such a way as to yield the desired development in this case, too.

We now assume that F and G are even functions of \dot{q} . This includes the case mentioned above (i.e., F and G are independent of \dot{q}) in which our procedure failed so far to give the desired development. A modified procedure is, however, easy to devise. Applying quite the same procedure as was done in the local non-linear theory,²⁾ we can reduce the periodicity conditions to the following one condition

$$\dot{q}(\tau/2) = \dot{q}(-\tau/2) = 0. \quad (2.3)''$$

The periodicity condition can, therefore, be written in the form

$$R(A, \eta; \epsilon) = A \epsilon^{-1} \sin(\omega \epsilon \eta/2)$$

$$-\int_0^{\pi/\omega + \epsilon\eta/2} dt' \cos \omega (\epsilon\eta/2 - t') F(q(t'), \dot{q}(t')) \int \tilde{f}(t' - t_1, \dots, t' - t_r) G(q(t_1), \dots, \dot{q}(t_r)) dt_1 \dots dt_r = 0 \quad (2.5)$$

and, in the case of $\epsilon=0$, the above equation reduces to

$$R_0(A) = \omega^2 \eta_0/2 - \int_0^{\pi/\omega} \cos \omega t' F(A \cos \omega t', -\omega A \sin \omega t') \cdot$$

$$\cdot \int \tilde{f}(t' - t_1, \dots, t' - t_r) G(A \cos \omega t, -\omega A \sin \omega t, \dots, A \cos \omega t_r, -\omega A \sin \omega t_r) \cdot$$

$$\cdot dt_1 \dots dt_r dt' = 0. \quad (2.6)$$

Now we are permitted to prescribe $A \neq 0$ arbitrarily, after which $R_0 = 0$ can be solved to yield

$$\begin{aligned} \gamma_0 = (2A^{-1}/\omega^2) \int_0^{\pi/\omega} F(A \cos \omega t', -\omega A \sin \omega t') \int \tilde{f}(t' - t_1, \dots, t' - t_r) \cdot \\ \cdot G(A \cos \omega t_1, -\omega A \sin \omega t_1, \dots, A \cos \omega t_r, -\omega A \sin \omega t_r) dt_1 \dots dt_r dt' = 0. \end{aligned}$$

Since $\partial R / \partial \gamma|_{\epsilon=0} = A\omega^2/2 \neq 0$, it follows that (2.5) can be solved for $\gamma = \gamma(\epsilon)$ with $\gamma(0) = \gamma_0$.

We can then conclude that (2.1)' has the periodic solutions for arbitrary values of A in the neighborhood of $\epsilon = 0$, if F and G are even functions of \dot{q} .

(c) *The equivalence to a differential equation*

The solution of (2.1) obtained in (a) is equivalent to the general solution of a second order differential equation, because it contains two arbitrary constants, and we can write down explicitly the equivalent differential equation.

For the sake of simplicity, we consider the following special case: $F = 1$, $r = 1$, then the equation becomes*)

$$\ddot{q} + \omega^2 q = \epsilon \int_{-\infty}^{\infty} f(t - t') G(q(t')) dt'. \quad (2.7)$$

(The extension to general equations can easily be done.)

The above equation is equivalent to the integral equation

$$\begin{aligned} q(t') = q(t', t) \\ + (\epsilon/\omega) \int_t^{t''} \sin \omega(t' - t'') \int f(t'' - t''') G(q(t''')) dt''' dt'', \end{aligned} \quad (2.8)$$

in which

$$q(t, t') = q(t) \cos \omega(t' - t) + (\dot{q}(t)/\omega) \sin \omega(t' - t). \quad (2.9)$$

Substituting the solutions of (2.8) given by the iteration method into (2.7), we obtain

$$\begin{aligned} \ddot{q}(t) + \omega^2 q(t) = \epsilon \int_{-\infty}^{\infty} f(t - t') F(q(t', t), t', t) dt' \\ = \epsilon K(q(t), \dot{q}(t); \epsilon). \end{aligned} \quad (2.10)$$

(2.10) is the differential equation equivalent to (2.7), so far as a development in the neighborhood of the linearized vibration problem is discussed.

Example 1

*) In what follows, we consider that $f = \tilde{f}$ for all t and the interaction is switched in and off adiabatically at the infinite past and future, instead of using $\mu(t)$.

Let us consider the following linear equation :

$$\ddot{q} + \omega^2 q = \epsilon \int_{-\infty}^{\infty} f(t-t') q(t') dt', \quad (2.11)$$

which can be given by taking G equal to $q(t')$ in (2.7).

Corresponding to (2.8), we have, in this case,

$$q(t') = q(t', t) + (\epsilon/\omega) \int_t^{t'} \sin \omega(t' - t'') \int_{-\infty}^{\infty} f(t'' - t''') q(t''') dt''' dt''.$$

If we introduce the Fourier amplitude $F(\omega)$ of $f(t)$, which is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(s) \cos \omega s ds,$$

(2.10) reduces to the differential equation :

$$\ddot{q} + \omega^2 q = \epsilon F(\omega) q(t) + \dots, \quad (2.12)$$

in which the higher order terms in ϵ do not contain \dot{q} . On the other hand, (2.11) can be solved exactly by applying the Fourier transformation. We can then reexamine that among the general solutions of (2.11), (2.12) represents the special solutions which approach those of the harmonic oscillator as ϵ tends to zero.

Example 2

We consider the equation

$$\ddot{q} + \omega^2 q = \epsilon q \int f(t-t') q^2(t') dt' \quad (2.15)$$

for $q(t)$, which may be derived from the following Lagrangian L ;

$$L = \frac{1}{2} \int \{ \dot{q}^2(t) - \omega^2 q^2(t) \} dt + (\epsilon/4) \iint q^2(t) f(t-t') q^2(t') dt' dt.$$

Corresponding to (2.9), we can obtain, in this case, the differential equation

$$\ddot{q} + \omega^2 q = \epsilon q (u \dot{q}^2 + v \dot{q}^2) + \dots, \quad (2.16)$$

in which u and v are defined as follows :

$$u = \int_{-\infty}^{\infty} f(s) \cos^2 \omega s ds,$$

$$v = \int_{-\infty}^{\infty} f(s) \sin^2 \omega s ds.$$

(2.16) is equivalent to (2.15) whenever the perturbation theory is considered.

If $v \neq 0$, the differential equation (2.16) can not be derived easily from a Lagrangian with two independent variables q and \dot{q} .

If we employ the condition that the coefficients of \dot{q} should be zero, the higher harmonics with respect to ω in the Fourier amplitudes of $f(t)$ may be determined successively

as we proceed to the higher order approximation in ϵ . As is obvious in this example, it can be seen that the integro-differential equations considered in this section are generally equivalent to the second-order differential equations so far as a development in the neighborhood of the linearized vibration problems is concerned. It follows that in the mechanical theories the motion of systems with non-localized interactions can be described with the development of time and we can determine the behavior of a mass point in the past and future by prescribing its velocity and coordinate at an instant.

§ 3. Higher derivative couplings in mechanical theories

In this section, perturbation problems are investigated concerning such differential equations that they have the highest derivative term multiplied by a coupling constant. Following the terminology introduced by Friedliuchs and Wasow,³⁾ such perturbation problems will be called singular.

We start with the following type of differential equation as was already considered in I,

$$\ddot{q} + \omega^2 q = \epsilon f(q^{(n)}, q^{(n-1)}, \dots, q), \quad (3.1)$$

in which f is regular analytic with respect to all its arguments.

We are interested in the solutions of (3.1) which approach that of the harmonic oscillator as ϵ tends to zero. The existence of these solutions can not be proved by the iteration method which was successfully applied in § 2. The reason why the usual iterative method fails in this case is as follows:

The iteration method gives formally

$$q_\lambda = q_0 + (\epsilon/\omega) \int \sin \omega(t-t') f(q_{\lambda-1}^{(n)}(t'), \dots, q_{\lambda-1}(t')) dt'. \quad (3.2)$$

In the equation corresponding to $\lambda=1$, we can easily see that $q_1(t)$ is bounded for the suitable initial values of q and \dot{q} , if f is bounded in the certain domain of $q, \dot{q}, \dots, q^{(n)}$ space.

If we want to prove that q_2 is bounded in the equation corresponding to $\lambda=2$, we have to prove that $q_1, \dot{q}_1, \dots, q_1^{(n)}$, accordingly $q_0, \dot{q}_0, \dots, q_0^{(2n-2)}$ are bounded. In this manner, it has to be shown that all higher derivatives (up to the infinite order) of q_0, q_1 , etc., are bounded in order to prove that $\lim_{\lambda \rightarrow \infty} q_\lambda(t)$ is bounded.

This condition, however, can not be satisfied except in the case in which f is the linear functional of $q^{(k)}$ ($k=0, 1, 2, \dots, n$). In non-linear equations, even if q_λ can be in the prescribed domain for the certain values of λ , it may increase indefinitely as λ increases. In the present stage of our study, we can not, therefore, say anything about the validity of the perturbation problem of (3.1). However, it has been shown by several authors that the singular perturbation problem may be solved and we can obtain the periodic solutions tending to those of the harmonic oscillator as ϵ tends to zero, if the highest order derivative $q^{(n)}$ can uniquely be expressed in terms of q, \dot{q}, \dots and $q^{(n-1)}$ in (3.1) and depends analytically on them.*)

*) This condition is rather severe and may be replaced by the existence of two continuous derivatives³⁾.

In the equation (3.1), we can consider the limiting case: $n \rightarrow \infty$. It is easily seen, in this case, that the convergency of the iteration method is not guaranteed because of the same reason as was explained in the case of the finite n .

If we assume that the infinite order derivative coupling is equivalent to some non-local coupling, this type of non-local coupling does not, in general, lead to the results obtained in § 2 (a) in which the non-localizability was expressed by the function $f(t-t')$.

This difference between two types of the non-local couplings results from the fact that the function $f(t-t')$ can not be replaced by some infinite order differential operator compatible with the condition (iii). In any case, the convergency of the perturbation method for the higher derivative coupling is worse than in the ordinary coupling and the infinite order derivative coupling may be inadequate to get rid of the so-called divergence difficulties. As was noted in the above discussion, the exceptional equations are the linear equations with constant coefficients. We can find some similar examples of these linear non-local couplings represented in terms of the infinite order derivatives, in the theories of servomechanisms and automatic control systems⁵⁾ and also in certain mathematical investigations in economics⁶⁾ and probability problems.⁷⁾

Aside from the convergency of the perturbation problem, the special solutions of (3.1) which were formally constructed in I and also may be obtained by the iteration method are equivalent to the general solutions of a second order differential equation, because they contain only two arbitrary constants. This can explicitly be shown as follows:

We can assume the initial conditions $q(0)=A$, $\dot{q}(0)=B$ at $t=0$, without loss of generality, because (3.1) is invariant under the time displacement transformation. Then, we can obtain the formal solution

$$q(t) = g(t; \epsilon, A, B). \quad (3.3)$$

By the differentiation with respect to t , we obtain also

$$\dot{q} = g_1(t; \epsilon, A, B), \quad (3.4)$$

$$\ddot{q} = g_2(t; \epsilon, A, B). \quad (3.5)$$

If ϵ is put equal to zero, (3.3), (3.4) reduce to the following relations:

$$q = A \cos \omega t + (B/\omega) \sin \omega t, \quad (3.3)'$$

$$\dot{q} = -A \omega \sin \omega t + B \cos \omega t.$$

Then it follows from the implicit function theorem that A and B can be expressed in terms of q and \dot{q} in the neighborhood of $\epsilon=0$. Inserting this expression into (3.5), we obtain the equivalent second order differential equation:

$$\begin{aligned} \ddot{q} &= g_2(t, \epsilon, A(q, \dot{q}, t); B(q, \dot{q}, t)) \\ &= h(\epsilon, q, \dot{q}). \end{aligned} \quad (3.6)$$

As was noted in I, the equation (3.1) corresponding to $n=2$ should be distinguished from

the equations corresponding to the values of n larger than 2. We can, in this case, prove the convergency of the iteration method, and the discussions on the characteristic features are similar to those of the ordinary couplings in which only q and \dot{q} are contained in the coupling terms, so far as the nearly linear solutions are concerned.

§ 4. The non-local couplings in field theories

In this section, we employ the following notations: Greek subscripts assume values ranging from 1 to 4, and repeated index is to be summed. The coordinate vector of a four dimensional point x_i is denoted by $x_{i,\mu}(x_i, it)^{*)}$. The real time coordinate $x_0 = (1/i)x_4 = t$ is also used. In particular, the four dimensional element of volume is defined as $(dx) = dx_0 dx_1 dx_2 dx_3$, and we define also a four-vector differential surface area: $d\sigma_\mu = (dx_2 dx_3 dx_0, dx_1 dx_3 dx_0, dx_1 dx_2 dx_0, dx_1 dx_2 dx_3/i)$.

(a) *The existence proof of iterative solutions*

We consider the equation

$$(\square - m^2)\phi(x) = -g \int_{-\infty}^{\infty} f(x, x_1, \dots, x_r) G(\phi(x), \phi(x_1), \dots, \phi(x_r), \partial_\mu \phi(x), \partial_\mu \phi(x_1), \dots, \partial_\mu \phi(x_r)) (dx_1) (dx_1) \dots (dx_r), \quad (4.1)$$

where g is a small parameter.

(4.1) can be written in the following integral form:

$$\phi(x) = \phi^{(0)}(x) + g \int J(x-x') \int_{-\infty}^{\infty} f(x', \dots, x_r) G(\phi(x'), \dots, \partial_\mu \phi(x_r)) (dx_1) \dots (dx_r) (dx') \quad (4.1)'$$

in which $\phi^{(0)}(x)$ satisfies the free equation: $(\square - m^2)\phi^{(0)} = 0$, and $J(x)$ is the Schwinger's Δ -function.⁸⁾

Theorem:

For the appropriate g and all t , (4.1) (or (4.1)') has the unique solution satisfying the initial conditions: $\phi(\hat{z}, t)|_{t=t_0} = R(\hat{z})$, $\phi_t(\hat{z}, t)|_{t=t_0} = S(\hat{z})$, on the three dimensional surface $t=t_0$; if the following conditions are satisfied.

$$(i) \quad If |\phi(x)| < A_1, \quad |\partial_\mu \phi(x)| < A_1 \quad \text{for all } x,$$

$$G \text{ is bounded, or } |G| < M.$$

$$(ii) \quad f(x', x_1, \dots, x_r) \text{ satisfies the following conditions,}$$

$$\int_{-\infty}^{\infty} \dots \int |f(x', t-r, x_1, \dots, x_r)/r| (dx_1) \dots (dx_r) (dx')$$

$$+ \int_{-\infty}^{\infty} \int |F(x-x') f(x', x_1, \dots, x_r)| (dx_1) \dots (dx_r) (dx') < 4\pi\alpha,$$

*) The unit $\hbar=c=1$, is used.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f_t(\mathbf{x}', t-r, x_1, \dots, x_r) / r| (dx_1) \dots (dx_r) (d\mathbf{x}')$$

$$+ \int_{-\infty}^{\infty} |F_t(x-x') f(x', x_1, \dots, x_r)| (dx_1) \dots (dx_r) (d\mathbf{x}') < 4\pi\alpha,$$

$$f(\mathbf{x}', t', x_1, \dots, x_r) \rightarrow 0 \quad (\mathbf{x}' \rightarrow \pm \infty), \quad r = \sqrt{|\mathbf{x} - \mathbf{x}'|^2},$$

$$F(x-x') = \begin{cases} m J(m((t-t')^2 - r^2))^{1/2} / \{(t-t')^2 - r^2\}^{1/2} & \text{for } (t-t')^2 - r^2 \geq 0 \\ 0 & \text{for } (t-t')^2 - r^2 < 0 \end{cases}$$

in which α is a positive definite number.

(iii) The Lipschitz condition for G :

$$|G(\bar{y}_1, \bar{y}_2, \dots; \bar{z}_1, \bar{z}_2, \dots) - G(y_1, y_2, \dots; z_1, z_2, \dots)| \\ \leq K(|\bar{y}_1 - y_1| + |\bar{y}_2 - y_2| + \dots + |\bar{z}_1 - z_1| + \dots).$$

(iv) $R(\hat{\xi})$ and $S(\hat{\xi})$ are continuous and differentiable and satisfy the following condition,

$$E = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{(\text{grad } R)^2 + S^2 + m^2 R^2\} d\hat{\xi} < A = A_1/2.$$

Proof:

Solving (4.1)' by the iteration method, we have, in general,

$$\phi^{(\lambda)}(x) = \phi^{(0)}(x)$$

$$+ g \int_{\sigma(t_0)}^{\sigma(t)} \Delta(x-x') \int f(x', x_1, \dots, x_r) G(\phi^{(\lambda-1)}(x'), \phi^{(\lambda-1)}(x_1), \dots; \partial_\mu \phi^{(\lambda-1)}(x'), \partial_\mu \phi^{(\lambda-1)}(x_1), \dots) \\ \cdot (dx') (dx_1) \dots (dx_r). \quad (4.2)$$

In the above equation, $\phi^{(0)}$ satisfies the free equation, and consequently the following relation holds for all t ,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{(\text{grad } \phi^{(0)})^2 + (\partial_t \phi^{(0)})^2 + m^2 (\phi^{(0)})^2\} (d\mathbf{x}) = \text{constant}.$$

Moreover, $\phi^{(0)}(x, t)$ and $\partial \phi^{(0)} / \partial t$ are equal to $R(x)$ and $S(x)$ respectively at $t = t_0$.

We have, then, the relations

$$|\text{grad } \phi^{(0)}| < A, \quad |\partial_t \phi^{(0)}| < A, \quad |\phi^{(0)}| < A \quad (4.3)$$

for all t .

From (4.2) and (4.3), we obtain the inequality

$$|\phi^{(1)}(x)| < A + |g|\alpha M$$

or, if $g < A/\alpha M$, we have the relation: $|\phi^{(1)}(x)| < A_1$.

In quite a same way, we can prove also

$$|\partial_\mu \phi^{(1)}(x)| < A_1.$$

Applying the mathematical induction we have, in general, the inequality :

$$|\phi^{(\lambda)}| < A_1, \quad |\partial_\mu \phi^{(\lambda)}| < A_1$$

for all x and λ .

The uniform convergency of the above iteration method can be seen as follows :

At the first, we can obtain from (ii) and (iii) the relations :

$$|\phi^{(1)} - \phi^{(0)}| < a,$$

$$|\partial_\mu \phi^{(1)} - \partial_\mu \phi^{(0)}| < a,$$

in which a is given by $a = |g|\alpha M$.

Applying the mathematical induction we can, in general, prove that the following relations hold for all x and λ ,

$$|\phi^{(\lambda)}(x) - \phi^{(\lambda-1)}(x)| < \rho^{\lambda-1} a,$$

where ρ is given by $\rho = 2K(r+1)|g|\alpha$.

Then, finally we obtain the result that $\phi^{(\lambda)}(x)$ converges uniformly for $\lambda \rightarrow \infty$ and $\lim_{\lambda \rightarrow \infty} \phi^{(\lambda)}(x)$ gives the solution $\phi(x)$ satisfying the prescribed initial conditions, if g satisfies the condition :

$$|g| < \text{Min}(A/\alpha M, 1/2K(r+1)\alpha). \quad (4.4)$$

The above proof has referred to the flat surface $t = \text{const.}$ However, the generalized discussions referring to the arbitrary space-like surface may easily be done in a similar way.

The condition (ii) for the form factor is not compatible with the condition of the translation-invariancy in the same meaning as was explained in the mechanical non-local theories. Instead of using this type of the form factor, we shall employ the adiabatic processes for space as well as for time. In the following discussions, the form factor then depends only on one or more coordinate differences of pairs of points.

(b) *The equivalence to a differential equation and the propagation character*

The solutions obtained in (a) depend on the two arbitrary functions $R(\xi)$ and $S(\xi)$ in a way analogous to that in the second order hyperbolic differential equations.

Contrary to the circumstance in the mechanical theories, however, it can not be proved in general that those solutions are equivalent to the general solutions of a second order differential equation. This will be shown in the following discussions.

Using the result obtained in (a), we can represent the field variable at an arbitrary space-time point x_1 , in terms of the field variables on the arbitrary space-like surface, not containing the point x' .

$\phi(x_k)$ in the right hand side of (4.1), then, can be expressed in terms of $\phi(\hat{z})$ and $\phi_v(\hat{z})$ by the equation

$$\phi(x_k) = \lim_{\lambda \rightarrow \infty} \phi^{(\lambda)}(x_k),$$

$$\phi^{(\lambda)}(x_k) = \phi(x_k, \sigma) + g \int_{\sigma(x)}^{\sigma(x_k)} \Delta(x_k - x') \int f(x', x_1, \dots, x_r).$$

$$\cdot G(\phi^{(\lambda-1)}(x'), \partial_\mu \phi^{(\lambda-1)}(x'), \dots, \phi^{(\lambda-1)}(x_r), \partial_\mu \phi^{(\lambda-1)}(x_r)) (dx') (dx_1) \dots (dx_r),$$

$$\phi(x_k, \sigma) = \int \{ \Delta(x_k - \xi) \phi_\nu(\xi) - \Delta_\nu(x_k - \xi) \phi(\xi) \} d\sigma_\nu(x_k, \xi) \quad (4.5)$$

in which both ξ and x are on the same space-like surface $\sigma(x, \xi)$, and the subscript ν denotes the differentiation with respect to the normal component of that surface.⁹⁾

Inserting these expressions of $\phi(x_k)$, $\partial_\mu \phi(x_k)$, etc., into (4.1), we can transform it into the following form,

$$(\square - m^2) \phi(x) = \int F(\phi(\xi), \partial_\nu \phi(\xi)) d\sigma(x, \xi) \quad (4.6)$$

in which F is the functional of $\phi(\xi)$ and $\partial_\nu \phi(\xi)$ and is given as the result of the integrations with respect to x_1, x_2, \dots, x_r , and the integral with respect to ξ should, in general, be extended multiply over all the space-like surface σ .

(4.6) is equivalent to (4.1) when a development in the neighborhood of $g=0$ is discussed, however, it can not be equivalent to any differential equation.

In (4.6), it may easily be seen that the non-localizability in the time-like direction disappears, but the space-like non-localizability remains because of the integral over the surface σ and this involves the infinite domain of dependence.

It should also be noted that in (4.6) there does not exist any connection between the discontinuity surface of higher derivatives and the domain of dependence, contrary to the case of the partial differential equations. In fact, the discontinuity surface of the second order derivatives in (4.6) may be given by the Minkowski's light cone and it may be due to the existence of this discontinuity surface that both ϕ and $\partial_\nu \phi$ may arbitrarily be prescribed on the initial space-like surface.

Example 1

We consider the linear equation

$$(\square - m^2) \phi(x) = g \int f(x - x') \phi(x') (dx')^*. \quad (4.7)$$

Applying the iteration method, we obtain⁹⁾

$$\phi^{(\lambda)}(x_k) = \phi(x_k, \sigma) + g \int_{\sigma(x)}^{\sigma(x_k)} \Delta(x_k - x') \int f(x' - x'') \phi^{(\lambda-1)}(x'') (dx') (dx''), \quad (4.7)'$$

$$\phi(x_k, \sigma) = \int \{ \Delta(x' - \xi) \phi_\nu(\xi) - \Delta_\nu(x' - \xi) \phi(\xi) \} d\sigma_\nu(x, \xi).$$

Inserting (4.7)' into (4.7), we get the following equation,

$$(\square - m^2) \phi(x) = g \int f(x - x') \int \{ \Delta(x' - \xi) \phi_\nu(\xi) - \Delta_\nu(x' - \xi) \phi(\xi) \} d\sigma_\nu(x) (dx')$$

$$+O(g^2). \quad (4.8)$$

If the form factor $f(x)$ is the function of $x_\mu x_\mu$ only, it results immediately that

$$\begin{aligned} \int f(x-x') \Delta(x'-\xi, x'_0-x_0) (dx') &= 0, \\ \int f(x-x') \partial_t \Delta(x'-\xi, x'_0-x_0) (dx') &= (2\pi)^2 F(-m^2) \delta(x-\xi), \end{aligned}$$

in which $f(x)$ is defined by $f(x) = \int F(k_\mu k_\mu) \exp(ik_\mu x_\mu) (dk)$.

The equation (4.8) then reduces to

$$(\square - m^2) \phi(x) = g(2\pi)^2 F(-m^2) \phi(x) + O(g^2). \quad (4.8)'$$

Following the similar procedure, we can prove that (4.8) is equivalent to a differential equation up to the infinite order in g . On the other hand, we can solve (4.7) exactly and from among these general solutions we can always choose the special solution which can be expressed in terms of the power series in g . It can easily be seen then that the special solution thus obtained is just the same as the general solution of (4.8)'.

Example 2

Let us consider the equation derived from the Lagrangian L ,

$$\begin{aligned} \bar{L} &= (1/2) \int \{ (\partial_\mu \phi(x))^2 + m^2 \phi^2(x) \} (dx) \\ &+ (1/4) g \int \phi^2(x) f(x-x') \phi^2(x') (dx') (dx), \end{aligned} \quad (4.9)$$

in which the form factor $f(x)$ satisfies the same condition as was assumed in example 1.

Euler's equation then becomes

$$(\square - m^2) \phi(x) = g \phi(x) \int f(x-x') \phi^2(x') (dx'). \quad (4.10)$$

If we refer to the special Lorentz system: $t = \text{const.}$ instead of the arbitrary space-like surface $\sigma(x)$, the expression for $\phi(x', \sigma)$ corresponding to (4.7)' reduces to

$$\begin{aligned} \phi(x', x'_0, x_0) &= \int \{ -\Delta(x'-x'', x'_0-x_0) \partial_t \phi(x'', x_0) - \partial_t \Delta(x'-x'', x'_0-x_0) \cdot \phi(x'', x_0) \} \\ &\quad \cdot dx''. \end{aligned} \quad (4.7)''$$

Inserting (4.7)'' into (4.10) and employing the abbreviations:

$$\begin{aligned} &\int f(x-x') \Delta(x'-x'', x'_0-x_0) \Delta(x'-x''', x'_0-x_0) (dx') \\ &= (1/8\pi^2) \int \{ F((\mathbf{k}' + \mathbf{k}'')^2 - (k'_0 - k''_0)^2) - F((\mathbf{k}' + \mathbf{k}''')^2 - (k'_0 + k''_0)^2) \} \\ &\quad \cdot \exp i \{ \mathbf{k}'(x-x'') + \mathbf{k}''(x-x''') \} \cdot d\mathbf{k}' d\mathbf{k}'' / k'_0 k''_0 \end{aligned}$$

$$= G^{(1)}(\mathbf{x} - \mathbf{x}'', \mathbf{x} - \mathbf{x}'''),$$

$$\int f(\mathbf{x} - \mathbf{x}') \Delta_i(\mathbf{x}' - \mathbf{x}'', x'_0 - x_0) \Delta_i(\mathbf{x}' - \mathbf{x}''', x'_0 - x_0) (d\mathbf{x}') \quad (4 \cdot 11)$$

$$= (1/8 \pi^2) \int \{ F((\mathbf{k}' + \mathbf{k}'')^2 - (k'_0 + k''_0)^2) - F((\mathbf{k}' + \mathbf{k}''')^2 - (k'_0 - k''_0)^2) \} \cdot$$

$$\cdot \exp i \{ \mathbf{k}'(\mathbf{x} - \mathbf{x}'') + \mathbf{k}''(\mathbf{x} - \mathbf{x}') \} d\mathbf{k}' d\mathbf{k}''$$

$$= G^{(2)}(\mathbf{x} - \mathbf{x}'', \mathbf{x} - \mathbf{x}'''),$$

we get, instead of (4.10), the following equation in the special reference system: $t = \text{constant}$,

$$\begin{aligned} (\square - m^2) \phi(x) = & g \phi(x) \int \{ G^{(1)}(\mathbf{x} - \mathbf{x}'', \mathbf{x} - \mathbf{x}''') \partial_i \phi(\mathbf{x}'', x_0) \partial_i \phi(\mathbf{x}''', x_0) \\ & + G^{(2)}(\mathbf{x} - \mathbf{x}'', \mathbf{x} - \mathbf{x}''') \phi(\mathbf{x}'', x_0) \phi(\mathbf{x}''', x_0) \} d\mathbf{x}'' d\mathbf{x}''' \\ & + O(g^2). \end{aligned} \quad (4 \cdot 12)$$

Example 3

We consider the charged scalar fields ψ , ψ^* and a neutral scalar field u with non-local interactions of Kristensen-Møller's¹⁰⁾ type:

$$\begin{aligned} \bar{L} = & \int \{ -(\partial_\mu \psi^*(x) \partial_\mu \psi(x) + M^2 \psi^*(x) \psi(x)) - (1/2) (\partial_\mu u(x) \cdot \partial_\mu u(x) + m^2 u^2(x)) \} (dx) \\ & + g \iint \psi^*(x) f(x, x', x'') u(x') \psi(x'') (dx) (dx') (dx''), \end{aligned} \quad (4 \cdot 13)$$

in which the form factor satisfies the Hermitian condition $f(x', x'', x) = f^*(x, x', x'')$ besides the translation invariancy.

Field equations then become

$$(\square - M^2) \psi(x) = -g \int u(x') \psi(x'') f(x, x', x'') (dx') (dx''), \quad (4 \cdot 14a)$$

$$(\square - m^2) u(x) = -g \int \psi^*(x') \psi(x'') f(x', x, x'') (dx') (dx''), \quad (4 \cdot 14b)$$

$$(\square - M^2) \psi^*(x) = -g \int \psi^*(x') u(x'') f(x', x'', x) (dx') (dx''). \quad (4 \cdot 14c)$$

The iterative solutions of the above equations are given by

$$u(x') = \lim_{\lambda \rightarrow \infty} u^{(\lambda)}(x'),$$

$$u^{(\lambda)}(x') = u(x', \sigma) + g \int_{\sigma(x')}^{\sigma(x')} \mathcal{A}_m(x' - \gamma) \int \psi^{(\lambda-1)}(x') \psi^{(\lambda-1)}(x'') f(x', \gamma, x'') (dx'') (dx') (d\gamma), \quad (4 \cdot 15a)$$

$$\psi(x') = \lim_{\lambda \rightarrow \infty} \psi^{(\lambda)}(x'),$$

$$\psi^{(\lambda)}(x') = \psi(x', \sigma) + g \int_{\sigma(x')}^{\sigma(x')} \mathcal{A}_M(x' - \gamma) \int u^{(\lambda-1)}(x') \psi^{(\lambda-1)}(x'') f(\gamma, x', x'') (dx') (dx'') (d\gamma), \quad (4 \cdot 15b)$$

in which, corresponding to (4.7), $u(x', \sigma)$ and $\psi(x', \sigma)$ are given by

$$u(x', \sigma) = \int \{ \mathcal{A}_m(x' - \xi) u_v(\xi) - \mathcal{A}_{m,v}(x' - \xi) u(\xi) \} d\sigma_v(x, \xi), \quad (4.16a)$$

$$\psi(x', \sigma) = \int \{ \mathcal{A}_M(x' - \xi) \psi_v(\xi) - \mathcal{A}_{M,v}(x' - \xi) \psi(\xi) \} d\sigma_v(x, \xi). \quad (4.16b)$$

The expression for $\psi^*(x)$ can be obtained by making the complex conjugation, (Hermite conjugation, if the quantized quantities are concerned) of $\psi(x)$.

Inserting (4.16) into the right hand side of (4.14) and referring to the special coordinate system $t = \text{const.}$, we get, instead of (4.14), the equation:

$$\begin{aligned} (\square - M^2) \psi(x) = & -g \int \{ G_{mM}^{(1)}(x, \xi_1, \xi_2) u_t(\xi_1, x_0) \psi_t(\xi_2, x_0) \\ & + G_{mM}^{(2)}(x, \xi_1, \xi_2) u(\xi_1, x_0) \psi(\xi_2, x_0) \\ & - G_{mM}^{(3)}(x, \xi_1, \xi_2) u_t(\xi_1, x_0) \psi(\xi_2, x_0) \\ & - G_{mM}^{(4)}(x, \xi_1, \xi_2) u(\xi_1, x_0) \psi_t(\xi_2, x_0) \} d\xi_1 d\xi_2 + O(g^2), \end{aligned} \quad (4.17a)$$

$$\begin{aligned} (\square - m^2) u(x) = & -g \int \{ G_{mM}^{(1)}(\xi_1, x, \xi_2) \psi_t^*(\xi_1, x_0) \psi_t(\xi_2, x_0) \\ & + G_{mM}^{(2)}(\xi_1, x, \xi_2) \psi^*(\xi_1, x_0) \psi(\xi_2, x_0) \\ & - G_{mM}^{(3)}(\xi_1, x, \xi_2) \psi^*(\xi_1, x_0) \psi_t(\xi_2, x_0) \\ & - G_{mM}^{(4)}(\xi_1, x, \xi_2) \psi_t^*(\xi_1, x_0) \psi(\xi_2, x_0) \} d\xi_1 d\xi_2 \\ & + O(g^2) \end{aligned} \quad (4.17b)$$

in which we introduced, for convenience sake, the quantities $G_{mM}^{(1)}(x, \xi_1, \xi_2)$ etc., defined by

$$\begin{aligned} G_{ab}^{(1)}(x, \xi_1, \xi_2) &= \int \mathcal{A}_a(x' - \xi_1) \mathcal{A}_b(x'' - \xi_2) f(x, x', x'') (dx') (dx'') \\ G_{ab}^{(2)}(x, \xi_1, \xi_2) &= \int \mathcal{A}_{a,t}(x' - \xi_1) \mathcal{A}_{b,t}(x'' - \xi_2) f(x, x', x'') (dx') (dx'') \\ G_{ab}^{(3)}(x, \xi_1, \xi_2) &= \int \mathcal{A}_a(x' - \xi_1) \mathcal{A}_{b,t}(x'' - \xi_2) f(x, x', x'') (dx') (dx'') \\ G_{ab}^{(4)}(x, \xi_1, \xi_2) &= \int \mathcal{A}_{a,t}(x' - \xi_1) \mathcal{A}_b(x'' - \xi_2) f(x, x', x'') (dx') (dx'') \\ G_{ab}^{(1)}(\xi_1, x, \xi_2) &= \int \mathcal{A}_a(x' - \xi_1) \mathcal{A}_b(x'' - \xi_2) f(x', x, x'') (dx') (dx'') \\ &\dots\dots\dots \end{aligned}$$

$$\text{etc., and } (\square - a^2) \mathcal{A}_a = 0 \quad \mathcal{A}_{a,t} = \partial \mathcal{A}_a / \partial t,$$

(in the above definitions, both points ξ_1, ξ_2 are on the same surface $t = \text{const.}$, i.e., $\xi_{10} = \xi_{20}$).

The equation for $\psi^*(x)$ can be obtained from (4.17a) by complex conjugation.

These equations (4·17) are equivalent to the original equations (4·14) whenever a development in the neighborhood of $g=0$ is concerned. However, they are not equivalent to a system of differential equations.

§ 5. Concluding remarks

From the foregoing investigations in this series, we have seen that in the c -number theory it is possible to construct the solutions of integro-differential equation by the perturbation method, provided that the form factor satisfies some conditions.*¹⁾ Moreover, it was shown that in the mechanical theories the integro-differential equation of this type was entirely equivalent to the differential equation of the same order as that of the free equation so far as the perturbation theory is concerned. If we could find the Lagrangian from which the equivalent equation can be derived directly, it would be possible to construct the Hamiltonian in the usual way, and the quantization of the system might be performed. In fact, according to the above procedure, we easily quantize the linear equation considered in the example 1 in § 2. In the mechanical theories, it seems to be probable that even in the systems with non-linear interaction, there exists a wide class of equivalent equations derivable from the corresponding Lagrangians.

However, in the field theories with non-local interactions, the equivalent equations are, in general, far more complicated and essentially normal dependent. Because of this character it seems to be doubtful whether there exist, in general, Lagrangians from which the equivalent field equations may be derived directly.

Hitherto the quantization of the system with non-local interactions has been attempted by several authors on the basis of the perturbation theory.¹¹⁾¹²⁾ However, in their theories it has never been explicitly written down in the mathematical formulae what motion should be quantized among the various motions contained in the original equation. This has made the physical meaning of their results obscure. Now, in this paper, the motions to be quantized are clearly expressed by our equivalent equations. In other words, these equivalent equations may be considered as the fundamental equations so far as the perturbation theory is concerned. Therefore, if the quantization is possible by some procedure, the quantized equations in the Heisenberg-picture are to be connected with these equivalent equations.**²⁾ In this paper, on the other hand, the discussions on these equivalent equations in the field theories were always referred to the space-like surfaces, and the initial value problems on various surfaces other than the space-like ones were not discussed.

Especially, the initial value problem on the Minkowski's light cone may correspond to the characteristic initial value problem in the hyperbolic differential equation.

Detailed investigations of Cauchy problems on those surfaces which are not necessarily space-like may present some suggestion on the description of non-localized action.

*) These conditions are severer than that of the "normal class".⁽¹²⁾

**) The relation between the solutions obtained in this paper and the canonical theory will be discussed elsewhere in more detail.

As for the higher derivative couplings, we could not conclude anything about the existence of the solution expressible in terms of the power series in the coupling constant, and we have seen that the non-local coupling expressed in terms of the derivative of the infinite order is rather worse than the ordinary coupling on the point of the convergency of the perturbation. In any case, the singular perturbation problems of the equations considered in this paper could not be solved and there remains a problem for mathematicians.

In the derivative couplings of the same order in field theories as discussed in I, there existed the remarkable problem connected with the propagation character of the wave field. We are, however, not so interested in the general theory of convergency for these systems and it should be solved case by case. The only note which we want to cite here is that the perturbational calculation on those systems should carefully be treated, if the characteristics of the original equation is different from that of the free equation and the disturbance concentrating near the wave front is in question. We can find some examples of the interaction of this type in the hydrodynamical approximation¹³⁾ of the ordinary wave equation.

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He-He Repulsive Potential, I

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The repulsive potential between two normal He atoms has been calculated theoretically. In order to include the effect of deformation of the atomic orbitals, a wave function similar to those used by Inui and Nordsieck (Ref. 1 and 2) in their calculations of the H_2 molecule is composed, from which the repulsive potential is obtained. The result is:

$$V(R) = 4.08755 \times 10^2 e^{-4.46893R} \text{ e.v. } (0.53 \text{ \AA} < R < 1.06 \text{ \AA}).$$

Comparison with potential curves obtained by various theories and by experiment is made. Discussions and illustrations concerning the nature of AO's here obtained are given.

§ 1. Introduction

Generally speaking, it is not a easy task to determine theoretically or experimentally the repulsive potential between two neutral atoms or molecules. But as to the repulsive potential between two normal He atoms, a rather accurate potential is obtained from the experiment of Amdur and Harkness.³⁾ On the other hand, purely theoretical investigations for two normal He system were performed hitherto by many authors. Slater,⁴⁾ N. Rosen⁵⁾ and P. Rosen⁶⁾ treated this problem by the valence bond (VB) method. Griffing and Wehner⁷⁾ calculated by the antisymmetrized molecular orbital (ASMO) theory, which is equivalent in this case to the simple Heitler-London (HL) theory. Margenau and P. Rosen⁸⁾ has also given an interesting discussion on this problem. These authors adopted various kinds of functions for normal He atom (the VB theory) or molecular orbital (the ASMO theory). The wave functions used and the energies of two He atoms at infinite internuclear distance calculated by these functions are shown in Table I, columns 2 and 3, respectively. In the reduction of the energy integrals, Slater and N. Rosen neglected higher order permutations, whereas Margenau and P. Rosen, and P. Rosen took into account all permutations.

But these authors have not taken into account explicitly the effect of the deformation of AO's due to the approach of another atom. On the other hand, Inui¹⁾ and Nordsieck²⁾, in their treatment of H_2 molecule, considered this effect by means of their modified AO's

Table I.

Authors	Wave Function used (r in a.u.)	$E_{R=\infty}$ in a.u.	Potential Functions $V(R)$ in e.v. (R in A)	Method of Calculation
Slater ⁴⁾	$\varphi_{\text{He}} = 1.392e^{-2(r_1+r_2)+0.5r_{12}}$ $\cdot e^{0.0107(r_1^2+r_2^2)}$ for $r_1, r_2 < 3$ $\varphi_{\text{He}} = 1.241e^{-2r_1-1.344r_2r_2^{-0.255}}$ $\cdot (1+0.0707/r_2+\dots)$ for $\begin{matrix} r_1 < r_2 \\ r_2 > 3 \end{matrix}$	a)	$4.81e^{-4.5875R} \times 10^2$ (Table I.1)	Interaction operator. Without higher order permutation.
N. Rosen ⁵⁾	Simple exponential form ^{b)}	c)	d)	As above.
P. Rosen ⁶⁾	$\varphi_{\text{He}} = \left(\frac{Z_1^3}{\pi}\right)^{\frac{1}{2}} e^{-Z_1 r_1} \left(\frac{Z_2^3}{\pi}\right)^{\frac{1}{2}} e^{-Z_2 r_2}$ $Z_1 = 2.15, Z_2 = 1.19$	-5.7508	$5.77e^{-4.40R} \times 10^2$ (Table I.2)	V.B.
Margenau & P. Rosen ⁸⁾	$\varphi_{\text{He}} = \left(\frac{Z_1^3}{\pi}\right)^{\frac{1}{2}} \left(\frac{Z_2^3}{\pi}\right)^{\frac{1}{2}} (2+2t_0^0)^{-\frac{1}{2}}$ $\cdot (e^{-Z_1 A_1 - Z_2 A_2} + e^{-Z_1 A_2 - Z_2 A_1})$	-5.7508	e)	V.B.
Griffing & Wehner ⁷⁾	M.O. $\varphi_{12} = N_{12}(\chi_a \pm \chi_b),$ $\chi_a = \left(\frac{Z^3}{\pi}\right)^{\frac{1}{2}} e^{-Z r_a} \quad Z = 27/16$	-5.69531	f)	A.S.M.O. (=H.L.)
Present paper	$\varphi_{\text{He}} = \chi_a(1)\chi_a(2)$ $\chi_a(1) = N_a e^{-(\alpha+\beta/R)r_{a1}}$ $\cdot N_b e^{-(\alpha-\beta/R)r_{b1}}$	-5.69531	$4.08755e^{-4.46393R} \times 10^2$ (Table I.3)	V.B.
Experiment Observed ³⁾		-5.80672 ¹⁰⁾	$2.884/R^{1.9}$ for $0.52 < R < 1.02$ $4.713/R^{5.94}$ for $1.27 < R < 1.59$ (Table I.4)	

a), c) : The corresponding values were not given in Ref. 4), 5).

b) : The explicit expression was not given in Ref. 5).

d), e) : $V(R)$ was given only as the numerical table in Ref. 5), 8).

f) : $V(R)$ was given only as the numerical table and graphical form.

(*vide infra*) and obtained good results. Hence we have applied their methods to the He-He system and investigated the aspect of deformation of AO's due to the existence of the neighbouring atom.

§ 2. Calculation of energy integral

We shall calculate the energy of the system composed of two helium atoms by making use of the following wave function :

$$\psi = \begin{vmatrix} a(1)\alpha(1) & a(1)\beta(1) & b(1)\alpha(1) & b(1)\beta(1) \\ a(2)\alpha(2) & a(2)\beta(2) & b(2)\alpha(2) & b(2)\beta(2) \\ a(3)\alpha(3) & a(3)\beta(3) & b(3)\alpha(3) & b(3)\beta(3) \\ a(4)\alpha(4) & a(4)\beta(4) & b(4)\alpha(4) & b(4)\beta(4) \end{vmatrix}. \quad (1)$$

Here α and β are the usual spin functions, and, $a(i)$ and $b(i)$ are the modified AO's introduced by Inui¹⁾ and Nordsieck²⁾, namely

$$a(i) = e^{-\alpha \xi_i - \beta \eta_i}, \\ b(i) = e^{-\alpha \xi_i + \beta \eta_i}, \quad (i=1, 2, 3, 4) \quad (2)$$

where ξ_i and η_i are the spheroidal coordinates defined by

$$\xi_i = (r_{ai} + r_{bi})/R, \\ \eta_i = (r_{ai} - r_{bi})/R, \quad (3)$$

where R denotes the internuclear distance and r_{ai} and r_{bi} are the distances of the i -th electron from the nuclei a and b , respectively.

Now in terms of the atomic units, the Hamiltonian of our system is given by

$$H = -1/2 \cdot \sum_i \Delta_i - 2 \sum_i (1/r_{ai}) - 2 \sum_i (1/r_{bi}) + \sum_{i>j} (1/r_{ij}) + 4/R \\ = 4/R + \sum_i H(i) + \sum_{i>j} V_{ij}, \quad (4)$$

where r_{ij} is the distance between the i -th and j -th electrons, $V_{ij} = 1/r_{ij}$, Δ_i is the Laplace operator with respect to the coordinates of the i -th electron, namely,

$$\Delta_i \equiv \frac{4}{R^2(\xi_i^2 - \eta_i^2)} \left[\frac{\partial}{\partial \xi_i} \left\{ (\xi_i^2 - 1) \frac{\partial}{\partial \xi_i} \right\} + \frac{\partial}{\partial \eta_i} \left\{ (1 - \eta_i^2) \frac{\partial}{\partial \eta_i} \right\} + \frac{(\xi_i^2 - \eta_i^2)}{(\xi_i^2 - 1)(1 - \eta_i^2)} \frac{\partial^2}{\partial \varphi^2} \right],$$

and

$$H_i = -\frac{1}{2} \Delta_i - \frac{8\xi_i}{\xi_i^2 - \eta_i^2} \cdot \frac{1}{R}.$$

Our problem is to find the appropriate values of the parameters α and β which minimize the total energy

$$E(\alpha, \beta; R) = \int \bar{\psi} H \psi d\tau / \int \bar{\psi} \psi d\tau, \quad (5)$$

for fixed values of R .

In order to reduce eq. (5), we define the following integrals,

$$S = \int a(i) a(i) dv_i = \int b(i) b(i) dv_i,$$

$$T = \int a(i) b(i) dv_i = \int b(i) a(i) dv_i,$$

$$\begin{aligned}
M &= \int a(i) H_i a(i) dv_i = \int b(i) H_i b(i) dv_i, \\
N &= \int a(i) H_i b(i) dv_i = \int b(i) H_i a(i) dv_i, \\
K &= \iint a(i) a(j) V_{ij} a(i) a(j) dv_i dv_j = \iint b(i) b(j) V_{ij} b(i) b(j) dv_i dv_j, \\
L &= \iint a(i) a(j) V_{ij} a(i) b(j) dv_i dv_j = \iint a(i) a(j) V_{ij} b(i) a(j) dv_i dv_j \\
&= \iint a(i) b(j) V_{ij} a(i) a(j) dv_i dv_j = \iint b(i) a(j) V_{ij} a(i) a(j) dv_i dv_j \\
&= \iint b(i) b(j) V_{ij} b(i) a(j) dv_i dv_j = \iint b(i) b(j) V_{ij} a(i) b(j) dv_i dv_j \\
&= \iint b(i) a(j) V_{ij} b(i) b(j) dv_i dv_j = \iint a(i) b(j) V_{ij} b(i) b(j) dv_i dv_j, \\
C &= \iint a(i) b(j) V_{ij} a(i) b(j) dv_i dv_j = \iint b(i) a(j) V_{ij} b(i) a(j) dv_i dv_j, \\
J &= \iint a(i) b(j) V_{ij} b(i) a(j) dv_i dv_j = \iint b(i) a(j) V_{ij} a(i) b(j) dv_i dv_j \\
&= \iint a(i) a(j) V_{ij} b(i) b(j) dv_i dv_j = \iint b(i) b(j) V_{ij} a(i) a(j) dv_i dv_j, \tag{6}
\end{aligned}$$

where

$$dv_i = (R/2)^3 (\zeta_i^2 - \eta_i^2) d\zeta_i d\eta_i d\phi_i.$$

In terms of these integrals the total energy (5) can be expressed as follows,

$$\begin{aligned}
E(\alpha, \beta; R) &= \int \bar{\psi} H \psi d\tau / \int \bar{\psi} \psi d\tau \\
&= \frac{1}{\int \bar{\psi} \psi d\tau} \left\{ \frac{4}{R} \int \bar{\psi} \psi d\tau + \sum_i \int \bar{\psi} H(i) \psi d\tau + \sum_{i>j} \int \bar{\psi} V_{ij} \psi d\tau \right\} \\
&= \frac{4}{R} + \frac{4(MS - NT)}{S^2 - T^2} + \frac{2(C - J)}{S^2 - T^2} + \frac{2\{(K + C)S^2 - 4LST + 2JT^2\}}{(S^2 - T^2)^2}. \tag{7}
\end{aligned}$$

The derivation of eq. (7) is given in Appendix I. The integrals (6) are functions of α , β and R , and can be computed by the use of Kotani, Amemiya and Simose's table⁶⁾ (KAS's table) for fixed values of α , β and R . The method of calculations is shown in Appendix II. Thus the total energy $E(\alpha, \beta; R)$ can be evaluated for fixed values of α , β and R by means of eqs. (7), (A·3) and KAS's table. If we put $\alpha = \beta = R(2 - 5/16)/2$, our treatment reduces, as is easily seen, into the simple Heitler-London method or the ASMO treatment of Griffing and Wehner.⁷⁾

§ 3. Minimization of energy

The energy expression (7) can be shown to have the form :

$$E(R; \alpha, \beta) = -A(\alpha, \beta)/R + B(\alpha, \beta)/R^2 \quad (8)$$

where $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are certain functions of parameters α and β given by eqs. (7), (A.3) and (A.4). But the functional forms of $A(\alpha, \beta)$ and $B(\alpha, \beta)$ are so complicated that it is not practical to determine the values of α and β by an analytical procedure. Accordingly we have calculated the numerical values of $A(\alpha, \beta)$ and $B(\alpha, \beta)$ for forty nine pairs of the parameters α, β as shown in Tables II and III.

Table II. $A(\alpha, \beta)$

$\beta \backslash \alpha$	0.750	0.875	1.000	1.125	1.250	1.375	1.500	1.625	1.750
0.750	11.02297	12.75459	14.31919						
0.875	11.10243	12.85892	14.44924						
1.000	11.19071	12.97524	14.59466	16.06829					
1.125		13.10163	14.75317	16.25953	17.63720				
1.250		13.23596	14.92225	16.46411	17.87751	19.17647	20.37332		
1.375			15.09914	16.67885	18.13045	19.46753	20.70207		
1.500					18.39224	19.76957	21.04404	22.22593	
1.625						20.07833	21.39450	22.61760	23.75648
1.750						20.38972	21.74886	23.01457	24.19547
1.875							22.10291	23.41214	24.63608
$\alpha = \beta$	2.000	2.125	2.250	2.375	2.500	2.625	3.250	3.500	3.750
$A(\alpha, \beta)$	27.38043	28.99100	30.61382	32.24808	33.89253	35.54580	43.89830	47.26215	50.63112

Table III. $B(\alpha, \beta)$

$\beta \backslash \alpha$	0.750	0.875	1.000	1.125	1.250	1.375	1.500	1.625	1.750
0.750	6.80362	8.53426	10.33320						
0.875	6.87146	8.62047	10.43917						
1.000	6.96398	8.73681	10.58078	12.48049					
1.125		8.88759	10.76301	12.69551	14.67349				
1.250		9.07609	10.98971	12.96171	14.98038	17.03681	19.12417		
1.375			11.26323	13.28184	15.34824	17.45342	19.59042		
1.500					15.77815	17.93924	20.13301	22.35393	
1.625						18.49333	20.75095	23.03644	25.34533
1.750						19.11273	21.44107	23.79795	26.17884
1.875							22.19839	24.63313	27.09244
$\alpha = \beta$	2.000	2.125	2.250	2.375	2.500	2.625	3.250	3.500	3.750
$B(\alpha, \beta)$	33.22717	37.14905	41.34216	45.80825	50.54738	55.55265	84.61624	93.05481	112.53518

From these data, we obtained the minimum value of the energy and the corresponding values of the parameters α and β for R of 1 (0.2) 2 and 4 a.u. We shall denote these values as E_R , α_R and β_R . The method of determination of these values is explained in detail in Appendix III. The values of α_R , β_R and E_R for $R=1.0$ (0.2) 2.0 and 4.0 a.u. are given in Table IV.

Table IV.

R in a.u.	1.0	1.2	1.4	1.6	1.8	2.0	4.0
α_R	0.81747	0.99425	1.16577	1.33668	1.50718	1.67521	3.37623
β_R	0.92350	1.09277	1.25888	1.42507	1.58825	1.75059	3.37623
E_R in a.u.	-4.26560	-4.82033	-5.15308	-5.35509	-5.48224	-5.56170	-5.69414

§ 4. Results and discussion

At infinite internuclear distance, the AO's are free from the deformation, so at $R=\infty$ the following relations hold,

$$2\alpha/R=2\beta/R=27/16, \quad (9)$$

$$E_{R=\infty}=2 \times (27/16)^2 = 5.69531 \text{ a.u.} \quad (10)$$

The potential $V(R)$ between two normal He atoms is obtained by subtracting (10) from E_R given in Table IV. From the values $V(R)$ thus obtained for each R , we find by the method of least squares the following approximate formula for $V(R)$

$$\begin{aligned} V(R) &\doteq 15.02222e^{-2.36478R} \text{ a.u. } (R \text{ in a.u.}) \\ &= 4.08755 \times 10^2 e^{-4.46893R} \text{ e.v. } (R \text{ in } \text{\AA}) \\ &= 6.54850 \times 10^{-10} e^{-4.46893R} \text{ ergs } (R \text{ in } \text{\AA}). \end{aligned} \quad (11)$$

The mean deviation of this potential is about 0.007 a.u. (0.2 e.v.). In Figs. 1 and 2, the full line (4) is our potential curve and the dotted curve (3) is the one obtained by the HL method (or ASMO method⁷⁾, in which the relation (9) holds for each R . The difference of these two curves is also shown in Fig. 1. The amount of energy decrease due to the appropriate choice of α_R and β_R is about 0.8 e.v. at $R=1.0$ a.u. and decreases rapidly at larger R . In Figs. 1, 2, Tables I and V, we give $V(R)$ calculated by Slater⁴⁾, N. Rosen⁵⁾ and P. Rosen⁶⁾ together with the one obtained from the experiment of Amdur and Harkness.³⁾ At $R \sim 0.5 \text{ \AA}$ Amdur and Harkness's curve is remarkably lower than the theoretical ones and our curve is the deepest among the latter, although the latter curves lie more or less near each other. The agreement between theories and experiment at $R \sim 1.0 \text{ \AA}$ is satisfactory, but at $R \sim 0.5 \text{ \AA}$ the discrepancy between theories and experiment amounts to about 30 e.v., whereas the total energy at this distance is about 190 e.v. This discrepancy might be seen to be rather large to attribute it to the breaking down of Heitler-London approach at small internuclear distance, and in the succeeding paper, we shall give the further investigations for the experimental data.

Table V. Interatomic Potential for He₂. (in e.v.)

Authors <i>R</i> (in Å)	Slater ^(a)	P. Rosen ^(b)	Griffing ^(c) & Wehner	Present ^(d) Paper	exp. ^(e)
0.4704	55.582	72.955	51.843	49.944	10.959
0.52916	42.451	56.235	—	38.414	9.011
0.6275	27.082	36.530	24.862	24.783	6.647
0.63499	26.124	35.298	—	23.937	6.502
0.74082	16.077	22.161	—	14.916	4.934
0.84666	9.892	13.909	—	9.294	3.885
0.9408	6.423	9.261	6.212	6.115	3.216
0.95249	6.088	8.730	—	5.792	3.147
1.05832	3.747	5.481	—	3.609	2.606
1.16415	2.306	3.440	—	2.249	2.206
1.2544	1.526	2.328	1.5754	1.503	1.922

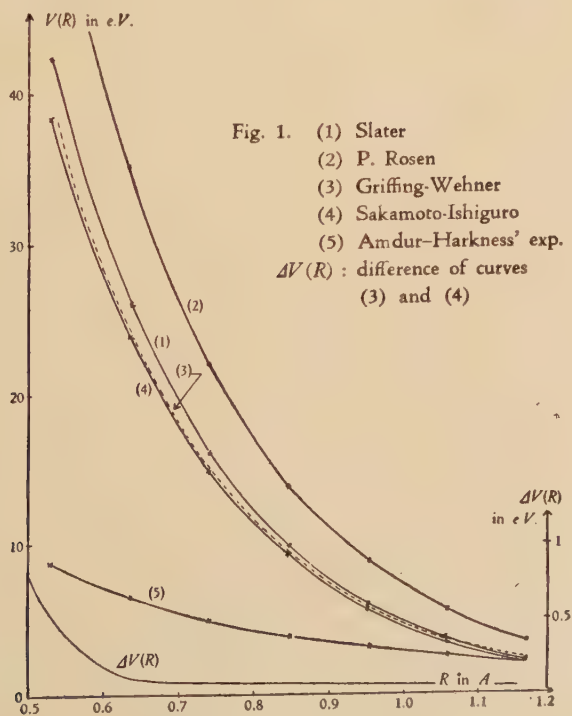
a) Calculated by eq. 1 of Table I.

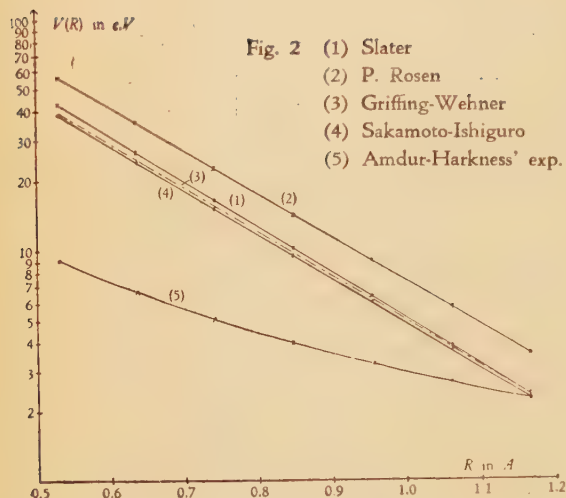
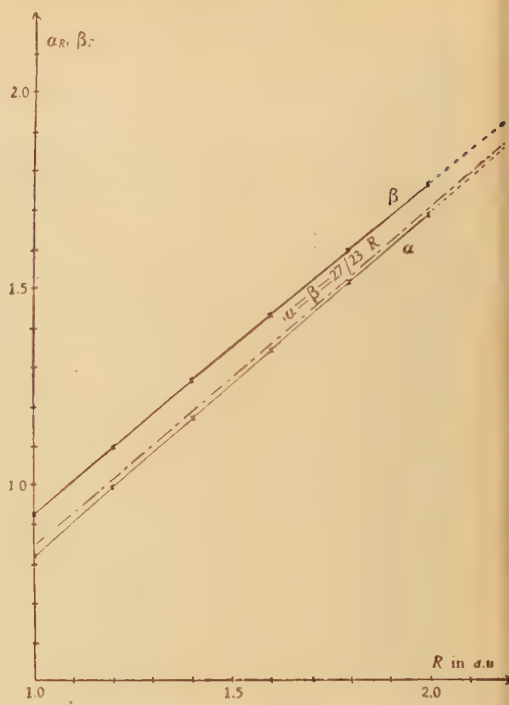
b) Calculated by eq. 2 of Table I.

c) Adopted from Table I of Ref. 7).

d) Calculated by eq. 3 of Table I.

e) Calculated by eq. 4 of Table I.

Fig. 1. He-He Repulsive potential versus *R*.

Fig. 2. He-He Repulsive potential versus R .Fig. 3. The curves of α_R and β_R as functions of R .

In Fig. 3, we give the curves of α_R and β_R as functions of R . In contrast to the case of H_2 molecule,¹⁾ the inequality

$$\beta_R > (27/32)R > \alpha_R$$

holds and this manifest the fact that the AO's are deformed so as to repel each other. Although the difference in energy from the simple HL theory is negligible at $R=2.0$ a.u.,

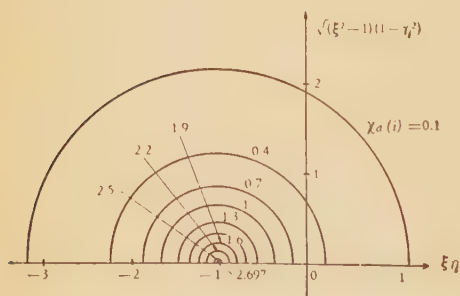
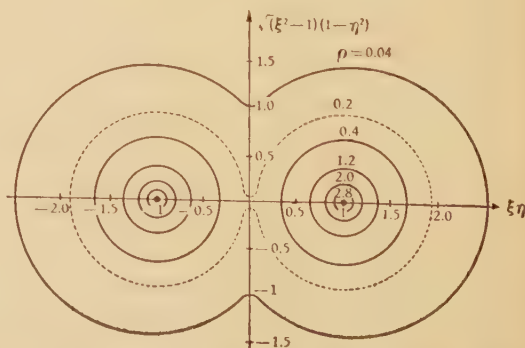


Fig. 4. The contours of normalized AO

$$\chi_a(i) = \frac{a(i)}{\sqrt{\int a(i)^2 dv}} \text{ for } R=1.8 \text{ a.u. in the plane through the nuclear axis.}$$

Fig. 5. The contour map of the electron density function $\rho(\xi, \eta) = 4(S^2 - T^2)^{-1} e^{-2\alpha \xi} (S \cosh 2\beta \eta - T)$, for $R=1.8$ a.u.

(Fig. 1), α and β differs markedly from $(27/32)R$ at this region. At $R=4.0$ a.u., we determined the values of α and β assuming $\alpha=\beta$. The value thus obtained is 3.37623 which is compared with $(27/32)R=3.375$. (Table IV)



Fig. 6. The contour map (the photograph of model) of the electron density function, at $R=1.8$ a.u.

To visualize this circumstance, we give in Fig. 4 contours of normalized AO', $\chi_a(i) = a(i) / \sqrt{\int a(i)^2 dv_i}$, for $R=1.8$ a.u. ($\alpha_{R=1.8}=1.50718$, $\beta_{R=1.8}=1.58825$) in the plane through the nuclear axis. Also, Figs. 5 and 6 are the contour maps of the electron density function,

$$\begin{aligned} \rho(\xi, \eta) &= \frac{4 \int \{\psi(1, 2, 3, 4)\}^2 d\tau_2 d\tau_3 d\tau_4}{\int \{\psi(1, 2, 3, 4)\}^2 d\tau_1 d\tau_2 d\tau_3 d\tau_4} \\ &= \frac{4 \cdot 24 (S^2 - T^2) e^{-2\alpha\xi} [S \cosh 2\beta\eta - T]}{24 (S^2 - T^2)^2} \\ &= \frac{4}{S^2 - T^2} e^{-2\alpha\xi} [S \cosh 2\beta\eta - T], \end{aligned}$$

for $R=1.8$ a.u. From these figures, we can know that the deformation of AO's are rather small at $R=1.8$ a.u. as one might imagine from the value of α_R and β_R . In Fig. 7 we give the curve of overlap integral

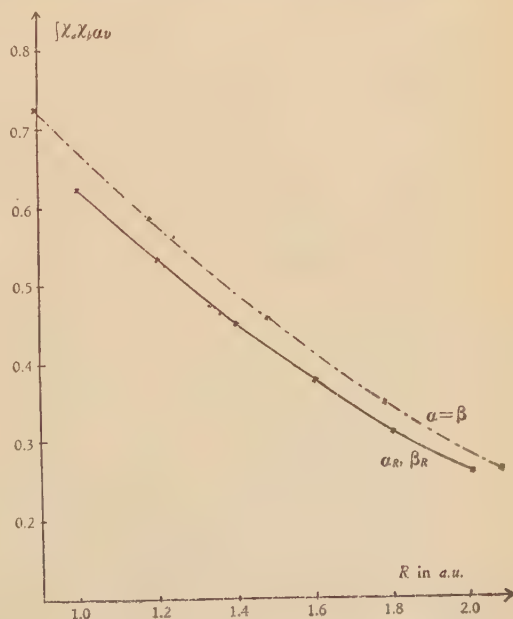


Fig. 7. The overlap integrals versus R function.

as a function of R , in which the full line is obtained by our wave function and the dotted line is the one obtained by the simple Heitler-London method. This figure shows clearly the above mentioned tendency of deformation of AO's.

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Appendix

I. Derivation of eq. (7).

The integrals in energy expression (7) can be expressed as

$$\begin{aligned}
 \int \bar{\psi} \psi d\tau &= 4! (S^4 - S^2 T^2 - S^2 T^2 + T^4) = 4! (S^2 - T^2)^2, \\
 \int \bar{\psi} H_i \psi d\tau &= 4! (MS^3 - MS^2 T^2 - NS^2 T + NT^3) = 4! (MS - NT) (S^2 - T^2), \\
 &\quad (i=1, 2, 3, 4) \\
 \int \bar{\psi} V_{12} \psi d\tau &= \int \bar{\psi} V_{34} \psi d\tau \\
 &= 4! (KS^2 - LST - LST + JT^2) = 4! (KS^2 - 2LST + JT^2), \\
 \int \bar{\psi} V_{13} \psi d\tau &= \int \bar{\psi} V_{23} \psi d\tau \\
 &= 4! (CS^2 - CT^2 - JS^2 + JT^2) = 4! (C - J) (S^2 - T^2), \\
 \int \bar{\psi} V_{14} \psi d\tau &= \int \bar{\psi} V_{23} \psi d\tau \\
 &= 4! (CS^2 - LST - LST + JT^2) = 4! (CS^2 - 2LST + JT^2). \quad (A \cdot 1)
 \end{aligned}$$

Then we have

$$\begin{aligned}
 \int \bar{\psi} H \psi d\tau &= 4! \times 4 (S^2 - T^2) \left(\frac{S^2 - T^2}{R} + MS - NT \right) \\
 &\quad + 2 \times 4! ((K + C) S^2 - 4LST + 2JT^2 + (C - J) (S^2 - T^2)). \quad (A \cdot 2)
 \end{aligned}$$

II. Method of Calculation of the Integrals (6).

The numerical values of integrals (6) are obtained by means of the following equations,

$$\begin{aligned}
S &= \pi R^3/4 \cdot \{A_2(2\alpha) B_0(2\beta) - A_0(2\alpha) B_2(2\beta)\}, \\
T &= \pi R^3/2 \cdot \{A_2(2\alpha) - 1/3 \cdot A_0(2\alpha)\}, \\
M &= \pi R/2 \cdot \{A_0(2\alpha) B_0(2\beta) (\alpha^2 - \beta^2) + A_1(2\alpha) B_0(2\beta) (2\alpha - 4R) - \alpha^2 A_2(2\alpha) B_0(2\beta) \\
&\quad + \beta^2 A_0(2\alpha) B_2(2\beta) - 2\beta A_0(2\alpha) B_1(2\beta)\}, \\
N &= \pi R/3 \cdot \{(3\alpha^2 - 2\beta^2) A_0(2\alpha) + 6(\alpha - 2R) A_1(2\alpha) - 3\alpha^2 A_2(2\alpha)\}, \\
K &= \pi^2 R^5/8 \cdot \sum_{\tau=0}^{\infty} (2\tau+1) \{W_{\tau}^0(2, 2; 2\alpha) G_{\tau}^0(0, 2\beta)^2 \\
&\quad - 2W_{\tau}^0(2, 0; 2\alpha) G_{\tau}^0(0, 2\beta) G_{\tau}^0(2, 2\beta) + W_{\tau}^0(0, 0; 2\alpha) G_{\tau}^0(2, 2\beta)^2\}, \\
L &= \pi^2 R^5/12 \cdot [G_0^0(0, 2\beta) \{3W_0^0(2, 2; 2\alpha) - W_0^0(2, 0; 2\alpha)\} \\
&\quad - G_0^0(2, 2\beta) \{3W_0^0(0, 2; 2\alpha) - W_0^0(0, 0; 2\alpha)\} \\
&\quad + 2\{W_2^0(0, 0; 2\alpha) G_2^0(2, 2; 2\beta) - W_2^0(2, 0; 2\alpha) G_2^0(0, 2\beta)\}], \\
C &= \pi^2 R^5/8 \cdot \sum_{\tau=0}^{\infty} (2\tau+1) (-1)^{\tau} \{W_{\tau}^0(2, 2; 2\alpha) G_{\tau}^0(0, 2\beta)^2 \\
&\quad - W_{\tau}^0(2, 0; 2\alpha) G_{\tau}^0(0, 2\beta) G_{\tau}^0(2, 2\beta) \\
&\quad - W_{\tau}^0(0, 2; 2\alpha) G_{\tau}^0(2, 2\beta) G_{\tau}^0(0, 2\beta) + W_{\tau}^0(0, 0; 2\alpha) G_{\tau}^0(2, 2\beta)^2\} \\
J &= \pi^2 R^5/90 \cdot \{45W_0^0(2, 2; 2\alpha) - 30W_0^0(2, 0; 2\alpha) \\
&\quad + 5W_0^0(0, 0; 2\alpha) + 4W_2^0(0, 0; 2\alpha)\}. \tag{A.3}
\end{aligned}$$

In the above formulae $A_n(\alpha)$, $B_n(\alpha)$, $G_{\tau}^{\nu}(m; \alpha)$ and $W_{\tau}^{\nu}(m, n; \alpha)$ are the functions defined and tabulated in KAS's table⁹⁾, namely,

$$\begin{aligned}
A_n(\alpha) &= \int_1^{\infty} e^{-\alpha \xi} \xi^n d\xi, \\
B_n(\alpha) &= \int_{-1}^1 e^{-\alpha \eta} \eta^n d\eta, \\
G_{\tau}^{\nu}(m; \alpha) &= \int_{-1}^1 e^{-\alpha \eta} P_{\tau}^{\nu}(\eta) \eta^m (1 - \eta^2)^{\frac{\nu}{2}} d\eta, \\
W_{\tau}^{\nu}(m, n; \alpha) &= \int_1^{\infty} \int_1^{\infty} Q_{\tau}^{\nu}(\xi_+) P_{\tau}^{\nu}(\xi_-) e^{-\alpha(\xi_1 + \xi_2)} \xi_1^m \xi_2^n (\xi_1^2 - 1)^{\frac{\nu}{2}} (\xi_2^2 - 1)^{\frac{\nu}{2}} d\xi_1 d\xi_2, \tag{A.4}
\end{aligned}$$

where P_{τ}^{ν} and Q_{τ}^{ν} are the associated Legendre functions of the 1st and 2nd kinds, respectively.

III. Determination of α_R and β_R for fixed R .

As the steps of α and β in Tables II, III are $1/8$, we put α and β as $x/8$ and $y/8$, respectively. In accordance with this, α_R and β_R are put as $x_R/8$ and $y_R/8$, respectively. We imagine that α_R and β_R are nearly equal to α_0 and β_0 respectively, where α_0 and β_0 are the pair of α , β in Table II, III. Values of x , y corresponding to α_0 , β_0 may

be denoted as a , b . We assume that $a-1 < x_R < a+1$ and $b-1 < y_R < b+1$ and the values of $E(a, b)$, $E(a+1, b)$, \dots , $E(a-1, b-1)$ are known. Energy function $E(\alpha, \beta; R)$ is assumed to be expressible by the following form of series expansions of x and y ,

$$\begin{aligned} E(x, y) = & C_0 + C_1(x-a) + C_2(x-a)(x-a-1) \\ & + (y-b)(C_3 + C_4(x-a) + C_5(x-a)(x-a-1)) \\ & + (y-b)(y-b-1)(C_6 + C_7(x-a) + C_8(x-a)(x-a-1)) \\ & + \dots \end{aligned} \quad (\text{A} \cdot 5)$$

If terms not written in eq. (A·5) are neglected, C 's can be determined easily by known $E(a, b)$'s, in the following manner,

$$\begin{aligned} C_0 &= E(a, b), \\ C_1 &= E(a+1, b) - C_0, \\ C_2 &= 1/2 \cdot (E(a-1, b) - C_0 + C_1), \\ C_3 &= E(a, b+1) - C_0, \\ C_4 &= E(a+1, b+1) - C_0 - C_1 - C_3, \\ C_5 &= 1/2 (E(a-1, b+1) - C_0 + C_1 - 2C_2 - C_3 + C_4), \\ C_6 &= 1/2 (E(a, b-1) - C_0 + C_3), \\ C_7 &= 1/2 \cdot (E(a+1, b-1) - C_0 - C_1 + C_3 + C_4 - 2C_5), \\ C_8 &= 1/4 \cdot (E(a-1, b-1) - C_0 + C_1 - 2C_2 + C_3 - C_4 + 2C_5 - 2C_6 + 2C_7). \end{aligned} \quad (\text{A} \cdot 6)$$

Now we put

$$\begin{aligned} x-a &= x', \\ y-b &= y', \end{aligned}$$

then eq. (A·5) reduces to

$$\begin{aligned} E(x', y') = & C_0 + C_1 x' + C_2 x' (x' - 1) + y' (C_3 + C_4 x' + C_5 x' (x' - 1)) \\ & + y' (y' - 1) (C_6 + C_7 x' + C_8 x' (x' - 1)). \end{aligned} \quad (\text{A} \cdot 7)$$

From this equation we can obtain the equation for x' , y' which give the minimum energy $E(x', y')$, viz.,

$$\begin{aligned} \partial E(x', y') / \partial x' = & C_1 - C_2 + (C_4 - C_5) y' + (C_7 - C_8) y' (y' - 1) \\ & + 2 (C_2 + C_5 y' + C_8 y' (y' - 1)) x' \\ = & 0 \end{aligned} \quad (\text{A} \cdot 8)$$

$$\begin{aligned} \partial E(x', y') / \partial y' = & C_3 - C_6 + (C_4 - C_7) x' + (C_5 - C_8) x' (x' - 1) \\ & + 2 (C_6 + C_7 x' + C_8 x' (x' - 1)) y' \\ = & 0. \end{aligned} \quad (\text{A} \cdot 9)$$

From eqs. (A·8) and (A·9), we shall obtain the best values of α' and β' for the values of fixed R . So that we shall obtain :

$$\alpha_R = \alpha_0 + \alpha',$$

$$\beta_R = \beta_0 + \beta',$$

where

$$\alpha' = \frac{x'}{8},$$

$$\beta' = \frac{y'}{8}.$$

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Extensions of Variational Methods, II

— Two Parameter Eigenvalue Problem for the Deuteron State —

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In the usual treatment of the deuteron problem, depths of both central and tensor potentials are to be determined so as to give the correct experimental values for the binding energy γ^2 and the electric quadrupole moment Q assuming a suitable function for the potential. We can consider these two depth values as "two parameter eigenvalue". In this connection the following approximate relation among Q , γ^2 and the mixing parameter ε of the D -state to the S -state is useful,

$$\tan \varepsilon = -\sqrt{2} Q \gamma^2, \quad \gamma^2 = (\text{binding energy of deuteron}) \times (\text{nucleon mass}) \div \hbar^2,$$

which is shown to be accurate within 10 percents.

The usual variational method is extended to this problem: We first find the stationary expressions for both depths of central and tensor potentials, and it is proved that the stationary requirement is equivalent to the fundamental equations of the deuteron state with specified mixing ratio ε . Actual applications are carried out for the square well potentials. The most advantageous point of the present method is that we can determine both depths simultaneously without resource to approximate method such as trial-and-error method.

Lastly the variation principle corresponding to the Rayleigh-Ritz variational method is generalized to many parameter eigenvalue problem.

§ 1. Introduction

The present paper deals with the second point among several advantageous features of variational methods, namely, simultaneous determination of eigenvalues and eigenfunction belonging to them, as described in the introduction of the first paper¹⁾. In § 1 the meaning of "two parameter eigenvalue problem" is explained, and in § 2 this problem is formulated in variationally correct way, illustrating it by an example: the deuteron problem. This method is used to determine depths of central and tensor potentials, and we discuss the effective range theory of nuclear forces with the tensor force in § 3. In § 4 an extension is given of the usual variational method for one eigenvalue to many parameter eigenvalues.

We will first explain so-called "two parameter eigenvalue problem". A classical example is the Lamé problem,²⁾ of which almost all the boundary value problems usually treated in mathematical physics may be considered as a special or limiting case. The differential equations in this problem contain two constants (μ and ν), but the imposed boundary conditions cannot be satisfied for an arbitrary choice of μ , ν . We must determine a pair of associated eigenvalues μ , ν , so we have "two parameter eigenvalue problem". Another example familiar in quantum mechanics is a motion of a particle in a field of

central force. Fortunately two examples cited above can be reduced to ordinary one parameter eigenvalue problem, so we have no serious trouble due to multiplicity of eigenvalues.

As an example of essentially two parameter eigenvalue problem, we may consider the coupled equations for the deuteron state,³⁾

$$\{d^2/dr^2 - \gamma^2 + \mu V(r)\} u(r) + \sqrt{8} \nu W(r) w(r) = 0, \quad (1a)$$

$$\{d^2/dr^2 - \gamma^2 - 6/r^2 + \mu V(r) - 2W(r)\} w(r) + \sqrt{8} \nu W(r) u(r) = 0, \quad (1b)$$

with the conditions $u(0) = w(0) = 0$, $u(r) \rightarrow 0$, $w(r) \rightarrow 0$, $(r \rightarrow \infty)$. $\mu V(r)$ and $\nu W(r)$ are central and tensor potentials respectively. If $\mu V(r)$ and $\nu W(r)$ are known, it is the usual eigenvalue problem with the eigenvalue γ^2 . However, our situation is quite different. The quantities available from experiments are the binding energy of deuteron γ^2 and the electric quadrupole moment Q , and we wish to find the values of μ and ν which reproduce the correct values of γ^2 and Q . Since there exists an approximate relation,⁴⁾

$$\tan \varepsilon = -\sqrt{2} Q \gamma^2, \quad (2)$$

we will specify the binding energy γ^2 and the mixing ratio ε of the D -state to the S -state instead of γ^2 and Q as

$$\phi(r) = \begin{cases} u(r) \rightarrow \cos \varepsilon e^{-\gamma r} & \text{for } r \rightarrow \infty, \\ w(r) \rightarrow -\sin \varepsilon \{3/(\gamma r)^2 + 3/(\gamma r) + 1\} e^{-\gamma r}, \end{cases} \quad \phi(0) = 0. \quad (3)$$

Imposing the condition (3), we obtain sets of eigenvalues (μ_i, ν_i) . If we restrict our attention to the ground state (μ_0, ν_0) only, and want to apply the usual variational method, we first assume an approximate value for one of μ and ν , for example ν , and calculate the value of μ by variational method. This process must be repeated until the solution has the correct ε specified in (3). In the next section it is shown that, if we extend the usual variational method in another direction, the both values of μ and ν can be calculated simultaneously without the help of troublesome trial-and-error method.

§ 2. A variation principle for the deuteron problem

Before entering into the main part, we will briefly mention the general plan of this section. In the usual eigenvalue problem concerning a Hermitian operator A ,

$$A\phi = \lambda\phi, \quad (4)$$

the stationary expression for the eigenvalue λ is given by

$$\lambda = (\phi, A\phi) / (\phi, \phi). \quad (5)$$

It is also well known that the stationary requirement for λ

$$\delta\lambda = \delta(\phi, (A - \lambda)\phi) / (\phi, \phi) = 0, \quad \lambda \equiv (\phi, A\phi) / (\phi, \phi), \quad (6)$$

is equivalent to the fundamental equation (4). It will be shown below that there exist similar situations in the two parameter eigenvalue problem for the deuteron state too.

a). Stationary expressions for μ and ν .

The fundamental equation for the deuteron state (1) is rewritten into the following 2×2 matrix form,

$$\begin{bmatrix} -d^2/dr^2 + \gamma^2, & 0, \\ 0, & -d^2/dr^2 + \gamma^2 + 6/r \end{bmatrix} \begin{bmatrix} u(r) \\ w(r) \end{bmatrix} = \left\{ \mu \begin{bmatrix} V(r), & 0 \\ 0, & V(r) \end{bmatrix} \right. \\ \left. + \nu \begin{bmatrix} 0, & \sqrt{8} W(r) \\ \sqrt{8} W(r), & -2W(r) \end{bmatrix} \right\} \begin{bmatrix} u(r) \\ w(r) \end{bmatrix}, \quad (7)$$

which is abbreviated in

$$A\phi = (\mu B + \nu C)\phi, \quad \text{or} \quad L\phi = 0; \quad L \equiv A - \mu B - \nu C. \quad (8)$$

For special values of μ and ν , there exists a solution having the form (3) with given values of γ and ε , namely; two quantities γ and ε determine the two parameter eigenvalues (μ_i, ν_i) and corresponding eigenfunction ϕ_i . The mixing parameter ε is related to the experimental values as in (2) for the ground state.

Since eq. (7) is simultaneous differential equations of second order there are four independent solutions in general. If the two conditions: $\phi(0) = 0$ are imposed, we have two solutions: one of these corresponds to (3) and the other solution can be specified by the conditions

$$\phi(r) = \begin{cases} u(r) \rightarrow \sin \varepsilon (\sinh \gamma r + \lambda_1 \cosh \gamma r), \\ w(r) \rightarrow \cos \varepsilon (j(\gamma r) + \lambda_2 n(\gamma r)), \end{cases} \quad r \rightarrow \infty, \quad u(0) = w(0) = 0, \quad (9)$$

where

$$j(\gamma r) \equiv (1 + 3/(\gamma r)^2) \sinh \gamma r - (3/(\gamma r)) \cosh \gamma r, \\ n(\gamma r) \equiv (1 + 3/(\gamma r)^2) \cosh \gamma r - (3/(\gamma r)) \sinh \gamma r.$$

The two quantities λ_1 and λ_2 can be determined by solving the eq. (7). We will first show that $\lambda_1 = \lambda_2$. Denoting the solutions ϕ in (3) and (9) as $\phi^{(1)}$ and $\phi^{(2)}$, and using the notation $L \equiv A - \mu B - \nu C$, we have

$$(\phi^{(1)}, L\phi^{(2)}) \equiv \int_0^\infty \phi^{(1)} \cdot (L\phi^{(2)}) dr = \int_0^\infty \{ u^{(1)} (-d^2/dr^2 + \gamma^2 - \mu V) u^{(2)} - u^{(1)} \sqrt{8} W w^{(2)} \\ + w^{(1)} (-d^2/dr^2 + \gamma^2 + 6/r^2 - \mu V + 2\nu W) w^{(2)} - w^{(1)} \sqrt{8} W u^{(2)} \} \cdot dr = 0, \quad (10a)$$

and similarly

$$\int_0^\infty \phi^{(2)} \cdot (L\phi^{(1)}) dr = 0. \quad (10b)$$

Subtracting (10b) from (10a), and integrating partially, we get

$$\phi^{(1)} \cdot d\phi^{(2)}/dr - \phi^{(2)} \cdot d\phi^{(1)}/dr \Big|_0^\infty = 0. \quad (11)$$

By (3) and (9), eq. (11) is reduced to

$$\gamma(\lambda_1 - \lambda_2) \cos \varepsilon \sin \varepsilon = 0,$$

hence, if $\gamma \sin 2\varepsilon \neq 0$, it follows that $\lambda_1 = \lambda_2 (= \lambda)$. Next we define A_{ij} , B_{ij} and C_{ij} as follows

$$A_{ij} \equiv (\phi^{(i)}, A\phi^{(j)}), \quad B_{ij} \equiv (\phi^{(i)}, B\phi^{(j)}), \quad C_{ij} \equiv (\phi^{(i)}, C\phi^{(j)}),$$

where the parenthesis denotes, as in Eq. (10a), the inner product for 2×2 matrix and moreover the integration $\int_0^\infty dr$. If $B_{11}C_{21} - B_{12}C_{11}$ holds, which is assumed hereafter, it can be seen that

$$\mu = (A_{11}C_{21} - A_{21}C_{11}) / (B_{11}C_{21} - B_{21}C_{11}), \quad \nu = (A_{11}B_{21} - A_{21}B_{11}) / (C_{11}B_{21} - C_{21}B_{11}). \quad (12)$$

For, $\phi^{(1)}$ satisfies eq. (8), hence

$$(\phi^{(1)}, (A - \mu B - \nu C)\phi^{(1)}) = 0, \quad (\phi^{(2)}, (A - \mu B - \nu C)\phi^{(1)}) = 0. \quad (13)$$

Solving μ and ν from (13) we get eq. (12). Similarly $\phi^{(2)}$ satisfies Eq. (8), we also have

$$(\phi^{(1)}, (A - \mu B - \nu C)\phi^{(2)}) = 0, \quad (13')$$

and

$$\mu = (A_{11}C_{12} - A_{12}C_{11}) / (B_{11}C_{12} - B_{12}C_{11}), \quad \nu = (A_{11}B_{12} - A_{12}B_{11}) / (C_{11}B_{12} - C_{12}B_{11}). \quad (12')$$

We will now prove that these expressions for μ and ν are stationary with respect to any infinitesimal variations of $\phi^{(1)}$ and $\phi^{(2)}$, which satisfy the boundary conditions (3) and (9). From now on we consider μ and ν as the functional of $\phi^{(1)}$ and $\phi^{(2)}$ as defined in (12). Similar discussions are all valid also for (12'). Eq. (12) and eq. (11) are equivalent to each other. Eq. (13) is also abbreviated in

$$A_{11} - \mu B_{11} - \nu C_{11} = 0, \quad A_{21} - \mu B_{21} - \nu C_{21} = 0. \quad (14)$$

Taking variations for (14) we get

$$\delta A_{11} - \mu \delta B_{11} - \nu \delta C_{11} = B_{11} \delta \mu + C_{11} \delta \nu, \quad (15a)$$

$$\delta A_{21} - \mu \delta B_{21} - \nu \delta C_{21} = B_{21} \delta \mu + C_{21} \delta \nu. \quad (15b)$$

Let us calculate the left side terms. These are equal to the variations of $(\phi^{(i)}, (A - \mu B - \nu C)\phi^{(i)})$ provided that μ and ν are kept constant. This variation is

$$\begin{aligned} \delta(\phi^{(i)}, (A - \mu B - \nu C)\phi^{(i)}) &= (\delta\phi^{(i)}, (A - \mu B - \nu C)\phi^{(i)}) + (\phi^{(i)}, (A - \mu B - \nu C)\delta\phi^{(i)}) \\ &= (\phi^{(i)}, (A - \mu B - \nu C)\delta\phi^{(i)}). \end{aligned} \quad (16)$$

Since B and C are symmetric, it follows that $(\phi^{(i)}, B\delta\phi^{(i)}) = (\delta\phi^{(i)}, B\phi^{(i)})$, $(\phi^{(i)}, C\delta\phi^{(i)}) = (\delta\phi^{(i)}, C\phi^{(i)})$. For the operator A , we have

$$(\phi^{(i)}, A\delta\phi^{(i)}) - (\delta\phi^{(i)}, A\phi^{(i)}) = (d\phi^{(i)}/dr) \cdot \delta\phi^{(i)} - \phi^{(i)} \cdot d(\delta\phi^{(i)})/dr \Big|_0^\infty. \quad (17)$$

$\phi^{(1)}, \phi^{(2)}$ and these derivatives are specified according to (3) and (9) in the outside region of nuclear forces $V(r)$ and $W(r)$, so we have

$$\partial\phi^{(1)} = \begin{cases} \partial u^{(1)} \rightarrow 0 (\epsilon^{-\gamma r}), \\ \partial w^{(1)} \rightarrow 0 (\epsilon^{-\gamma r}), \end{cases} \quad \partial\phi^{(2)} = \begin{cases} \partial u^{(2)} \rightarrow \partial\lambda_1 \sin \epsilon \cosh \gamma r + 0 (\sinh \gamma r), \\ \partial w^{(2)} \rightarrow \partial\lambda_2 \cos \epsilon n(\gamma r) + 0 (j(\gamma r)), \end{cases} \quad (18a)$$

and

$$\partial\phi^{(1)}(0) = \partial\phi^{(2)}(0) = 0. \quad (18b)$$

Substituting (18) into (17) we get

$$\begin{aligned} (\phi^{(1)}, A\partial\phi^{(1)}) &= (\partial\phi^{(1)}, A\phi^{(1)}), \\ (\phi^{(1)}, A\partial\phi^{(2)}) &= (\partial\phi^{(2)}, A\phi^{(1)}) + (\partial\lambda_2 - \partial\lambda_1)\gamma \cos \epsilon \sin \epsilon. \end{aligned}$$

Hence it follows for the variations restricted by $\lambda_1 = \lambda_2$ that,

$$(\phi^{(i)}, (A - \mu B - \nu C)\partial\phi^{(1)}) = (\partial\phi^{(1)}, (A - \mu B - \nu C)\phi^{(i)}) = 0.$$

Consequently the left side terms of (15) vanish by (16). Eqs. (15, a, b) are reduced to

$$B_{11}\partial\mu + C_{11}\partial\nu = 0, \quad B_{21}\partial\mu + C_{21}\partial\nu = 0. \quad (19)$$

Since we have assumed that $B_{11}C_{21} \neq B_{21}C_{11}$, it follows that

$$\partial\mu = \partial\nu = 0. \quad (20)$$

Thus we have proved that the expressions (12) for μ and ν are stationary* for any variations of $\phi^{(i)}$, provided that variations do not violate the boundary conditions (3), (9) and $\lambda_1 = \lambda_2$.

The same is true for the expressions (12').**

b). Equivalence of (20) and (1).

We now prove that the stationary requirement of expression (12) with respect to arbitrary variations, which are restricted by the conditions (3), (9) and $\lambda_1 = \lambda_2$, is equivalent to eq. (1) with the same boundary conditions for $\phi^{(1)}$ and $\phi^{(2)}$. We have proved that the expressions (12) are really stationary for arbitrary variations with the conditions (3), (9) and $\lambda_1 = \lambda_2$. Then we only need to prove the converse of this proposition. Since the equations (12) and (14) are equivalent, the requirement that the expressions (12) be stationary is that the right side terms of (15) vanish. Thus the stationary requirement can be reduced to

$$\delta A_{11} - \mu\delta B_{11} - \nu\delta C_{11} = 0, \quad \delta A_{21} - \mu\delta B_{21} - \nu\delta C_{21} = 0. \quad (20a)$$

* We do not know yet under what conditions the expression (12) gives an upper (or lower) limit for μ and ν .

** Since A is Hermitian provided that $\lambda_1 = \lambda_2$, we can put

$$-\int_0^\infty \phi^{(i)} \cdot (d^2\phi^{(j)}/dr^2) dr = \int_0^\infty (d\phi^{(i)}/dr) \cdot (d\phi^{(j)}/dr) dr$$

in the usual way.

Writing explicitly, we have

$$(\delta\phi^{(i)}, (A - \mu B - \nu C)\phi^{(1)}) + (\phi^{(i)}, (A - \mu B - \nu C)\delta\phi^{(1)}) = 0, \quad (i=1, 2).$$

Since L is symmetric for variations which are compatible with (3), (9) and $\lambda_1 = \lambda_2$, as shown in (18), it follows that,

$$(\phi^{(i)}, (A - \mu B - \nu C)\delta\phi^{(1)}) = (\delta\phi^{(1)}, (A - \mu B - \nu C)\phi^{(i)}).$$

Hence we obtain

$$(\delta\phi^{(i)}, (A - \mu B - \nu C)\phi^{(1)}) + (\delta\phi^{(1)}, (A - \mu B - \nu C)\phi^{(i)}) = 0.$$

If we put $i=2$, it follows that

$$(\delta\phi^{(2)}, (A - \mu B - \nu C)\phi^{(1)}) + (\delta\phi^{(1)}, (A - \mu B - \nu C)\phi^{(2)}) = 0. \quad (21)$$

Since $\delta\phi^{(1)}$ and $\delta\phi^{(2)}$ are independent arbitrary variations, the following equations are derived

$$(A - \mu B - \nu C)\phi^{(1)} = 0, \quad (A - \mu B - \nu C)\phi^{(2)} = 0.$$

c). Variational procedure

The actual procedure in the usual variational method in most cases is as follows.

Since eq. (6) and eq. (4) are equivalent to each other, we first calculate

$$\delta(\psi, (A - \lambda)\psi) = 0. \quad (22)$$

for possible variations of a trial function ψ . If ψ has the form

$$\psi = \sum_{i=1}^n a_i \varphi_i, \quad (23)$$

(22) is written as follows,

$$\sum_{j=1}^n (A_{ij} - \lambda N_{ij}) a_j = 0, \quad (i=1, \dots, n), \quad A_{ij} = (\varphi_i, A\varphi_j), \quad N_{ij} = (\varphi_i, \varphi_j). \quad (24)$$

λ is determined by the existence condition of the solution: $\text{Det. } (A_{ij} - \lambda N_{ij}) = 0$, and the ratio of a_j can be found correspondingly. As already seen above, in the two parameter eigenvalue problem eq. (1) (or eq. (7)) is equivalent to (20), so we can consider the similar method for actual procedure. To find out $\phi^{(i)}$ ($i=1$ and 2) for which the relations (20) hold is to find out $\phi^{(i)}$ which satisfy (20a). Assuming the trial functions $\phi^{(1)}$, $\phi^{(2)}$ with adjustable parameters* a_i and b_i

$$\phi^{(1)} = \sum_{i=1}^n a_i \varphi_i^{(1)}, \quad \phi^{(2)} = \sum_{i=1}^n b_i \varphi_i^{(2)}, \quad (25)$$

we can determine μ and ν by the following conditions

$$\text{Det. } (A_{ij}^{(1)} - \mu B_{ij}^{(1)} - \nu C_{ij}^{(1)}) = 0, \quad \text{Det. } (A_{ij}^{(2)} - \mu B_{ij}^{(2)} - \nu C_{ij}^{(2)}) = 0, \quad (26)$$

* We use the word "parameter" in two different meaning. In trial functions it means "variationally adjustable parameter", while in eigenvalue problems it indicates a kind of multiplicity of eigenvalues as explained in § 1

where $A_{ij}^{(1)} = (\varphi_i^{(1)}, A\varphi_j^{(1)})$, $B_{ij}^{(1)} = (\varphi_i^{(1)}, B\varphi_j^{(1)})$, $A_{ij}^{(2)} = (\varphi_i^{(1)}, A\varphi_j^{(2)})$, etc. Each of the equations in (26) is of the n -th degree in the unknown μ and ν , and they together determine the values of these unknowns. This procedure, however, becomes prohibitively cumbersome as n increases. But it seems that there is no other systematic procedure except several expedient means. The first device is to minimize the number of adjustable parameters involved in the trial function to three or four at most, using auxiliary calculations, for example, the perturbational or iterative method, as far as possible. The second is to adopt some successive methods to solve (26). We may take a method devised by Hulthen and Laurikainen⁵⁾ in solving (26), or may add adjustable parameters step by step, and take variations for restricted number of parameters only. A method used in Appendix II may also belong to this category. The third is more realistic procedure: We assume the approximate quadratic relation between μ , ν and the substantial adjustable parameter $C_i (i=1, 2, \dots, n)$. For example writing down for $n=1$ and $n=2$, the quadratic functions are

$$\begin{aligned} \mu &= \mu_{11}C_1^2 + \mu_1C_1 + \mu_0; \text{ for } n=1, \\ \mu &= \mu_{11}C_1^2 + \mu_{12}C_1C_2 + \mu_{22}C_2^2 + \mu_1C_1 + \mu_2C_2 + \mu_0; \text{ for } n=2, \end{aligned} \quad (27)$$

and corresponding equations for ν . We calculate the values of μ , ν of (12) at $(n+1) \times (n+2)/2$ sets of parameter values. We can determine $(n+1)(n+2)/2$ coefficient μ_{ij} , μ_i , μ_0 for μ , and ν_{ij} , ν_i , ν_0 for ν . Then the stationary point (C_i) and stationary value of μ are determined using this approximate function (27). This procedure is also valid for ν . This is accurate if we choose $(n+1)(n+2)/2$ sets of parameter near at the stationary value. One of the advantageous points of this procedure is that this is applicable to general type of trial function not necessarily of the form (25).

§ 3. Applications to square well

In the present section we will discuss the problem about the deuteron state and the neutron-proton scattering for the low energy region. The argument is confined to S -predominant waves in the spin triplet state only. The differential cross section due to this state is given by⁶⁾

$$d\sigma = (3/4) (\sin^2 \eta / k^2) [1 + (1 + \tan \varepsilon \sqrt{8})^2 \sin^2 2\varepsilon \cdot P_2(\cos \theta)] d\Omega, \quad (28)$$

where the phase-shift η and the mixing parameter ε are expressed, in the first approximation, in terms of "shape independent" parameters: scattering length a , effective range r_0 and q .

$$k \cot \eta = -1/a + (r_0/2)k^2 + O(k^4), \quad (29a)$$

$$\tan \varepsilon = qk^2 + O(k^4). \quad (29b)$$

If we set $k = -i\gamma$, the corresponding values for the deuteron are obtained. It is well known that the data below 20 Mev available at present are explained fairly well by three quantities a , r_0 and q without detailed discussions through the formula (28). The scattering is no longer isotropic. Taking into account the contribution of the spin singlet state, and using the approximate relation for q : $q = \sqrt{2} Q$ (see Appendix I), we have the result

$d\sigma(\pi)/d\sigma(\pi/2) = 1.046$ for 20 Mev neutron of lab. system. If we consider P or D -waves, these waves contribute to the anisotropy of differential cross section in the same order of magnitude as the anisotropy due to ε of (29).

For example we have at 20 Mev for the even theory (no force in odd states)

$$d\sigma(\pi)/d\sigma(\pi/2) = 1.01 \text{ (square well)}, = 1.06 \text{ (Yukawa well)},$$

and for the repulsive force in P -state which is $1/5$ times of the one in S -state

$$d\sigma(\pi)/d\sigma(\pi/2) = 1.03 \text{ (square well)}, = 1.08 \text{ (Yukawa well)}.$$

provided that the interaction is pure central force.

Thus the effects of P and D -waves are remarkably shape-dependent, while the anisotropy due to the tensor force is not too much dependent on the assumed shape of potentials.

It is the purpose of this section to see how accurately we can find the depth values of μ and ν so as to give the correct constants γ^2 , Q of the deuteron state by the present variational method, and to investigate to what extent the relation $q = \sqrt{2} Q$ is accurate.

In order to adjust the force constant to the values of the binding energy γ^2 and the electric quadrupole moment Q of the deuteron, the large scale digital computer Illiac (Univ. of Illinois, U.S.A.) is working since 1952. The computing mechanism of Illiac is an iterative procedure which adjusts the force constants to the binding energy only. After that the trial-error method is adopted to produce the correct value for Q . It seems to be very enormous computations, and, to my present knowledge, the final report is not published yet. We hope it will be completed in near future to make clear the effects of the tensor force to the low energy phenomena. We will try to attack the same problem by the present variational method.* The type of potential is assumed to be square well, for which Illiac is not working on account of its discontinuous property.

a). Equal range for both potentials

Let us first consider the case where the ranges of central and tensor well are equal. Let the range be unity, and

$$V(r) = \begin{cases} \mu (0 < r < 1), \\ 0 (1 < r), \end{cases} \quad W(r) = \begin{cases} \nu (0 < r < 1), \\ 0 (1 < r). \end{cases}$$

We will take the following form as the trial functions for the deuteron state.⁷⁾

$$\phi^{(1)} = \begin{cases} u^{(1)} = c_1 r + c_2 r^3 (r < 1), & = ce^{-\gamma r} (1 < r) \\ w^{(1)}(r) = c_3 r^3 + c_4 r^5 (r < 1), & = -c \tan \varepsilon (3/(\gamma r)^2 + 3/(\gamma r) + 1) e^{-\gamma r} (1 < r), \end{cases} \quad (30)$$

* The brief report of this variation principle was published in Prog. Theor. Phys. **11** (1954), 493. I would like to thank to Prof. J. M. Blatt who advised me that this research should not be stopped half way. The Illiac calculation seems not easy unless one understands qualitative trend, so the present variational method may also serve this qualitative purpose.

where c_1, c_2, c_3, c_4 and c are all determined by the continuity condition at $r=1$ and by the normalization constant. Outside the force range, the trial function (30) represents the correct solution. This trial function does not contain the adjustable parameter and may be considered as a crude approximation for the inside. For the second solution, we take analogously as

$$\phi^{(2)} = \begin{cases} u^{(2)}(r) = d_1 r + d_2 r^3 (r < 1), & = d \tan \varepsilon \sinh \gamma r (1 < r), \\ w^{(2)}(r) = d_3 r^3 + d_4 r^5 (r < 1), & = d \{ (1 + 3/(\gamma r)^2) \sinh \gamma r - (3/\gamma r) \cosh \gamma r \} (1 < r), \end{cases} \quad (31)$$

which corresponds to $\lambda_1 = \lambda_2 = \lambda = 0$ in (9). Since λ is a variationally adjustable parameter and a quantity of $O(\gamma^5)$, it seems to be justifiable to set $\lambda = 0$. d_i and d are all determined as c_i and c . These crude but simple trial functions make the computations very easy. The adopted binding energy⁹⁾ and the mixing⁹⁾ ratio are

$$\begin{aligned} B.E. &= 2.227 \text{ Mev}, & Q &= 2.74 \times 10^{-27} \text{ cm}^2, \\ \gamma &= 2.317 \times 10^{12} \text{ cm}^{-1}, & \gamma^2 &= 5.367 \times 10^{24} \text{ cm}^{-2}, & \tan \varepsilon &= -\sqrt{2} Q \gamma^2 = -0.02080. \end{aligned} \quad (32)$$

Using (30) and (31), we get the results tabulated in Table I by the expression (12) or equivalently (12') for μ and ν .

Table I. Variationally computed values of μ, ν by (12), and approximate values of Q and $D\%$ obtained from the trial functions (30), (31). The ranges of the central and tensor forces are assumed to be equal. (+) values of depth correspond to attractive well.

Potential range in 10^{-13} cm	μ in Mev	ν in Mev	Q in 10^{-27} cm	D -probability
2.0	-118.	120.	2.51	7.6%
2.4	4.7	26.1	2.70	4.9%
2.8	14.8	12.0	2.83	3.3%

It may be interesting to compare this with the accurate calculation by Rarita-Schwinger¹⁰⁾ to examine the accuracy of the results. They took the following deuteron data: $B.E. = 2.185$ Mev, $Q = 2.73 \times 10^{-27} \text{ cm}^2$, and got the depth values $\mu = 13.89$ Mev, $\nu = 10.76$ Mev, $D\% = 3.9\%$ for the range 2.8×10^{-13} cm. Then it may be supposed that μ and ν in Table I contains the errors of the order of magnitude 10% . The sources of errors are two-fold, one is due to the crudeness of the trial functions and the other is due to the approximate nature of (2). An estimation of the latter effect may be possible by computing Q actually. The computed values using (30) are written in the fourth column of Table I. The contribution to Q from the inside of the force range is about 13% (20%) for the force range 2.0×10^{-13} cm (2.8×10^{-13} cm), and the values Q in Table I are inferred to be not very inaccurate. The corresponding values of D -state probability are also added in Table I, which are not reliable, because the contribution of inside the force range amounts to 55% (57%) for the range 2.0×10^{-13} cm (2.8×10^{-13} cm). If

the relation (2) is correct, ε specified as (32) will produce the values near $2.74 \times 10^{-27} \text{ cm}^2$. Actually we see from Q of Table I that the answer is affirmative. Christian and Hart⁽¹¹⁾ computed roughly the values of μ and ν . According to their Fig. 14 the ratios μ/ν are $-0.93, 0.21, 1.3$ for the well range $2.0, 2.4, 2.8 \times 10^{-13} \text{ cm}$ respectively, while from our Table I we find $\mu/\nu = -0.98, 0.18, 1.23$. The discrepancies are again within 10%. An estimation of errors due to only the crudeness of the trial functions is given in Appendix II with improved trial functions, which shows the errors are about 3%. Then we may conclude that the relation $\tan \varepsilon = -\sqrt{2} Q_i^{-2}$ is correct within 10% for possible wide choice of the well range. This relation is very useful for the qualitative interpretation of low energy data, (see Appendix I), which shows that the outside behavior of wave function is given only by Q and γ .

b). Different ranges for the central and tensor forces.

Next we will deal with the cases where the ranges of tensor well are $3/2$ and $2/3$ times the ranges of central well. Let the larger range of potentials be unity. The trial functions are the same as (30) and (31). This trial function may be worse approximation for the different ranges than for the equal range. The results are given in Table II.

Table II. Variationally computed values for μ and ν . The ranges of central and tensor forces are taken to be different.

Range of central potential, in 10^{-13} cm	Range of tensor potential, in 10^{-13} cm	μ in Mev	ν in Mev
1.33	2.0	-276.	111.
1.6	2.4	11.8	26.0
1.87	2.8	36.2	10.3
2.0	1.33	-17.	234.
2.4	1.6	10.3	93.
2.8	1.87	14.4	47.

§ 4. Generalization of the theory

A brief note is given in this section about the generalization of the theory discussed in §2 and §3. Suppose that the number of eigenvalues is n and we have m independent solutions. Let A and $B_i (i=1, 2, \dots, n)$ be Hermitian.

$$A\psi^{(i)} = (\lambda_1 B_1 + \lambda_2 B_2 + \dots + \lambda_n B_n) \psi^{(i)}, \quad (j=1, 2, \dots, m) \quad (33)$$

To construct the stationary expressions of $\lambda_1, \lambda_2, \dots, \lambda_n$, we consider the quantity: $(\psi^{(k)}, A\psi^{(j)}) \equiv A_{kj}$, $(\psi^{(i)}, B_i \psi^{(j)}) \equiv B_{ij}^{(i)}$. They satisfy the following relation by (33).

$$A_{kj} = \lambda_1 B_{kj}^{(1)} + \dots + \lambda_n B_{kj}^{(n)}, \quad (k \leq j; 1, 2, \dots, m). \quad (34)$$

The independent number of eq. (34) is $m(m+1)/2$. If $m(m+1)/2 \geq n$ holds, we can choose n equations among above $m(m+1)/2$ equations suitably. We will write them in the following way.

$$A_l = \lambda_1 B_l^{(1)} + \lambda_2 B_l^{(2)} + \cdots + \lambda_n B_l^{(n)}, \quad (l=1, 2, \cdots n). \quad (35)$$

It is possible to prove analogously as in §2 that the stationary expression of λ_i is given by

$$\lambda_i = \frac{\begin{array}{c} \text{\scriptsize i-th column} \\ \downarrow \\ \left| \begin{array}{ccc} B_1^{(1)}, \cdots A^{(1)}, \cdots B_n^{(1)} \\ B_1^{(2)}, \cdots A^{(2)}, \cdots B_n^{(2)} \\ \vdots \\ B_1^{(n)}, \cdots A^{(n)}, \cdots B_n^{(n)} \end{array} \right|}{\left| \begin{array}{ccc} B_1^{(1)}, \cdots B_n^{(1)} \\ \vdots \\ B_1^{(n)}, \cdots B_n^{(n)} \end{array} \right|} \quad (i=1, 2, \cdots n). \quad (36)$$

This is just the Cramer formula for the simultaneous linear equations (35). One of the most important applications of (36) is that, when we have n knowledges on the S -matrix of the system (including the knowledge on bound state), n interaction parameters are determined by (36) simultaneously.

The author would like to thank heartily to Professor T. Yamanouchi for his kind interest and helpful corrections to this work.

Appendix I.

Derivation of $\tan \varepsilon = -\sqrt{2}Q\gamma^2$

The approximate relation between the mixing ratio ε and the experimental values of Q and γ^2 of the deuteron state is probably first due to J. Schwinger¹², and only the result is given by Blatt and Weisskopf¹⁾, p 112. After that, Biedenharn and Blatt¹²⁾ have discussed this relation and gave a modification according to the effective range theory of nuclear forces as,

$$\tan \varepsilon = -\sqrt{2} (1-\gamma r_0)^2 Q \gamma^2, \quad (37)$$

where the new added factor $(1-\gamma r_0)^2$ is $0.35 \sim 0.40$ in actual cases. Their derivation of (37) seems reasonable, but according to our analysis in §3, this factor is surely unnecessary. Hence it will be somewhat worthwhile to follow the way of derivation again.

To do this we need to assume that the wave functions of the deuteron spread over enough compared with the range of nuclear forces. If this is the case, we can estimate by (29) that the D -state is small compared with the S -state outside the range of nuclear forces. The wave functions of the deuteron are given by (3) outside the range of nuclear forces. Since the most important contribution to Q comes from the outer region, we can neglect the effects of nuclear forces in the first approximation. Then

$$\begin{aligned} Q &= (\sqrt{2}/10) \int_0^\infty r^2 (u w - w^2 / \sqrt{8}) dr \rightarrow (\sqrt{2}/10) \int_0^\infty r^2 u w dr \\ &\rightarrow -(\sqrt{2}/10N) \tan \varepsilon \int_0^\infty r^2 e^{-2\gamma r} \{3/(\gamma r)^2 + 3/(\gamma r) + 1\} dr \\ &= -(\sqrt{2} \tan \varepsilon / 40 N \gamma^3) (6 + 3 + 1), \end{aligned}$$

where N is a normalization constant and $N \rightarrow \int_0^\infty e^{-2\gamma r} dr = 1/(2\gamma)$. Thus we have $Q \rightarrow -\tan \varepsilon / (\sqrt{2} \gamma^2)$. We have used the asymptotic form (3) not only for S -state but also for

D-state. But this form for *D*-state has singular nature at the origin as $3/(\gamma r)^2$ and $3/(\gamma r)$, and actually these terms give main contributions to *Q* as seen above (60 % and 30 % respectively). This seems to be far from the actual situation. While, we observe that the normalization constant *N* is also overestimated by neglecting the effect of nuclear forces. Nevertheless, $Q = -\tan \varepsilon / (\sqrt{2} \gamma^2)$ has turned out to be good approximation, hence it will be inferred that the two over-estimations in the numerator and denominator are cancelled out to each other. Biedenharn and Blatt have corrected a part of these errors.

Appendix II. Improved trial functions

In order to estimate the errors due to the crudeness of trial functions (30), (31), let us take improved trial functions and consider the problem at zero energy.* In this case, the depths μ and ν are to be determined by the scattering length $a(=1/\alpha)$ and the mixing parameter *q* defined in (29b).

The assumed trial functions are

$$\phi^{(1)} = \begin{cases} u^{(1)} = c_1 r + c_2 r^3 + c_3 r^5, & (r < 1), & = 1 - \alpha r, \\ w^{(1)} = c_3 r^3 + c_4 r^5 + c_5 r^3 \log r, & & = 3q/r^2, \end{cases} \quad (1 < r), \quad (38)$$

$$\phi^{(2)} = \begin{cases} u^{(2)} = d_1 r + d_2 r^3 + d_3 r^5, & (r < 1), & = -15qr, \\ w^{(2)} = d_3 r^3 + d_6 (7r^3 - 5r^5), & & = r^3, \end{cases} \quad (1 < r). \quad (39)$$

The outside forms of $\phi^{(1)}$ and $\phi^{(2)}$ are correct, except that an approximation is made corresponding to $\lambda=0$ in (9). If we put $c_5=c_6=d_5=d_6=0$, (38) and (39) are approximate functions corresponding to (30) and (31). (38) and (39) contain four essential parameters, which are conveniently determined as follows. First using the trial functions with no adjustable parameter, crude μ' and ν' are calculated by (12). Using μ' , ν' and the trial functions $\phi^{(i)}$ with four parameters, we set

$$\int_0^\infty (A - \mu' B - \nu' C) \phi^{(i)} dr = 0; \text{ for the equation (1a),} \quad (i=1, 2).$$

$$\int_0^\infty r^2 (A - \mu' B - \nu' C) \phi^{(i)} dr = 0; \text{ for the equation (1b).}$$

These four equations settle the values of four parameters. The computations are carried out for $(\alpha=0.5, q=0)$ and $(\alpha=0.5, q=0.02)$. The results thus obtained are summarized in Table III. We can see from this table that the trial functions with no parameter produce fairly good results which contain the errors only 3 %.

*Since this calculation was carried out in an early stage of this research, the zero energy state was dealt with for simplicity. The chosen value of *q* is too small, because we referred to eq. (37) instead of eq. (2).

Table III. Variationally computed values using no parameter and four parameter trial functions (38), (39) are compared with the exact one.

Trial function	$q=0$		$q=0.02$	
	μ	ν	μ	ν
No parameter function	4.14	0	3.98	0.673
Four "	4.118	0	3.929	0.6951
Exact solution	4.116	0	?	?

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On the Exchange Magnetic Moment of the Two Nucleon System*

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The two body exchange magnetic moment operators are calculated in the pseudoscalar meson theory with non-relativistic pseudovector coupling. The method of canonical transformation is used and results are obtained to the order of eg^4 in the power series expansion. The radial parts of all possible operators in the static approximation are given. The contributions of the exchange moment to the magnetic moment of the deuteron are calculated and it is found that in order to adjust the experimental moment $\mu r = 0.74$ should be taken as the lower limit in the radial integral.

§ 1. Introduction

It has been recognized for a long time that the presence of exchange forces between nucleons implies the existence of currents in the space about interacting nucleons. These currents will contribute to the magnetic properties of nuclei. In fact, a contribution of this origin has been known in the magnetic moments of H^3 and He^3 and also should be taken account of in the problem of the photodisintegration of deuteron. Phenomenological attempts have been made by Osborne-Foldy and Berger-Foldy¹⁾ to obtain the possible forms of the exchange magnetic moment operators, by making use of various invariance requirements, such as those with respect to space translation, space rotation, time reversal and so on. Physically, these currents are due to the charged mesons which are being exchanged between the nucleons and give rise to the exchange forces. Therefore, their explicit forms must depend upon the choice of the particular meson theory to be used. In fact, the radial parts of all but one term among these exchange moment operators can not be defined unambiguously in these phenomenological discussions.

On the other hand, calculations by many authors^{2), 3)} on nuclear forces, meson-scattering, magnetic moment of nucleon and photoproduction of mesons, in the static approximation which neglects recoil of nucleons and nucleon pair-formation and assumes p-wave mesons only, show good agreements with the experiments. The purpose of the present work is to obtain the radial parts of the exchange magnetic moment operators in the same approximation as in the case of the fourth order adiabatic nuclear potential assuming p-wave mesons only.

In Sections 2 and 3, the exchange magnetic moment operators are obtained to the order of eg^4 . Here we use the method of canonical transformation, which was employed

* Some parts of the results of this paper have been reported in the letter to the editor of this journal, S. Hatano, Prog. Theor. Phys. 14 (1955), 170

by Nishijima¹⁾ to obtain the fourth order nuclear potential, to eliminate successively the parts of Hamiltonian which are non-diagonal with respect to meson occupation numbers. Of course, various divergent contributions arise due to the self meson field. Even after the renormalization of the coupling constant has been made, divergencies still remain⁵⁾. These divergencies are dealt with by the cut-off of meson momentum above 6μ . As to the value of the coupling constant we adopt here $g^2/4\pi = 0.08$. No cut-off procedures are applied to the high frequency parts of the virtual mesons which are being exchanged between two nucleons. In Section 4, we use the exchange moment operators obtained in Section 3 to evaluate the exchange moment contribution to the magnetic moment of deuteron and discuss the results.

§ 2. Derivation of the exchange magnetic moment operator

We shall start from the following non-relativistic Hamiltonian for a system which consists of nucleon field, meson field and external magnetic field. This Hamiltonian is

$$H = H_N + H_M + H_{MN} + H_{RN} + H_{RM} + H_{RMN} \quad (2.1)$$

where

$$H_N = m \int d\mathbf{x} \phi^*(\mathbf{x}) \phi(\mathbf{x}),$$

$$H_M = \frac{1}{2} \int d\mathbf{x} \{ \pi_\alpha^2(\mathbf{x}) + (\nabla \phi_\alpha(\mathbf{x}))^2 + \mu^2 \phi_\alpha^2(\mathbf{x}) \},$$

$$H_{MN} = \frac{g}{\mu} \int d\mathbf{x} \phi^*(\mathbf{x}) \tau_\alpha (\boldsymbol{\sigma} \cdot \nabla \phi_\alpha(\mathbf{x})) \phi(\mathbf{x}),$$

$$H_{RN} = -\frac{e}{2m} \int d\mathbf{x} \phi^*(\mathbf{x}) \left[\mu_p \frac{1+\tau_3}{2} + \mu_n \frac{1-\tau_3}{2} \right] (\boldsymbol{\sigma} \cdot \mathbf{H}(\mathbf{x})) \phi(\mathbf{x}), \quad (2.2)^*$$

$$H_{RM} = -e \int d\mathbf{x} (\phi_2(\mathbf{x}) \nabla \phi_1(\mathbf{x}) - \phi_1(\mathbf{x}) \nabla \phi_2(\mathbf{x})) \cdot \mathbf{A}(\mathbf{x}),$$

$$H_{RMN} = -\frac{eg}{\mu} \int d\mathbf{x} \phi^*(\mathbf{x}) (\boldsymbol{\sigma} \cdot \mathbf{A}(\mathbf{x})) (\tau_1 \phi_2(\mathbf{x}) - \tau_2 \phi_1(\mathbf{x})) \phi(\mathbf{x}).$$

In the above expressions, μ and m are meson and nucleon masses, respectively, \mathbf{A} is the vector potential of the external magnetic field, $\mathbf{H} = \text{rot } \mathbf{A}$, and other notations have usual meanings. As we consider in this paper exchange magnetic moment operators independent of nucleon momentum, the terms which depend on the nucleon momentum are omitted from the outset in the Hamiltonian. If we assume a constant external magnetic field \mathbf{H} and take $\mathbf{A} = -\frac{1}{2}[\mathbf{x} \times \mathbf{H}]$, we get from (2.2) the following magnetic moment operator μ in unit of $e/2m$.

* We take $\hbar=c=1$.

$$\mu = \mu_N + \mu_M + \mu_{MN},$$

$$\mu_N = \int d\mathbf{x} \psi^*(\mathbf{x}) \left[\mu_p \frac{1+\tau_3}{2} + \mu_n \frac{1-\tau_3}{2} \right] \sigma \psi(\mathbf{x}),$$

$$\mu_M = m \int d\mathbf{x} (\phi_2(\mathbf{x}) [\mathbf{x} \times \mathbf{V}] \phi_1(\mathbf{x}) - \phi_1(\mathbf{x}) [\mathbf{x} \times \mathbf{V}] \phi_2(\mathbf{x})), \quad (2.3)$$

$$\mu_{MN} = \frac{gm}{\mu} \int d\mathbf{x} \psi^*(\mathbf{x}) [\mathbf{x} \times \boldsymbol{\sigma}] (\tau_1 \phi_2(\mathbf{x}) - \tau_2 \phi_1(\mathbf{x})) \psi(\mathbf{x}).$$

For our purpose it is convenient to introduce the Fourier transforms of $\phi_\alpha(\mathbf{x})$, $\pi_\alpha(\mathbf{x})$, $\psi(\mathbf{x})$ and $\psi^*(\mathbf{x})$,

$$\phi_\alpha(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} (2\omega_k)^{-1/2} (a_\alpha(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a_\alpha^*(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}),$$

$$\pi_\alpha(\mathbf{x}) = \frac{-i}{(2\pi)^{3/2}} \int d\mathbf{k} (\omega_k/2)^{1/2} (a_\alpha(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} - a_\alpha^*(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}}),$$

$$\psi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p} \psi(\mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}}, \quad (2.4)$$

$$\psi^*(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p} \psi^*(\mathbf{p}) e^{-i\mathbf{p} \cdot \mathbf{x}},$$

where

$$\omega_k = (\mu^2 + k^2)^{1/2},$$

$$[a_\alpha(\mathbf{k}), a_\beta^*(\mathbf{k}')] = \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{k}'),$$

and

$$\{\psi_\alpha(\mathbf{p}), \psi_\beta^*(\mathbf{p}')\} = \delta_{\alpha\beta} \delta(\mathbf{p} - \mathbf{p}').$$

In treating the original Schrödinger equation

$$H\Psi = E\Psi, \quad (2.5)$$

we shall regard the interactions of meson and nucleon with the external magnetic field as small perturbation and at first we shall restrict ourselves to the equation

$$(H_N + H_M + H_{MN})\Psi' = E'\Psi'. \quad (2.6)$$

Then we transform this equation by a sequence of canonical transformations, and eliminate non-diagonal parts with respect to the meson occupation numbers successively in the power series expansion in the coupling constant g . Among these transformations the first one eliminates the linear term in g . If we write

$$\Psi' = e^{-iS_1} \Psi''_1, \quad (2.7)$$

S_1 can be determined by the condition

$$[iS_1, H_N + H_M] = -H_{NM}. \quad (2.8)$$

The second transformation eliminates from the resulting equation the terms in g^2 which

correspond to the double meson production and annihilation (e^{-iS_2}). From the transformed Hamiltonian the terms in g^3 which contain the single meson production and annihilation are removed by the third transformation (e^{-iS_3}). Finally, the terms in g^4 corresponding to the double meson production and annihilation are eliminated in the same manner (e^{-iS_4} and e^{-iS_5}). We shall summarize the generating functions of the sequence of canonical transformations which are necessary to obtain the exchange magnetic moments to the order of eg^4 ,

$$\begin{aligned}
 S_1 &= -\frac{g}{\mu} \frac{1}{(2\pi)^{3/2}} \int d\mathbf{q} d\mathbf{p} d\mathbf{k} \frac{1}{\sqrt{2\omega_k \omega_k}} \phi^*(\mathbf{q}) \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}) \times \\
 &\quad \{ \tau_\alpha(\mathbf{k}) \delta(\mathbf{q} - \mathbf{p} - \mathbf{k}) + a_\alpha^*(\mathbf{k}) \delta(\mathbf{q} - \mathbf{p} + \mathbf{k}) \} \phi(\mathbf{p}), \\
 S_2 &= \frac{i}{4} \left(\frac{g}{\mu} \right)^2 \frac{1}{(2\pi)^3} \int d\mathbf{q} d\mathbf{p} d\mathbf{k} d\mathbf{k}' \frac{1}{\sqrt{\omega_k^3 \omega_{k'} \omega_k (\omega_k + \omega_{k'})}} \times \\
 &\quad \phi^*(\mathbf{q}) [\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}), \tau_\beta(\boldsymbol{\sigma} \cdot \mathbf{k}')] \{ a_\alpha(\mathbf{k}) a_\beta(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}') \\
 &\quad + a_\alpha^*(\mathbf{k}) a_\beta^*(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} + \mathbf{k}') \} \phi(\mathbf{p}), \\
 S_3 &= \left(\frac{g}{\mu} \right)^3 \frac{1}{(2\pi)^{9/2}} \int d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' \frac{1}{\omega_k^2 \sqrt{2\omega_{k'} \omega_{k'}}} \times \\
 &\quad \phi_k^*(\mathbf{q}') \phi_i^*(\mathbf{q}) [\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}), \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}')] j_j(\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}))_{kl} \times \\
 &\quad \{ a_\alpha(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k}) \\
 &\quad - a_\alpha^*(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} + \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' - \mathbf{k}) \} \phi_j(\mathbf{p}) \phi_l(\mathbf{p}') \\
 &\quad - \frac{1}{3} \left(\frac{g}{\mu} \right)^3 \frac{1}{(2\pi)^{9/2}} \int d\mathbf{q} d\mathbf{p} d\mathbf{k} d\mathbf{k}' \frac{1}{\omega_k^3 \sqrt{2\omega_{k'} \omega_{k'}}} \times \\
 &\quad \phi^*(\mathbf{q}) [\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}), \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}')] \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}) \{ a_\alpha(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k}') \\
 &\quad + a_\alpha^*(\mathbf{k}') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k}') \} \phi(\mathbf{p}), \tag{2.9} \\
 S_4 &= \frac{i}{16} \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' d\mathbf{k}'' \frac{(\omega_k - \omega_{k'}) (\omega_k + \omega_{k'})}{\omega_k^3 \sqrt{\omega_{k'} \omega_{k'}} (\omega_k + \omega_{k'}) \sqrt{\omega_{k''} \omega_{k''}} (\omega_{k'} + \omega_{k''})} \\
 &\quad \times \phi_k^*(\mathbf{q}') \phi_i^*(\mathbf{q}) [\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}), \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}')] j_j [\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}), \tau_\beta(\boldsymbol{\sigma} \cdot \mathbf{k}'')] j_l \times \\
 &\quad \{ a_\alpha(\mathbf{k}') a_\beta(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k} - \mathbf{k}'') \\
 &\quad - a_\alpha^*(\mathbf{k}') a_\beta^*(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} + \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' - \mathbf{k} + \mathbf{k}'') \} \phi_j(\mathbf{p}) \phi_l(\mathbf{p}') \\
 &\quad + \frac{i}{4} \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' d\mathbf{k}'' \frac{1}{\omega_k^2 \sqrt{\omega_{k'} \omega_{k'}} \sqrt{\omega_{k''} (\omega_{k'} + \omega_{k''})^2}} \times \\
 &\quad \phi_k^*(\mathbf{q}') \phi_i^*(\mathbf{q}) [[\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}'), \tau_\beta(\boldsymbol{\sigma} \cdot \mathbf{k}'')], \tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k})] j_j (\tau_\alpha(\boldsymbol{\sigma} \cdot \mathbf{k}))_{kl} \times \\
 &\quad \{ a(\mathbf{k}') a_\beta(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}' - \mathbf{k}'') \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k}) \\
 &\quad - a_\alpha^*(\mathbf{k}') a_\beta^*(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} + \mathbf{k}' + \mathbf{k}'') \delta(\mathbf{q}' - \mathbf{p}' - \mathbf{k}) \} \phi_j(\mathbf{p}) \phi_l(\mathbf{p}'),
 \end{aligned}$$

$$S_5 = \frac{i}{16} \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' d\mathbf{k}'' \frac{1}{\omega_k^2 \sqrt{\omega_{kl}} \omega_{kl'} \sqrt{\omega_{kl''}} \omega_{kl''} (\omega_{kl} + \omega_{kl''})} \times$$

$$\phi_k^*(\mathbf{q}') \phi_i^*(\mathbf{q}) [3[\tau_\alpha(\sigma \cdot \mathbf{k}), \tau_\alpha(\sigma \cdot \mathbf{k}')]_{ij} [\tau_\alpha(\sigma \cdot \mathbf{k}), \tau_\beta(\sigma \cdot \mathbf{k}'')]_{kl} \times$$

$$\{a_\alpha(\mathbf{k}') a_\beta(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k} - \mathbf{k}'')$$

$$- a_\alpha^*(\mathbf{k}') a_\beta^*(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} + \mathbf{k}') \delta(\mathbf{q}' - \mathbf{p}' - \mathbf{k} + \mathbf{k}'') \}$$

$$- 4[\tau_\alpha(\sigma \cdot \mathbf{k}'), [\tau_\alpha(\sigma \cdot \mathbf{k}), \tau_\beta(\sigma \cdot \mathbf{k}'')]_{ij} (\tau_\alpha(\sigma \cdot \mathbf{k}))_{kl} \times$$

$$\{a_\alpha(\mathbf{k}') a_\beta(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} + \mathbf{k} - \mathbf{k}' - \mathbf{k}'') \delta(\mathbf{q}' - \mathbf{p}' - \mathbf{k})$$

$$- a_\alpha^*(\mathbf{k}') a_\beta^*(\mathbf{k}'') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} + \mathbf{k}' + \mathbf{k}'') \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k}) \}] \phi_j(\mathbf{p}) \phi_l(\mathbf{p}').$$

Now returning to the original Schrödinger equation (2.5), we can obtain the exchange magnetic moment operators from the perturbed Hamiltonian which are transformed by the above canonical transformations. Thus we apply these transformations to μ and compute the following commutators:

$$[iS_1, \mu_{MN}], \frac{1}{2!} [iS_1, [iS_1, \mu_N + \mu_M]], \frac{1}{3!} [iS_1, [iS_1, [iS_1, \mu_{MN}]]],$$

$$\frac{1}{4!} [iS_1, [iS_1, [iS_1, [iS_1, \mu_N + \mu_M]]]], [iS_2, [iS_2, \mu_{MN}]],$$

$$\frac{1}{2!} [iS_2, [iS_1, [iS_1, \mu_N + \mu_M]]], \frac{1}{2!} [iS_2, [iS_2, \mu_N + \mu_M]],$$

$$[iS_3, \mu_{MN}], [iS_3, [iS_1, \mu_N + \mu_M]], [i(S_4 + S_5), \mu_M]. \quad (2.10)$$

Then we obtain the expression for the transformed μ of the form

$$\int d\mathbf{q} d\mathbf{q}' d\mathbf{p} d\mathbf{p}' d\mathbf{k} d\mathbf{k}' \phi_k^*(\mathbf{q}') \phi_i^*(\mathbf{q}) M(\mathbf{k}, \mathbf{k}')_{ij,kl} \phi_j(\mathbf{p}) \phi_l(\mathbf{p}') \delta(\mathbf{q} - \mathbf{p} - \mathbf{k} - \mathbf{k}') \times \delta(\mathbf{q}' - \mathbf{p}' + \mathbf{k} + \mathbf{k}'), \quad (2.11)$$

where $M(\mathbf{k}, \mathbf{k}')$ is the Fourier transform of the exchange magnetic moment operator. Now we shall return to the configuration space and assign two spinor wave functions to two nucleons. In this way the expression (2.11) can be rewritten as

$$\int d\mathbf{r}_1 d\mathbf{r}_2 \phi_k^*(\mathbf{r}_2) \phi_i^*(\mathbf{r}_1) \left[\int d\mathbf{k} d\mathbf{k}' M(\mathbf{k}, \mathbf{k}')_{ij,kl} e^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}} \right] \phi_j(\mathbf{r}_1) \phi_l(\mathbf{r}_2), \quad (2.12)$$

where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

Therefore, if we evaluate $\int d\mathbf{k} d\mathbf{k}' M(\mathbf{k}, \mathbf{k}') e^{i(\mathbf{k} + \mathbf{k}') \cdot \mathbf{r}}$ by the usual method used in the calculation of nuclear potential¹⁾, we can obtain the exchange magnetic moment operators.

§ 3. Radial dependence of the magnetic moment operators

In the phenomenological theory based on differential charge conservation and reasonable symmetry restrictions on the nuclear Hamiltonian, Berger and Foldy¹⁾ have found the

possible forms of the two-body exchange current operators. Here we list the resultant magnetic moment operators for two-body system.

$$\begin{aligned}
 M_I &= m[\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 [\mathbf{r}_1 \times \mathbf{r}_2] V(\mathbf{r}), \\
 M_I &= 1/4 \cdot [\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 [\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}] F_I(\mathbf{r}), \\
 M_{II} &= 1/4 \cdot [\boldsymbol{\tau}^{(1)} \times \boldsymbol{\tau}^{(2)}]_3 \{3(\mathbf{r} \cdot [\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}]) \mathbf{r}/r^2 - [\boldsymbol{\sigma}^{(1)} \times \boldsymbol{\sigma}^{(2)}]\} F_{II}(\mathbf{r}), \\
 M_{II} &= 1/4 \cdot (\tau_3^{(1)} - \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)}) F_{II}(\mathbf{r}), \\
 M_{III} &= 1/4 \cdot (\tau_3^{(1)} - \tau_3^{(2)}) \{3(\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)})) \mathbf{r}/r^2 - (\boldsymbol{\sigma}^{(1)} - \boldsymbol{\sigma}^{(2)})\} F_{III}(\mathbf{r}), \\
 M_{III} &= 1/4 \cdot (\tau_3^{(1)} + \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) F_{III}(\mathbf{r}), \\
 M_{IIII} &= 1/4 \cdot (\tau_3^{(1)} + \tau_3^{(2)}) \{3(\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})) \mathbf{r}/r^2 - (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\} F_{IIII}(\mathbf{r}), \\
 M_{IV} &= 1/4 \cdot (1 + \tau_3^{(1)} \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) F_{IV}(\mathbf{r}), \\
 M_{IV'} &= 1/4 \cdot (1 + \tau_3^{(1)} \tau_3^{(2)}) \{3(\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})) \mathbf{r}/r^2 - (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\} F_{IV'}(\mathbf{r}), \quad (3 \cdot 1) \\
 M_V &= 1/4 \cdot (1 - \tau_3^{(1)} \tau_3^{(2)}) (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) F_V(\mathbf{r}), \\
 M_{V'} &= 1/4 \cdot (1 - \tau_3^{(1)} \tau_3^{(2)}) \{3(\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})) \mathbf{r}/r^2 - (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\} F_{V'}(\mathbf{r}), \\
 M_{VI} &= -1/4 \cdot (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)}) (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) F_{VI}(\mathbf{r}), \\
 M_{VI'} &= -1/4 \cdot (\tau_1^{(1)} \tau_1^{(2)} + \tau_2^{(1)} \tau_2^{(2)}) \{3(\mathbf{r} \cdot (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})) \mathbf{r}/r^2 - (\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)})\} F_{VI'}(\mathbf{r}),
 \end{aligned}$$

where

$$\mathbf{r} = |\mathbf{r}_1 - \mathbf{r}_2|.$$

In the above expressions M_I is a so-called longitudinal or space exchange moment which has been investigated by Sachs⁶⁾. Fortunately the radial dependence of M_I can be determined unambiguously to the above form, when the exchange forces of type $(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) V(\mathbf{r})$ act between the nucleons⁷⁾. However, in the remaining twelve exchange moment operators $M_I - M_{VI'}$, so-called transverse exchange moments, radial functions $F_I(\mathbf{r}) - F_{VI'}(\mathbf{r})$ remain arbitrary functions of r from the phenomenological considerations.

We will list the radial functions $F_I(\mathbf{r}) - F_{VI'}(\mathbf{r})$, which are calculated to the order of e/f^4 by the method described in Section 2, using pseudoscalar meson theory with pseudovector coupling in the non-relativistic limit. As mentioned in the introduction, after renormalization of the coupling constant in parallel with the case of the nuclear potential⁵⁾, the remaining divergent contributions are treated by the cut-off procedures. According to our calculation the radial functions $F_I(\mathbf{r}) - F_{VI'}(\mathbf{r})$ are (we use the notation $x = \mu r$ and the unit $e/2m$)

$$V(x) = V^{(2)}(x) + V^{(4)}(x),$$

$$V^{(2)}(x) = \frac{g^2}{4\pi} \mu \left\{ \frac{1}{3} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + \left(\frac{1}{3} + \frac{1}{x} + \frac{1}{x^2} \right) S_{12} \right\} \frac{e^{-x}}{x},$$

$$V^{(4)}(x) = - \left(\frac{f^2}{4\pi} \right)^2 \frac{8\mu}{\pi} \left\{ \left(\frac{1}{x} + \frac{23}{4x^3} \right) K_0(2x) + \left(\frac{3}{x^2} + \frac{23}{4x^4} \right) K_1(2x) \right\},$$

$$F_I(x) = -\left(\frac{g^2}{4\pi}\right)^2 \frac{4m}{3\mu} \left(-2 + \frac{1}{x}\right) e^{-x} + \left(\frac{g^2}{4\pi}\right)^2 \left(-\frac{32m}{3\pi\mu} f_0(\mu) + \mu^{-64} \frac{f_1(\mu)}{9\pi}\right) \frac{e^{-x}}{x}, \quad (3.2)$$

$$F_{II}(x) = -\left(\frac{g^2}{4\pi}\right)^2 \frac{4m}{3\mu} \left(1 + \frac{1}{x}\right) e^{-x} + \left(\frac{g^2}{4\pi}\right)^2 \left(\frac{16m}{3\pi\mu} f_0(\mu) - \mu^{-32} \frac{f_1(\mu)}{9\pi}\right) \times \\ \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x},$$

$$F_{III}(x) = \left(\frac{g^2}{4\pi}\right)^2 \left\{ -\frac{8m}{3\pi\mu} \left(8 + \frac{10}{x^2}\right) K_1(2x) + \frac{2}{3} \mu^{-2} \left(\frac{2}{x^4} + \frac{4}{x^3} + \frac{2}{x}\right) e^{-2x} \right. \\ \left. - \frac{32}{9\pi} \mu^{-2} f_1(\mu) \frac{e^{-2x}}{x} \right\},$$

$$F_{IV}(x) = \left(\frac{g^2}{4\pi}\right)^2 \left\{ -\frac{8m}{3\pi\mu} \left(\frac{6}{x^3} K_0(2x) - \left(4 - \frac{7}{x^2}\right) K_1(2x)\right) \right. \\ \left. + \frac{2}{3} \mu^{-2} \left(\frac{2}{x^4} + \frac{7}{x^3} + \frac{6}{x^2} + \frac{2}{x}\right) e^{-2x} - \frac{32}{9\pi} \mu^{-2} f_1(\mu) \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x} \right\},$$

$$F_{III'}(x) = \left(\frac{g^2}{4\pi}\right)^2 \left\{ -\frac{8m}{3\pi\mu} \left(8 + \frac{10}{x^2}\right) K_1(2x) - \frac{2}{3} \mu^{-2} \left(\frac{2}{x^4} + \frac{4}{x^3} + \frac{2}{x}\right) e^{-2x} - \frac{32}{9\pi} \mu^{-2} f_1(\mu) \frac{e^{-2x}}{x} \right\},$$

$$F_{III''}(x) = \left(\frac{g^2}{4\pi}\right)^2 \left\{ \frac{8m}{3\pi\mu} \left(\frac{6}{x^3} K_0(2x) - \left(4 - \frac{7}{x^2}\right) K_1(2x)\right) - \frac{2}{3} \mu^{-2} \left(\frac{2}{x^4} + \frac{7}{x^3} + \frac{6}{x^2} + \frac{2}{x}\right) e^{-2x} \right. \\ \left. - \frac{32}{9\pi} \mu^{-2} f_1(\mu) \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x} \right\},$$

$$F_{IV'}(x) = -\left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ \frac{2}{3} \left(\frac{2}{x^4} + \frac{4}{x^3} - \frac{4}{x^2} + \frac{4}{x}\right) e^{-2x} + \frac{16}{9\pi} f_1(\mu) \frac{1}{x} e^{-x} \right\},$$

$$F_{IV''}(x) = -\left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ \frac{2}{3} \left(\frac{14}{x^4} + \frac{13}{x^3} + \frac{2}{x^2} - \frac{2}{x}\right) e^{-2x} + \frac{16}{9\pi} f_1(\mu) \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x} \right\},$$

$$F_V(x) = \left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ \frac{2}{3} \left(\frac{14}{x^4} + \frac{28}{x^3} + \frac{20}{x^2} + \frac{4}{x}\right) e^{-2x} + \frac{16}{9\pi} f_1(\mu) \frac{e^{-x}}{x} \right\},$$

$$F_{V'}(x) = \left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ -\frac{2}{3} \left(\frac{10}{x^4} + \frac{17}{x^3} + \frac{10}{x^2} + \frac{2}{x}\right) e^{-2x} + \frac{16}{9\pi} f_1(\mu) \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x} \right\},$$

$$F_{VI}(x) = \left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ \frac{8}{3} \left(\frac{4}{x^4} + \frac{8}{x^3} + \frac{4}{x^2} + \frac{2}{x}\right) e^{-2x} + \frac{32}{9\pi} f_1(\mu) \frac{e^{-x}}{x} \right\},$$

$$F_{VI'}(x) = \left(\frac{g^2}{4\pi}\right)^2 \mu^+ \left\{ -\frac{8}{3} \left(-\frac{1}{x^4} + \frac{1}{x^3} + \frac{2}{x^2} + \frac{1}{x}\right) e^{-2x} + \frac{32}{9\pi} f_1(\mu) \left(\frac{3}{x^3} + \frac{3}{x^2} + \frac{1}{x}\right) e^{-x} \right\},$$

where

$$\mu^+ = \frac{\mu_p + \mu_n}{2}, \quad \mu^- = \frac{\mu_p - \mu_n}{2},$$

$$f_0(\mu) = \int_0^{k_{\max}} dk \frac{k^4}{(k^2 + \mu^2)^{5/2}},$$

$$f_1(\mu) = \frac{1}{\mu} \int_0^{k_{\max}} dk \frac{k^4}{(k^2 + \mu^2)^2}.$$

In fact, as shown in the general argument given by Dalitz⁷, $V(x)$ is the part which is proportional to $(\tau^{(1)} \cdot \tau^{(2)})$ in the second plus fourth order nuclear potentials¹⁾.

In Figs. 1 and 2 the functions $F_I(x) - F_{IV}(x)$ are plotted in unit of $e/2m$, using the values $g^2/4\pi = 0.08$ and $k_{\max} = 6\mu$. Among them $F_I^{(2)}(x)$ and $F_{IV}^{(2)}(x)$ have been given by Villars⁸⁾. Except $F_I(x)$ and $F_{IV}(x)$ other radial functions contain terms of order e/g^4 only.

The power series expansion used here is not so good an approximation and, moreover, in the present meson theory very large contributions come from the high frequency components of virtual mesons^{3),9)}. Therefore, in our calculation in which no cut-off procedure is taken for the high frequency part of the virtual mesons which are being exchanged between two nucleons, the resulting radial functions could not be believed to be quite reasonable for small values of x . Considerations in this respect are given in Section 4.

In the case of the deuteron, exchange moment operators M_I , M_{III} , $M_{III'}$, M_{IV} and $M_{IV'}$ do not contribute, because M_I does not contribute to any two-body systems and for other four terms we can see that they give rise to identically vanishing contributions, when their isotopic spin parts operate upon the singlet isotopic spin wave function of deuteron.

M_V , $M_{V'}$, M_{VI} and $M_{VI'}$, which give rise to the exchange contributions to the magnetic moment of deuteron, arise from H_{RN} only in the Hamiltonian, as shown by Miyazawa¹⁰⁾. $M_{VI} - M_{VI'}$ contain only e/g^4 -terms and this fact was pointed out by Sato and Itabashi.^{5),11)} $F_{IV}(x) - F_{IV'}(x)$ in (3.2) coincide with those obtained by Sato and Itabashi¹¹⁾, and their strengths are small by an order of magnitude compared with the strengths of $F_I(x) - F_{III'}(x)$ which arise also from the meson current. (See Figs. 1 and 2.)

The exchange contributions to the photodisintegration of the deuteron come mainly from M_V , $M_{V'}$, M_{VI} and $M_{VI'}$. As they contribute to this phenomenon with equal weight, the ratios of the e/g^4 terms in their radial functions to the e/g^2 terms, i.e. those given by Villars⁸⁾, will supply the measure of the relative importance of e/g^4 terms. The values of these ratios at $x=1.0$ and $x=1.5$ are given in Table 1.

Table 1. Ratio of eg^4 terms to eg^2 terms at $x=1.0$ and 1.5 .

x	$ F_I^{(4)}/F_I^{(2)} $	$ F_{II}/F_I^{(2)} $	$ F_{IV}^{(4)}/F_{IV}^{(2)} $	$ F_{IV'}/F_{IV}^{(2)} $
1.0	5.5%	40.1%	9.6%	35.9%
1.5	2.8%	12.5%	4.8%	15.3%

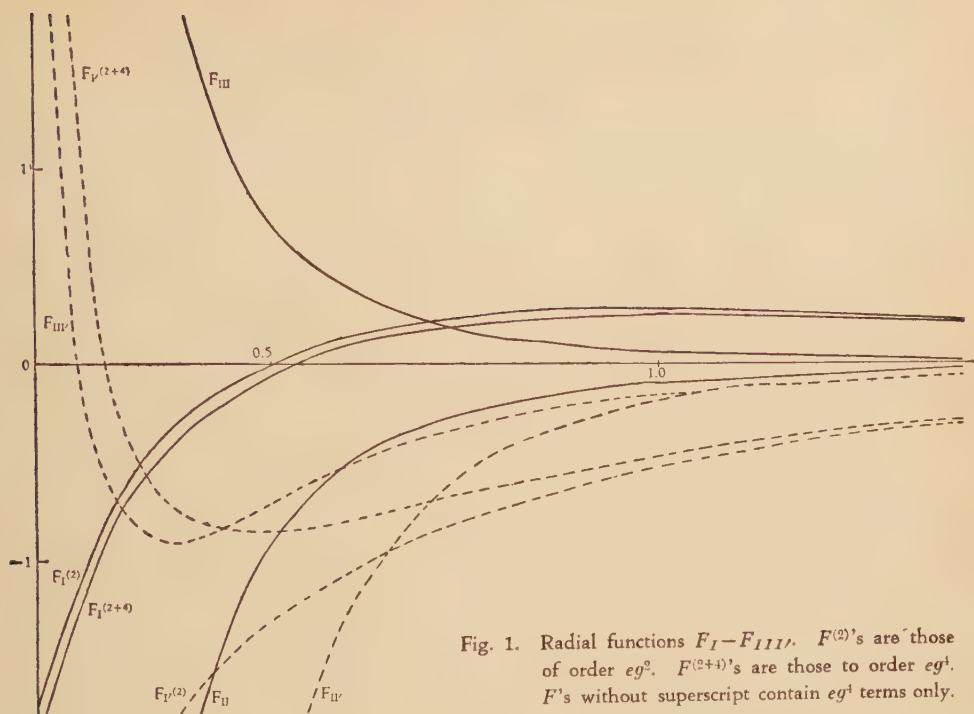


Fig. 1. Radial functions $F_I - F_{III}$. $F^{(2)}$'s are those of order eg^2 . $F^{(2+4)}$'s are those to order eg^4 . F 's without superscript contain eg^1 terms only.

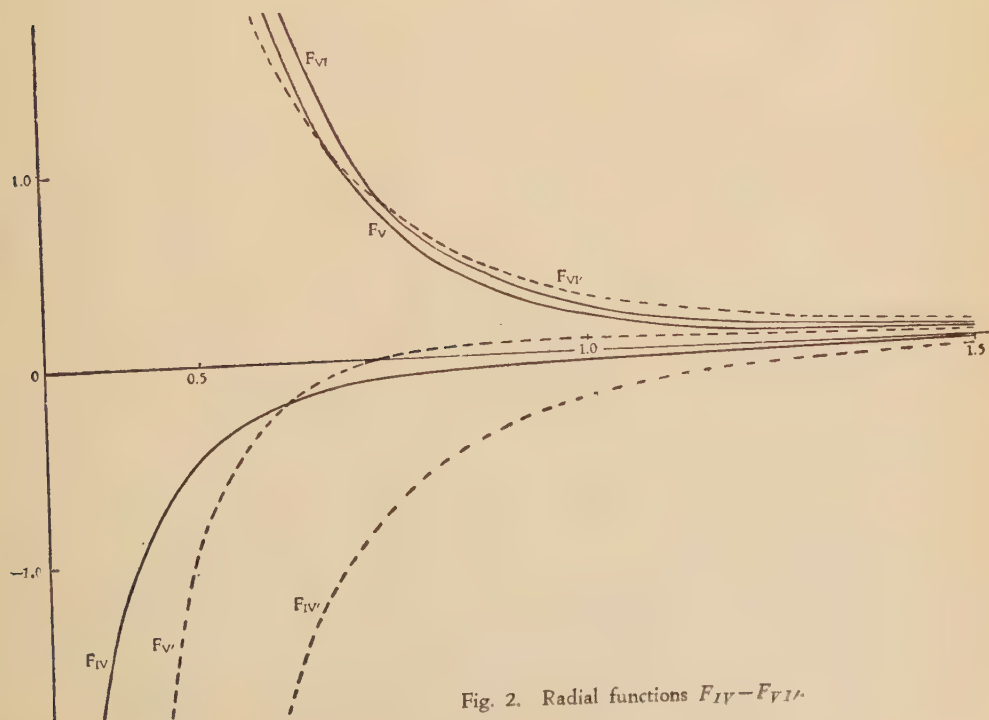


Fig. 2. Radial functions $F_{IV} - F_{VII}$.

§ 4. Exchange contributions to the magnetic moment of the deuteron and the criticism of the present calculation

Taking account of the exchange moment contributions, the magnetic moment of deuteron is given by the following expression

$$\mu_D = \mu_n + \mu_p - \frac{3}{2} \left(\mu_n + \mu_p - \frac{1}{2} \right) \int w^2 dx + \Delta\mu_D, \quad (4.1)$$

where

$$\begin{aligned} \Delta\mu_D &= \int u^2 A(x) dx + \sqrt{2} \int u w B(x) dx + \int w^2 C(x) dx, \\ A(x) &= F_V(x) + F_{V'}(x), \\ B(x) &= F_{V'}(x) + F_{V''}(x), \\ C(x) &= -\frac{A(x)}{2} + B(x). \end{aligned} \quad (4.2)$$

In the above expressions $u(x)$ and $w(x)$ are the S - and D -state wave functions of the ground state of deuteron, respectively. As to these wave functions we adopt here the wave functions obtained by Iwadare, Otsuki, Tamagaki and Watari¹²⁾. To solve the deuteron problem they have used the meson theoretical potential in the region $x > 1.0$ and, in the region $x < 1.0$ determined the wave functions phenomenologically so as to reproduce the effective range, and the binding energy and quadrupole moment of deuteron. These wave functions give seven percent D -state probability.

Inserting the experimental values $\mu_D = 0.8565$, $\mu_p = 2.7896$ and $\mu_n = -1.9103$ into equation (4.1), we see that $\Delta\mu_D > 0$ is necessary in order to fit the experimental results. In fact, referring to Fig. 2, we can see at once that $\Delta\mu_D > 0$, using the exchange moment operators obtained in Section 3. As to this sign, our results are in good agreement with experimental results. In the present calculation, however, extremely large contributions arise, in the region of small values of x , from the fact that no cut-off procedure is taken to the high frequency parts of virtual mesons which are being exchanged between two nucleons. Therefore, taking account of these circumstances we have investigated at what value of x one should cut off the lower limit of the radial integrals in equation (4.2) in order to fit the experimental results numerically. We found that $x = 0.74$ is a suitable cut-off point. If one cuts off the high frequency region of the virtual mesons, which are being exchanged between two nucleons, in evaluating the exchange moment operators and makes $F(x)$'s small enough from the neighbourhood of the above point down to the origin, then the magnetic moment of deuteron could be explained in a consistent manner with the nuclear force problem on the basis of meson theory. It may be said that $x = 0.74$ gives the critical point above which the radial functions obtained in the present work are significant. Also, in the application of the various radial functions of this paper to the evaluation of

the exchange moment contributions to the other phenomena, such as the photodisintegration of deuteron, one could expect to obtain results of the correct order, if the cut-off procedures are applied to the radial integrals in the neighbourhood of this point.

Recently Sugawara¹³⁾ has calculated the exchange corrections to the magnetic moment of deuteron. He assumed the ps-ps meson theory and employed Tamm-Dancoff formalism, retaining only the one meson amplitude. Then he identified the two nucleon amplitude with the deuteron wave function and found that the correction is negative and $P_D = 3\% \pm 1\%$. But his conclusion would be doubtful. The reason is: in his treatment of Tamm-Dancoff method he neglected fourth order terms corresponding to two meson amplitude and calculated normalization integral up to the second order. Then he expanded the normalization integral in the denominator and estimated the mesonic correction using normalized deuteron wave function. These treatments seem not to be justifiable. Also, his D-state wave function has not a correct asymptotic form.

The authors are indebted to Professors M. Kobayasi and S. Takagi for their continual encouragement and helpful discussions. They are also grateful to Messrs. J. Iwadare, S. Otsuki, R. Tamagaki and W. Watari for giving us the use of the wave function tables and for valuable discussions.

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Letters to the Editor

The Photodisintegration of Deuteron at High Energy

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September 27, 1955

Recently, the photodisintegration of deuteron has been investigated by many authors in high energy region over the

meson threshold. Keck, Tollestrup and Smythe¹⁾ clarified that the minimum of total cross section occurred around meson threshold 150 Mev and after passing through a broad maximum around 250 Mev, it decreased monotonically with the increasing energy up to 450 Mev. Bruno and Dempken²⁾ calculated the total cross section by taking into account the effect of virtual pion exchange, and they have obtained rather small, monotonically decreasing total cross section.

In this short note we shall show that the maximum of the total cross section around 250 Mev is an evidence for the resonance of a nucleon state, which has played essential role in the photoproduction of pion and the pion nucleon scattering. As in the case of Bruno and Dempken, we have calculated



Fig. 1a



Fig. 1b

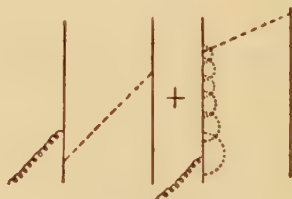


Fig. 1c

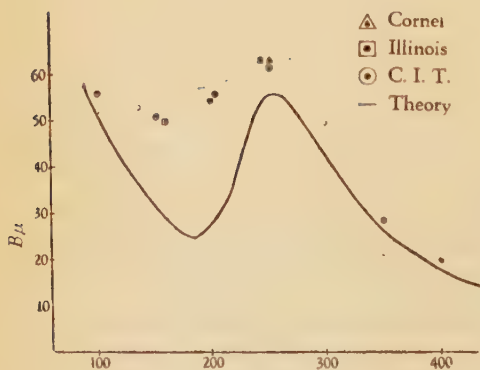


Fig. 2 Mev in Lab System

ed the total cross section in $ps-ps$ meson theory by taking into account the process in which one pion is exchanged, the diagrams of which are illustrated in Fig 1a-1c.

In our case, however, the isobar effect is taken into account as illustrated in Fig 1c. They are calculated by Tamm-Dancoff approximation. In this case we have restricted ourselves to the process in which a pion is exchanged in the free particle state. Of course, this is not a good approximation especially in the near threshold region. Never-

theless, by adopting this approximation, we have examined the qualitative behavior of our process. The calculated total cross section curves are illustrated in Fig 2. Our result shows qualitative agreement with experiment. In spite of too large value of coupling constant ($\frac{g^2}{4\pi} = 22$), the calculated total cross section is rather small in lower energy region. This may probably be due to our neglect of virtual meson exchange. One of the most important problem of the process consists in the angular distribution. However, our approximation is too crude to discuss it. More detailed calculation is now in progress. We wish to express our cordial thanks to Prof. Tollestrup for sending us his unpublished results of experiments and for his kind advices.

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Relation between the Quadrupole Moments and the Widths of the Giant Resonance of Photonuclear Reaction

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The widths of the giant resonance of photonuclear reaction have recently been measured for various nuclei,¹⁾ and the

results show that the widths decrease very slowly with increasing A . So far there have appeared very few theories concerning the widths and none of them has succeeded in explaining the general trend mentioned above. It has also been pointed out experimentally that the widths become very narrow at the magic numbers. Wilkinson has suggested that this fluctuation of the widths can be explained using shell model²⁾, but his detailed calculation has not yet been published.

In order to show more clearly that this fluctuation is closely related to the nuclear quadrupole moments, let us plot the widths in the graph together with the nuclear shape-parameters calculated in the following way from the experimental values of quadrupole moments.

If a nucleus can be regarded as a spheroid, its intrinsic quadrupole moment Q_0 is given by the formula; $Q_0 = 2Z/5 \cdot (R_1^2 - R_2^2)$, where $R_1(R_2)$ is the longer (shorter) axis of the spheroid. Then the value of ϵ , the eccentricity of the ellipse, can be obtained from the value of Q_0 , and we plot the value of ϵ against A in Fig. 1 together with the value of Γ , the width of the giant resonance. The similarity between the general trends of these two quantities suggests us an intimate relation between them.

A possible explanation of this fact may be given by the following consideration. The resonance energy of this dipole vibration is closely related to nuclear radius, i.e., $E_0 \propto R_0^{-n}$, where $n=1$ from SJJ-model³⁾, $1/2$ from GT-model⁴⁾, and about $1/2$ from experiments¹⁾.

If a nucleus is a spheroid instead of a sphere, because of the quadrupole coup-

ling its resonance level corresponding dipole vibration splits into two parts whose energy values may be approximated as $E_1 \propto R_1^{-n}$, $E_2 \propto R_2^{-n}$. This two resonances overlap and may be observed as broad total width. The ratio of this splitting ΔE to the original value E_0 is given by

$$\Delta E/E_0 = (R_2^{-n} - R_1^{-n})/R_0^{-n} = 5nQ_0/4ZR_0^2.$$

If we further know the experimental value of E_0 , the absolute value of ΔE can be obtained. Putting $\Delta E \simeq \Delta \Gamma$, and denoting the experimental values of the width of a neighbouring nucleus whose Q_0 is zero, as Γ_0 , the width of any nuclei whose Q_0 are large can be expressed approximately by the formula: $\Gamma = \Gamma_0 + \Delta \Gamma$. Values of Γ_{calc} listed in Table I are obtained in this way. Agreement with experiment is fairly well if we take $n=1$, namely SJJ-model³⁾.

In order to obtain ΔE more exactly, let us take SJJ-model and transform Helmholtz equation which describes the nucleon

density fluctuation into the spheroidal coordinates and take the difference of eigenvalues between the case of $l=1, m=0$ and that of $l=1, m=1$. Numerical calculations from Mn⁵⁵, In¹¹⁵, Ta¹⁸¹ yield the results about 3/4 times the values obtained above. So the values in Table I can be regarded as approximately correct.

In the above calculation, values of Q_0 are derived from spectroscopic Q -value. Of course these values of Q_0 cannot be regarded as the parameters that describe the nuclear shape exactly. We should rather prefer the value of Q_0 obtained by Coulomb excitation experiments⁵⁾. For instance, as is clear in Fig. 1, Zn⁶⁴, Rh¹⁰³ are the cases of serious discrepancies from the general trend. Their Q_0 are unknown, but from the known values of their spin, their Q are taken to be zero, although their Γ are large. Careful investigation of such nuclei using exact nuclear wave function would be very interesting. Moreover, there are many nuclei whose Q_0

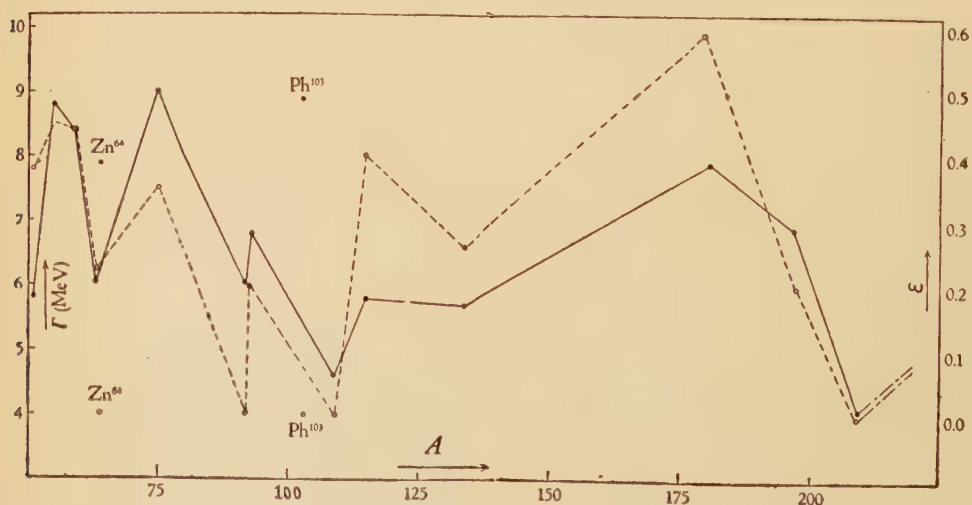


Fig. 1. Plot of eccentricity and resonance width against mass number. The solid line is for the widths and the dotted line is for eccentricities. Abscissa is mass number. Ordinate is width in Mev (on the left hand), and eccentricity (on the right hand). Zn⁶⁴ and Rh¹⁰³ seem to be exceptions, hence they are not included in the curves.

Table I. Values of eccentricity and width of various elements which have large quadrupole moments.

element	ε	ΔF (Mev)		Γ_0	$\Gamma_{calc.}$	$\Gamma_{obs.}$
		$n=1/2$	$n=1$			
$^{51}_{23}\text{V}$	0.38	0.8	1.6	4.2 (Ca ⁴⁰)	5.8	5.8
$^{55}_{25}\text{Mn}$	0.45	1.1	2.2	6.3 (Fe ⁵⁴)	8.5	8.8
$^{59}_{27}\text{Co}$	0.44	1.1	2.1	"	8.4	8.4
$^{75}_{33}\text{As}$	0.35	0.4	0.8	"	7.1	9.0
$^{115}_{49}\text{I}$	0.40	0.7	1.4	4.6 (Ag ¹⁰⁹)	6.0	5.5~5.8
$^{181}_{73}\text{Ta}$	0.59	1.7	3.4	5.7	9.1	7.9

Nuclear radius is taken to be $1.2 \times 10^{-13} A^{1/3} \text{cm}$ both in Fig. 1 and Table I.

are unknown, but their Γ are known, or vice versa. Experiments for those nuclei are very desirable, especially, if possible, for rare earth nuclei.

The author is greatly indebted to Prof. Inui for his kind help in lending him the tables of spheroidal functions.

A Method of Calculation of Electrical Conductivity

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November 14, 1955

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We have recently studied a new method of calculation of electrical conductivity.¹⁾ This method is an application of the theory of relaxation phenomena, developed by Kubo and Tomita in the problem of magnetic absorption.²⁾ We can express the current density of the system in the presence of arbitrarily time-dependent electric field, $\mathbf{F}(t)$, in the form

$$J_{\mu}(t) = \sum_{\nu} \int_{-\infty}^t dt_0 F_{\nu}(t_0) \varphi_{\nu\mu}(t-t_0),$$

$$(\mu=x, y, z, \nu=x, y, z). \quad (1)$$

The property of electric conductivity of the system is essentially condensed in the function, $\varphi_{\nu\mu}(t)$. Then our task reduces to find the expression of this function, which is found to be the following by investigating the response of the system to the pulse field:

$$\phi_{\nu\mu}(t) = \int_0^\beta d\lambda \langle e^{\lambda H} j_\nu e^{-\lambda H} j_\mu(t) \rangle, \quad (\beta = 1/kT) \quad (2)$$

where j_ν and $j_\mu(t)$ are the operators of current component respectively in the Schrödinger picture and in the Heisenberg picture, H is the Hamiltonian of the system in the absence of electric field, and $\langle a \rangle$ means the thermal average of the quantity a , viz.,

$$\langle a \rangle = T_r(e^{-\beta H} a) / T_r(e^{-\beta H}). \quad (3)$$

In the representation of diagonalizing H , the expression (2) is written also as,

$$\varphi_{\nu\mu}(t) = \frac{\sum_{m,n} \exp(-\beta E_n) - \exp(-\beta E_m)}{E_m - E_n} \frac{(m|j_\nu|n)(n|j_\mu|m) \exp\left(i \frac{E_n - E_m}{\hbar} t\right)}{\sum_n \exp(-\beta E_n)} \quad (4)$$

where E_m , E_n and so on are the eigenvalues of H and $(m|j_\nu|n)$ represents the element of the matrix corresponding to j_ν . The conductivity tensor can be calculated from this. Above all the static conductivity, σ , is given by³⁾

$$\begin{aligned} \sigma &= \int_0^\infty dt \varphi_{\mu\mu}(t) \\ &= \frac{\pi\hbar}{kT} \sum_n \exp(-\beta E_n) \sum_m |(m|j_\mu|n)|^2 \\ &\quad \cdot \delta(E_m - E_n) / \sum_n \exp(-\beta E_n). \end{aligned} \quad (5)$$

The exact calculation of this formula is, however, impossible at present, and we calculate approximately for a typical case. We assume that the Hamiltonian H consists of the components, H_0 and H' , where H_0 is the sum of the Hamiltonians of two or more subsystems each moving indepen-

dently and H' is the part representing the interaction energy between every pair of these subsystems. In effect

$$H = H_0 + H', \quad (2)$$

and we write the eigenvalues of H_0 as $E_m^{(0)}$, $E_n^{(0)}$ and so on. We assume further that H' may be treated as a perturbation. By making use of development of $j_\mu(t)$ in terms of H' , we then obtain

$$\begin{aligned} \varphi_{\mu\mu}(t) &= \beta \langle j_\mu^2 \rangle \cdot \{1 - \int_0^t d\tau (t - \tau) f(\tau)\}, \\ f(\tau) &= \hbar^{-1} \langle j_\mu^2 \rangle^{-1} \sum_\omega \\ &\quad \times \langle [H'_\omega [H_{-\omega}', j_\mu]] j_\mu \rangle \cos(\omega\tau), \end{aligned} \quad (7)$$

where H'_ω means the part of H' the elements $(m|H'_\omega|n)$ of which vanish except for $E_m = E_n + \hbar\omega$; and then

$$H' = \sum_\omega H'_\omega, \quad e^{\frac{\hbar}{i} H_0 t} H' e^{-\frac{\hbar}{i} H_0 t} = \sum_\omega H'_\omega e^{i\omega t}.$$

We here make the assumption that the relaxation behaviour of $\varphi_{\nu\mu}(t)$ can be described in terms of one relaxation time as usual. Actually we assume that $\varphi_{\nu\mu}(t)$ tends to zero as t tends to infinity:

$$\varphi_{\mu\mu}(t) \cong \varphi e^{-\alpha t} \quad (t \rightarrow \infty). \quad (8)$$

Comparing Eq. (8) with (7), we can determine φ and α ; and we can then have the concrete asymptotic expression of $\varphi_{\mu\mu}(t)$. By substituting this in the formula (5) we can have the expression of conductivity as follows;

$$\begin{aligned} \sigma &= \{\hbar \langle j_\mu^2 \rangle\}^2 \\ &\quad \times \{\pi kT \sum_\omega \langle [H'_\omega, j_\mu] [j_\mu, H_{-\omega}'] \rangle \delta(\omega)\}^{-1}. \end{aligned} \quad (9)$$

Basing upon this formula, we can calculate the electrical conductivity of different kind of different systems.

We express our cordial thanks to Professor K. Ariyama, Dr. S. Nakajima and

other colleagues of this Institute for their kind interests and discussions. We wish also to extend our sincere thanks to Professors R. Kubo and R. P. Feynman who enlightened us very much.

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- 3) R. P. Feynman, discussions with us. He told that he had obtained the second expression in (5) by means of the similar method to us. This of course has quite the same content as the first expression we had obtained.

Nature of Nuclear Force Indicated by the Photodisintegration of the Deuteron, I

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Recently many experiments to examine the polarization in p - p scattering have been done¹⁾, and the phase shifts at 213 Mev are obtained by Garren.²⁾ Hence it seems to be worth-while to see if these phase shifts can account for the photodisintegration of the deuteron.

In the photodisintegration of the deuteron, if we neglect the contribution from the D -state of the deuteron and waves with $L > 3$, the possible transitions at energies below 140 Mev are³⁾

$$\text{m.d.: } {}^3S_1 \rightarrow {}^1S_0, \quad \text{e.d.: } {}^3S_1 \rightarrow {}^3P_0, {}^3P_1, {}^3P_2,$$

$$\text{m.q.: } {}^3S_1 \rightarrow {}^3P_1, {}^3P_2,$$

$$\text{e.q.: } {}^3S_1 \rightarrow {}^3S_1 + {}^3D_1, {}^3D_2, {}^3D_3.$$

Here m.d., e.d., m.q., and e.q. indicate the magnetic dipole, the electric dipole, the magnetic quadrupole, and the electric quadrupole transitions respectively. These transitions depend on the wave function of the deuteron (U_d) as the initial state. On the final states, m.d. depends chiefly on δ_0^0 , and e.d. and m.q. depend chiefly on δ_1^0 , δ_1^1 and δ_1^2 . (Here the affix indicates the total angular momentum, and the suffix the orbital angular momentum.) Besides the magnetic cross sections depend on M^{int} (the exchange moment). Because, at energies below 140 Mev, the calculated e.q. cross section (σ_{eq}) is much smaller than the electric dipole cross section (σ_{ed}), we may say that the total cross section (σ_T) depends chiefly on U_d , M^{int} , δ_0^0 , δ_1^0 , δ_1^1 and δ_1^2 . It is also easy to show that the isotropic part (σ_{ip}) in the angular distribution also chiefly depends on U_d , M^{int} , δ_0^0 , δ_1^0 , δ_1^1 and δ_1^2 . However δ_0^0 , δ_1^0 , δ_1^1 , and δ_1^2 can be determined from p - p scattering, if we neglect the waves with $L > 1$. We considered if δ_0^0 , δ_1^0 , δ_1^1 and δ_1^2 at 213 Mev chosen by Garren²⁾ can account for the experimental σ_T and σ_{ip} of the photodisintegration of the deuteron at 108.7 Mev (in the center of mass system). Some possible sets of phase shifts chosen by Garren²⁾ are listed in Table 1. δ_0^0 is limited between 40° and -40° to account for the Coulomb interference at small angles.

For the wave function of the deuteron, we used

$$\frac{N}{\sqrt{4\pi}} \cdot \frac{(e^{-\mu(r-D_0)} - e^{-\nu(r-D_0)})}{r} \quad (1)^{4)}$$

and changed D_0 from 0 to $0.3/\kappa$, and also

κ and N to satisfy the effective range theory. (Here $\kappa = m_{\pi}c/\hbar$.) For the final states corresponding to m.d., e.d., and m.q., we used 1S and 3P waves having the given phase shifts (See Table 1.) at all points. (We neglected the modification in the inner region where potential is acting.) This approximation is very good for the case of e.d. and m.q. (because there is a factor r in the corresponding integrals,) and not so bad even for the case of m.d.⁵⁾ In the calculation of e.d. and m.d., we included the series expansion of $j_1(k_r r)$ up to the 2nd term, instead of only the 1st term in previous calculations⁶⁾⁷⁾ (The inclusion of the 2nd term makes the dipole cross sections increase by about 25%, but the 3rd and other terms hardly change the results.)

The calculated σ_T ⁸⁾ without including M^{int} is shown in Fig. 1. The experimental σ_T at this energy is about 5.8×10^{-29} cm², with experimental error of 10% or larger.⁹⁾ From Fig. 1 we see that for large δ_0^0 the calculated σ_T is too small, (In this case $\sigma_{m.d.}$ is negligible.), and also that the variation due to D_0 is not appreciable. $\delta_0^0 = -30^\circ$, -33° and -36° with $D_0 \approx 0.2/\kappa$ are a better set of choice. However the recent experiment of Rose suggests that the polarization is positive¹⁰⁾, and therefore $\delta_0^0 = -36^\circ$ with $D_0 \approx 0.2/\kappa$ is the best choice. In this case $\delta_1^0 = 10^\circ$, $\delta_1^1 = -30^\circ$, $\delta_1^2 = 10^\circ$, and σ_T is about 90% of the experimental value.

A similar conclusion was obtained in the calculation on σ_{ip} ⁸⁾. Without including M^{int} and the D -state of the deuteron, we obtained 50~60% of the experimental value for $\delta_0^0 = -30^\circ$, -33° or -36° , and $D_0 \approx 0.2/\kappa$. For other cases σ_{ip} is much smaller. (About this point we shall explain in a forthcoming paper.) The inclusion of

M^{int} and the D -state of the deuteron tends to increase the isotropic part, but, generally speaking, the inclusion of the D -state of the deuteron does not appreciably change σ_T ¹¹⁾. Therefore it seems to be that $\delta_0^0 = -36^\circ$, $\delta_1^0 = 10^\circ$, $\delta_1^1 = -30^\circ$ and $\delta_1^2 = 10^\circ$ better explain the experimental values at 213 Mev.

Calculation referring to a more recent analysis on the phase shifts of p - p scattering¹²⁾ is being made, and similar results are being obtained. Calculation including the D -state of the deuteron will be done in near future. More detailed explanation will be published in due time.

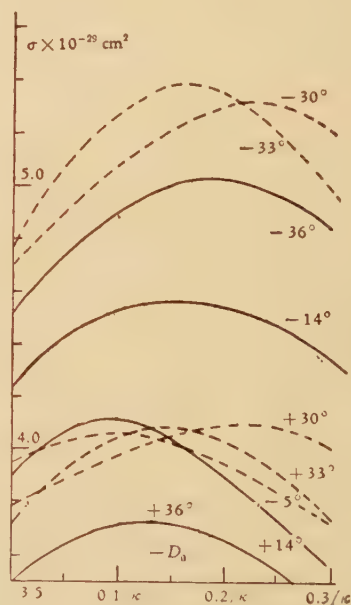


Fig. 1. σ_T as functions of δ_0^0 and D_0 . Figures written above are values of δ_0^0 .

— indicates the case when the polarization is negative,
 indicates the case when the polarization is positive.

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283.

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- 5) If we connect the outer wave of 1S with the inner one smoothly at $r=1/\kappa$, taking a hard core of radius less than $0.4/\kappa$, the error is within 15%.
- 6) The formulas of σ_{el} and σ_{had} usually used⁷⁾ are not correct at high energies, because they only contain the first term in the expansion of $j_1(kr)$.
- 7) For example, see J. F. Marshall and E. Guth, Phys. Rev. **78** (1950), 738.
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- 11) For example, see N. Austern, Phys. Rev. **85** (1951), 283.
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Table 1. Phase shifts $\delta_{L,J}$ chosen to account for p - p scattering at 213 Mev(L.S.) The last column gives $\eta = [k^2 \sigma(k, \theta) / \sin 2\theta] P(k, \theta)$ at 27° in L. S.¹⁾

δ_1^0	δ_1^1	δ_1^2	δ_0^0	η
10	-30	10	36	0.249
10	7	-25	33	-0.256
20	4	-25	30	-0.368
0	-37	10	14	0.339
95	16	-5	5	-0.297
95	16	-5	-5	-0.297
0	-37	10	-14	0.339
20	4	-25	-30	-0.368
10	7	-25	-33	-0.256
10	-30	10	-36	0.249

Mass Reversal and Space-Time Inversions

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A few remarks will be presented in connection with Ouchi-Senba-Yonezawa's paper¹⁾ on mass-reversal which appeared in a recent issue of this periodical.

From a purely mathematical point of view of spinor analysis, inversion of each coordinate x_μ ($\mu=1, 2, 3, 0$) corresponds to multiplication by $\gamma_5 \gamma_\mu$ (or only by γ_μ according to another assumption) of a spinor. Hence, the total space-and-time inversion (in either case) corresponds to multiplication by γ_5 . This is, as far as the spinor transformation is concerned, identical with the transformation considered under the name of mass-reversal. Thus, the column under M_1 on the Table of O-S-Y's paper is identical with the transformation rule (which determines the "class") given under the heading "c-number theory" in Table VI of Part I of the author's previous paper²⁾. See also Formula (2.43) of Part II.²⁾ The column under M_2 in O-S-Y's table can readily be obtained by combining Table VI, Part I with Table IV, Part II. Incidentally, it is of some interest to note that the same classification reappears in the discussion of the Yang-Tiomno type nucleon-lepton interaction. See Table VI, Part II.²⁾

The reason why the transformation :

$$\psi(\mathbf{r}, t) \rightarrow \gamma_5 \psi(-\mathbf{r}, -t), \quad (1)$$

and O-S-Y's transformation :

$$\psi(\mathbf{r}, t) \rightarrow \gamma_5 \psi(\mathbf{r}, t), \quad m \rightarrow -m, \quad (2)$$

do not lead to the same conclusion lies essentially in the fact that in point of view (1) the sign-change of the source due to the ψ -field is automatically compensated by the sign-change of the coupling term due to the interacting field. The consistent interpretation of (1) in quantized field theory is obtained by the product of three transformations considered in Part II, namely, time-reversal, space-inversion and charge conjugation. The transformation rule thus obtained of tensorial quantities is exactly opposite to the M_1 -transformation rule. Similarly, the space-and-time inversion without charge conjugation leads to the rule given under the heading "q-number theory" in Table VI, Part I which is the exact opposite of the M_2 -rule.

In any event, it should be pointed out that O-S-Y's selective prescription of mixture is by no means conclusive for the following reason. O-S-Y's theory involves a change of sign of a natural constant. If such a sign-change is in principle permissible, then there is no reason whatsoever to preclude the sign-change of some other natural constants, in particular, some of the interaction constants. One can then very well allow mixture of one of the (V, PV)-group with one of the (S, PS, T, PT)-group simply assigning opposite rules of sign-change to the interaction constants of the different groups. Conversely, if O-S-Y's rule proves to hold for all the possible interactions, that will merely show that all the interaction constants happen to belong to the same transformation rule. On the other hand, O-S-Y's rule can be easily incorporated in the space-time interpretation (1) by assigning different transformation rules to the interaction constants belonging to different "classes."

If we once permit sign-changes of natural

constants, we can formulate time-reversal and space-and-time inversion in the quantized field theory in a different way from the usual formulation explained in Part II.²⁾ For time-reversal, we can adopt

$$\left. \begin{aligned} t &\rightarrow -t, \quad m \rightarrow -m, \quad h \rightarrow -h, \\ \psi(\mathbf{r}, t) &\rightarrow \gamma_5 \gamma_4 \psi(\mathbf{r}, -t), \end{aligned} \right\} \quad (3)$$

and for space-and-time inversion

$$\left. \begin{aligned} \mathbf{r} &\rightarrow -\mathbf{r}, \quad t \rightarrow -t, \quad m \rightarrow -m, \quad h \rightarrow -h, \\ \psi(\mathbf{r}, t) &\rightarrow \gamma_5 \psi(-\mathbf{r}, -t). \end{aligned} \right\} \quad (4)$$

These transformations (3) and (4) have at least two advantages over the customary formulation of time-reversal and space-and-time inversion. First, they lead to transformation of state vector $\Psi(t)$ into unitary transforms of $\Psi(-t)$, and not into unitary transforms of the complex conjugate of $\Psi(-t)$, as is the case in the usual formalism. The latter feature of the customary theory, as is well-known, causes a great deal of complication. Second, the transformation rules of spinors involved in (3) and (4) are exactly the same as what the straightforward mathematical definition of spinors prescribes. In the usual formalism, one has to apply charge conjugation on the spinor transformations of (3) and (4) to get time-reversal and space-and-time inversion. Transformations similar to (3) and (4) can be devised also for boson fields.

It is true that the product of transformations (2) and (3) and the product of transformations (2) and (4) can be used respectively for time-reversal and space-and-time inversion. But, these do not have the second of the two advantages mentioned above.

The product of transformation (1), which is space-and-time inversion plus charge-

conjugation, with transformation (4), which is an alternative formulation of space-and-time inversion, can be considered as an alternative formulation of charge conjugation. This however involves transformation of state vector $\Psi(t)$ into a unitary transform of $\Psi^*(t)$.

The underlying fact is that the transformation :

$$\left. \begin{aligned} m &\rightarrow -m, \quad h \rightarrow -h \\ \psi(\mathbf{r}, t) &\rightarrow Z\psi^T(\mathbf{r}, t)Z^{-1} \\ &= \psi^+(\mathbf{r}, t)C, \quad \Psi(t) \rightarrow Z\Psi^*(t) \end{aligned} \right\} (5)$$

is an identical transformation as far as its physical content is concerned. The symbols ψ^+ and C are used in the same meanings as in Eq. (2.54) of Part II. The transformation given in (5) is actually equivalent to substitution of $i = \sqrt{-1}$ explicitly appearing in operators by $-i$.

Generally speaking, one can distinguish two categories among the ordinary symmetry transformations. One of them does not require any change in argument variables. An illustration is charge-conjugation in which the coordinate variables remain unchanged. The interaction constant e need not change its sign either, since the sign-change of the source is compensated by the sign-change of the electromagnetic field. To the other category belong space-inversion, time-reversal and space-and-time-inversion which require change of sign of coordinate variables. From this point of view, it seems more natural to classify transformation (2) considered by O-S-Y as belonging to the second category, i.e., to consider the intrinsic mass as a dynamical variable. This does not necessarily mean that any theory which considers the mass as a variable and which can give a consistent interpretation to transformation (2)

leads to O-S-Y's selective prescription of mixture. The simplest example of such a theory is the one which regards the mass as the canonical conjugate of the fifth coordinate and which interprets (2) as the inversion of the fifth axis. Such a theory was previously considered by the author³⁾ and was shown to allow those mixtures forbidden by O-S-Y's theory.⁴⁾ This is because a compensation of sign-changes automatically takes place.

Nevertheless, it is worthwhile to attempt to construct a theory in which natural constants become variables, since such a theory may give a natural interpretation to the various transformations considered in this communication and also reveal a hidden relationship between natural constants and the geometrical continuum underlying physical laws.

- 1) Ouchi, Senba and Yonezawa, *Prog. Theor. Phys.* **14** (1955), 172.
Other references on the same subject are cited in O-S-Y's paper.
Some of the results accredited in the present communication to O-S-Y may also be found in earlier papers.
- 2) S. Watanabe, *Rev. Mod. Phys.* **27** (1955), 26, 40.
- 3) S. Watanabe, *Sci. Pap. I. P. C. R.* **39** (1941), 157, **42** (1944), 1.
- 4) S. Watanabe, *Phys. Rev.* **74** (1948), 1864.

Remarks on Heisenberg's Non-Linear Field Theory

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The purpose of this note is to show

an objection against the arguments of inner consistency at a new attempt proposed recently by Heisenberg.¹⁾

Before entering into this argument it will be appropriate to summarize first the essential feature of Heisenberg's theory so far as concerned with our problem of consistency. One of the main intentions in Heisenberg's programme for quantizing non-linear wave equation is to give the commutation function the same properties as those which the propagation function of the corresponding non-linear c-number wave equation has. As was shown in II, the latter function vanishes in the space-like region and oscillates infinitely rapidly near the light cone but does not show $\delta(x_\mu^2)$ - and $\delta'(x_\mu^2)$ -type singularities, which appear usually in the ordinary quantum field theory. In order that commutation function can have these properties, Heisenberg introduced the fictitious and symbolic "Hilbert space II" and assumed that the commutation function is the sum over all contributions from various mass states belonging to Hilbert space I, each contribution being the ordinary S -function associated with its corresponding mass subtracted by δ - and δ' -singularities on the light cone by virtue of contributions from "Hilbert space II". Then, from his axioms concerning "Hilbert space II", the vacuum expectation value of the time-ordered operator $T(\psi(x)\psi^+(y))$ behaves, in the approximation in which only the contribution from one fermion state with the lowest mass κ is taken into account, like

$$\begin{aligned} \langle \Omega | T(\psi(x)\psi^+(y)) | \Omega \rangle &\approx S_F(x-y)/2 \approx \\ &- (\kappa^2/8\pi) \gamma_\mu (x_\mu - y_\mu) / (x_\nu - y_\nu)^2 \end{aligned}$$

at large distances. The inclusion of other mass states can only change the value of

constant factor, but the reality of this constant factor must remain unchanged. The exact form of S_F -function is considered as obtained by raising the order of N -approximation¹⁾ successively, though it is still unclear whether this procedure converges or not.

Now, in order to look up the problem of consistency of this procedure of quantization, it is most important, as was stated in III, to examine whether the equation for S_F -function derived from the field equation itself has a solution that behaves like $-\mathcal{Z} \frac{\kappa^2}{4\pi} \frac{\gamma_\mu x_\mu}{x_\nu^2}$ at large distances. So the author repeated the derivation of the equation for S_F in $N=4$ approximation according to the prescription stated in III and found that in the simultaneous equations III (159) and (160) the latter had a wrong sign. The correct equations in the approximation under consideration are the following:

$$\left. \begin{aligned} (-1/2) \gamma_\mu \partial S_F(-z) / \partial z_\mu &= -l^2 \sigma(z) \\ \partial \sigma(z) / \partial z_\mu \cdot \gamma_\mu &= -l^2 / 8 [S_F(-z) \text{spur}(S_F(z) S_F(-z)) \\ &- (S_F(-z) S_F(z) S_F(-z))] \end{aligned} \right\} \quad (1)$$

where $\sigma_{\alpha\beta}(z) \equiv \langle \Omega | T(\psi_\alpha(x)\psi_\beta(x)\psi_\beta^+(y)\psi_\alpha^+(x)) | \Omega \rangle$ and $z=y-x$, and Ω means the vacuum state and $\psi^+ \equiv \psi^* \gamma_4$. The origin which induced the wrong sign in the eq. III (160) is found in the use of the incorrect equation,

$$\partial \psi^+(y) / \partial y_\mu \cdot \gamma_\mu = -l^2 (\psi^+(y)\psi(y)) \psi^+(y)$$

instead of the correct equation

$$\partial \psi^+(y) / \partial y_\mu \cdot \gamma_\mu = +l^2 (\psi^+(y)\psi(y)) \psi^+(y) \quad (2)$$

adjoint to the original field equation

$$\gamma_\mu \partial \psi(y) / \partial y_\mu = -l^2 \psi(y) (\psi^+(y)\psi(y)). \quad (3)$$

Now from the simultaneous equations (1) the value of Z turns out to be pure imaginary: $Z = \pm 1.67 i$. This consequence is in contradiction with the assumption taken at the beginning, because the appearance of the term $-(\kappa^2/4\pi) \cdot \gamma_\mu x_\mu/x_\nu^2$ with real constant factor Z in S_F has its origin in the very elimination of $\partial(x_\mu^2)$ - and $\partial'(x_\mu^2)$ -singularities from the ordinary S -function and because this elimination is the essential point in Heisenberg's programme of the quantization of the non-linear equation (3). It can also be ascertained that such contradiction can not be removed by taking other coupling type than scalar coupling, namely, vector-, tensor-, pseudoscalar-, pseudo-vector or pseudotensor-coupling. It seems to the author that such contradiction shows the inconsistency of Heisenberg's formalism, including the approximative nature of the way of construction of the theory.

The problem of consistency can also be seen from the more general viewpoint. The Fourier-transform of S_F in III in the lowest mass approximation is

$$S_F(p) = \lim_{\delta \rightarrow +0} 2 \left[\frac{\gamma_\mu p_\mu \kappa^4}{(p^2 - i\delta)^2 (p^2 + \kappa^2 - i\delta)} - \frac{ik^3}{(p^2 - i\delta)(p^2 + \kappa^2 - i\delta)} \right].$$

This can also be expressed as the sum of two terms,

$$S_F(p) = \lim_{\delta \rightarrow +0} \lim_{m^2 \rightarrow 0} 2 \left[\left(\frac{\kappa^2 - m^2}{m^2} \right) \frac{\gamma p}{p^2 - i\delta} + \left(-\frac{\kappa^2}{m^2} \right) \frac{\gamma p}{p^2 + m^2 - i\delta} - \frac{ik}{p^2 - i\delta} \right] + \lim_{\delta \rightarrow +0} 2 \left[\frac{\gamma p + ik}{p^2 + \kappa^2 - i\delta} \right],$$

the first of which corresponds to a massless dipole state²⁾ associated with negative proba-

bility. The essential difference between the regulator theory³⁾ and (4) is that in the former auxiliary masses are led to tend to infinity at the end, whereas in the latter the mass of ghost is zero and this ghost is intimately connected with the "photon solution".¹⁾ At any rate the existence of ghost states will bring negative probability, non-hermiticity or non-unitarity into the theory, and there will appear some inconsistency somewhere in Heisenberg's theory. Actually, as stated above, inconsistency manifests itself in that the equation of S_F derived from field equation has not a solution having properties assumed at the very beginning.

It may be possible, however, to remove these difficulties by changing the method of approximation or by introducing some new variables into the theory.

In conclusion the author expresses his sincere thanks to Professor H. Yukawa and the members of his seminar for their kind discussions.

- 1) W. Heisenberg, *Nachr. d. Gött. Akad. d. Wissensch.* (1953) 111 (I)
W. Heisenberg, *Z. Naturforschg.* **9a** (1954), 292 (II).
W. Heisenberg, F. Kortel und H. Mitter, *Z. Naturforschg.* **10a** (1955), 425 (III).
- 2) I have heard from Dr. N. Fukuda and Dr. H. Umezawa that Professor W. Pauli used at Pisa Conference (1955) the term "dipole-ghost" in his critique of Heisenberg's theory. I suppose that the meaning of Pauli's statement would be such as that shown here.
- 3) W. Pauli and F. Villars, *Rev. Mod. Phys.* **21** (1949), 434.

Determination of the Pion Coupling Constant in Nuclear Forces

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It was already shown by two of the authors that the negative phase shift in triplet P -state at energies lower than 4 Mev could be accounted for quantitatively by the outer potential derived from the pion theory¹⁾. This success made more attention of people focus on this pion theoretical potential. In fact, the Japanese research group for nuclear forces has been doing a series of works for a long time along the idea that the present pion theory would be reliable in deriving the form of nuclear forces only when two nucleons are distant apart, while the inner potential would be of more complicated nature and beyond the range of validity of the present theory.²⁾

Furthermore, the potential at long distances is expected to be due mainly to the exchange of one pion between two nucleons and thus has the unit range, which we shall adopt as a unit of length throughout this letter. Other processes in which two or more pions are exchanged will contribute to the potentials with the range a half or shorter, so that they will be negligible at longer distances than about 1.5 unit compared with the one-pion-exchange potential.

This one-pion-exchange potential includ-

ing all radiative corrections can be shown to coincide at long distances with that obtained by the lowest perturbation, provided that the coupling constant and masses of nucleon and pion are replaced by their renormalized values³⁾. Therefore, the $P_S(p\pi)$ theory gives the one-pion-exchange potential at long distances in the static limit as

$$V(x) \sim \frac{g_r^2}{4\pi} \frac{\tau_1 \cdot \tau_2}{3} \mu^2 \left\{ (\sigma_1 \cdot \sigma_2) + S_{12} \right. \\ \left. \times \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \right\} \frac{e^{-x}}{x},$$

where g_r is the renormalized coupling constant and μ is the pion mass. The same expression holds in the case of $P_S(p\pi)$ theory by replacing g_r^2 by $G_r^2(\mu/2M)^2$, G_r being the renormalized coupling constant and M the mass of nucleon.

It is highly desirable to determine the value of the renormalized coupling constant of this one-pion-exchange potential, a brief report of which is the content of this letter.

1) Deuteron problem

We solve the Schrödinger wave equation for the deuteron ground state with the one-pion-exchange potential, starting from the well-known asymptotic forms and proceeding inward up to $x=1$, taking the mixing ratio of S to D waves in their asymptotic forms as an adjustable parameter. In the inner region $1 > x > 0$, we do not solve the Schrödinger equation, since the potential in this region is unknown. Instead, we assume various forms of functions without nodes that vanish at the origin and join at $x=1$ smoothly to the outer wave functions. We seek for the inner wave functions that give a fit to the experimental values of the effective range 3r_e and the quadrupole moment Q for an assumed value of $g_r^2/4\pi$ of the one-pion-exchange

potential in the outer region $x > 1$. If we can find no such inner functions, we can reject this value for $g_r^2/4\pi$. The D -state probability P_D , whose reasonable value is supposed to lie between 0 and 15%, gives another check of the potential. If we would find, in this way, no values of $g_r^2/4\pi$ that could give a good fit, then the symmetrical pseudoscalar pion theory of nuclear forces would be concluded to be hopeless.

However, this is not the actual case. We can find wave functions of the deuteron ground state which reproduce the experimental data if and only if $g_r^2/4\pi = 0.075 \pm 0.010$. The details of the inner wave functions have little influence on determining the value of $g_r^2/4\pi$, provided they are so chosen as explained above.

When $g_r^2/4\pi$ is larger, wave functions which give correct values for 3r_e show too large Q . For example, in the case of $g_r^2/4\pi = 0.10$, Q is 35% larger than the correct value. On the other hand, when $g_r^2/4\pi$ is smaller, wave functions that are smooth at $x=1$ and vanish at the origin have too small 3r_e and too small Q .

A possible correction to the outer wave functions is estimated, which may come from the two-pion-exchange potentials. Even if the correction is as large as that derived by the perturbation method (TMO potential)⁴⁾, the coupling constant $g_r^2/4\pi = 0.080 \pm 0.010$ is found to give correct deuteron wave functions.

Thus we can conclude the value of the coupling constant $g_r^2/4\pi = 0.075 \pm 0.015$ from the deuteron problem.

2) Low energy parameters in the singlet even state

We investigate the wave function of the singlet even state at zero energy, assum-

ing the one-pion-exchange potential in the region $x > 1$. We solve the Schrödinger equation starting from the outside and get wave functions with the asymptotic form of $(1 - (r/a))$, the scattering length a being from -20 to -15×10^{-13} cm (estimated lower and upper limits from the experiments on p - p scattering).

When the one-pion-exchange potential in the outer region $x > 1$ is too weak, we get wave functions which give too small effective range 1r_e compared with the experimental value $2.65 \pm 0.07 \times 10^{-13}$ cm of the p - p system. The two-pion-exchange potential of the perturbation theory is attractive and large in the outer region $1.5 > x > 1$ ⁴⁾, and has a trend to give wave functions that have large 1r_e .

However, even if we take into account this possible correction to the wave functions, we can say that the one-pion-exchange potential with $g_r^2/4\pi$ smaller than 0.07 yields 1r_e smaller than the experimental value, hence it is to be rejected. We can also infer that a hard core like repulsion with the radius at least 0.20 is required to reproduce the experimental value of 1r_e , even if we assume the one-pion-exchange potential with $g_r^2/4\pi$ as large as 0.09.

Thus we can conclude

$$g_r^2/4\pi = 0.080 \pm 0.010.$$

Recently, Chew proposed $g^2/4\pi = 0.081$ from the analysis of pion reactions,⁵⁾ in good agreement with our result.

More detailed discussion will be published soon in this journal.

We wish to express our cordial thanks to the members of the research group of nuclear forces for their stimulating discussion and wise criticism. We are also indebted to Profs. M. Kobayasi, K. Sakuma and our

colleagues for their encouragement.

- 1) S. Otsuki and R. Tamagaki, *Prog. Theor. Phys.* **12** (1954), 806L; **14** (1955), 52.
- 2) This idea was first proposed by Taketani and others, M. Taketani, S. Nakamura and M.

Sasaki, *Prog. Theor. Phys.* **6** (1951), 581.

- 3) K. Hiida, J. Iwadare and S. Machida, *Prog. Theor. Phys.* to be published.
- 4) M. Taketani, S. Machida and S. Onuma, *Prog. Theor. Phys.* **7** (1952), 45.
- 5) G. F. Chew, *Proceedings of the Fifth Rochester Conference on High Energy Physics* (1955).

Populations and Evolution of Stars*

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A scheme was proposed for the whole evolutionary process of celestial objects and the observational characteristics between two populations, Populations I and II, are discussed in some detail. It is suggested that these characteristics can be accounted for by the present scheme.

§ 1. Introduction and summary

In 1944 W. Baade¹⁾ announced the existence of two populations, Populations I and II, among celestial objects. According to recent views the differences in characteristics between two populations are summarized as follows:

- i. The distributions of two populations on the Herzprung-Russell diagram or color-magnitude diagram.
- ii. The distributions in space.
- iii. The high-velocity stars belong to Population II, while the low-velocity stars to Population I.
- iv. Spectroscopic difference between two populations.

These differences between two populations received a more definite support by recent accurate determinations of color and magnitude of stars in globular and galactic clusters. It seems very certain that the two populations are made up of fundamentally different kinds

*) Most of the content of this paper was read at the symposium on Nuclear Astrophysics held in February, 1955, at the Research Institute for Fundamental Physics, Kyoto University.

of stars and it has been made clear that these differences in characteristics between two populations cannot be accounted for by a simple evolutionary process of stars. Models with composite structures have been applied to explain the evolutionary track for Population II stars²⁾. The evolution of Population I stars has also been investigated by several authors³⁾.

The whole evolutionary process including the stars of Populations I and II and the interstellar gas has been suggested by Schwarzschild and Spitzer¹⁾. They attributed the difference in the abundances of metallic elements in the stars of two populations to the nuclear reaction which took place in the Population II stars in the very early stage of the universe.

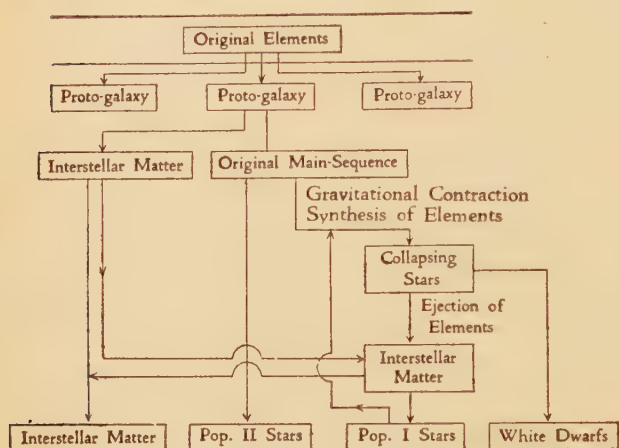


Fig. 1. Evolutionary scheme

It is the purpose of the present paper to extend their idea and propose the evolutionary scheme as shown in Fig. 1 which is now going on continuously in the universe. In this scheme we wish to explain the differences between the characteristics of stars belonging to the two different populations. Numerical calculations for successive stages are now in progress and will be published in subsequent papers. The general outline and the qualitative

discussion of our scheme will be the subject of the present paper.

§ 2. Stellar evolution and stellar populations

In its early stages in our universe, numberless spherical clouds of turbulent gas and dust have been separated and condensed from the primordial gas which consists of original elements. In these proto-galaxies less massive primordial clouds of turbulent gas and dust have again been separated and clusters or groups of stars have been condensed from their relatively dense clouds.

In earlier stages the temperature of central region of such a star undergoing pure Kelvin contraction is not high enough for any nuclear process to take place. The star then continues to contract by releasing the gravitational energy in the form of radiation from the surface, and then the central temperature continues to rise. During this stage nuclear species lighter than C^{12} are destroyed by thermonuclear reactions with hydrogen resulting in the formation of He^3 and He^4 with a considerable release of energy if original elements in our universe contained an appreciable amount of such elements⁵⁾. A rough estimate shows, however, that these reactions do not alter the time scale of the contracting stage appreciably. In later stages of the contraction, the temperature of the central region

becomes high enough to convert hydrogen into helium, namely by pp or CN reactions which take place stationary. Various characteristics of the star in this apparent stationary stage are uniquely fixed by its original chemical composition and mass. The original chemical composition is common in the present case and a star of a given mass represents a definite point in the H-R diagram. This is the original main-sequence. The time t_g necessary for a star to reach this stage of its evolution strongly depends on its mass and is exceedingly short compared with the time scale of the universe (5 or 6 billion years) for stars of large masses.

We next consider the subsequent evolutionary process of the original main-sequence star which is characterized with its mass and original chemical composition. As the hydrogen in its central region becomes converted into helium, the chemical composition will be altered. If there is no appreciable mixing between the central region and the outer envelope, the inhomogeneity of the chemical composition will be set up. On the other hand if there is much mixing, such a difference of the composition will be destroyed and the chemical composition remains homogeneous. Theoretical studies suggest that the star of the mass less than twice the solar mass is very unlikely to mix its whole mass, but instead develops itself into a non-homogeneous structure⁽⁶⁾. Hence hydrogen in the core is gradually transformed into helium, but outside the core the material retains its original composition. After all the hydrogen in the core has been used up, the energy generating zone moves outwards and the size of the helium core grows gradually.

The evolutionary sequence of models with exhausted hydrogen core was computed by various authors⁽²⁾⁽⁷⁾⁽⁸⁾. Sandage and Schwarzschild⁽⁸⁾, and more recently Hoyle and Schwarzschild⁽²⁾ have shown that at this later stage the helium core starts to contract by releasing the gravitational energy and at the same time by increasing its temperature to such a high value as 10^8 °K. At this stage of evolution it was suggested by Salpeter and Öpik⁽⁹⁾ that helium can be converted into carbon supplying a new source of energy. The rate of this reaction was found to be faster by a factor of 10^6 than that given by Salpeter as will be published in our subsequent paper⁽¹⁰⁾. As the energy yield from this process is, however, only about one-tenth of the energy yielded by the conversion of hydrogen into helium, after a very short time helium must become exhausted and perhaps the more and more rapid contraction of the core will result in the higher temperature of the core to start further nuclear reactions involving carbon. It will be also shown that at a little higher temperature nuclear reactions involving synthesis of heavier elements are possible in its deep interior of increasing temperature and density of the contracting star⁽¹⁰⁾.

We assume with Hoyle⁽¹¹⁾ that, after further contraction of the star along the horizontal branch across the main-sequence in the H-R diagram, the star collapses into a white dwarf perhaps shedding matter as a supernova or a nova or an other collapsing star. Hoyle has discussed in detail the processes of nuclear synthesis at this collapsing stage under a rather arbitrary condition. It seems interesting to rediscuss these processes by using more recent nuclear data⁽¹²⁾ and also to compare them with similar processes which take place in the course of stellar contraction.

The time necessary for the stars of various masses to evolve into this collapsing stage

is roughly proportional to their mass-to-luminosity ratio at their main-sequence stage. Again the time is very short compared with the time scale of the universe for stars of large masses and at the present time there exists no original main-sequence star of the mass larger than 1.2 solar mass. The original main-sequence stars now existing are those which are less massive than 1.2 solar mass and presumably non-mixing. We assume that they are stars of Population II.

More massive stars of original main-sequence had evolved to their collapsing stage and ejected most of their material into surrounding space in the very early stage of the universe.

By this scattering process heavy elements generated in the preceding stage mixed with the original interstellar matter (which has nearly negligible contents of heavy elements and composed mainly of hydrogen). The clouds of this material again condensed to the stars of main-sequence. We assume these stars as those of Population I. Again massive stars repeat the same type of stellar evolution, namely the development of helium core, synthesis of heavy elements, explosion and shedding of material into interstellar space in a relatively short period of time. Therefore massive stars may have repeated this cycle over and over again. Hot and massive stars now existing (in our neighbor) must be such stars of Population I which have recently condensed from the cloud of gas and dust after several such cycles.

§ 3. Discussion

We will now discuss whether or not the characteristics of two populations mentioned in section I can be explained by the evolutionary sequence proposed in the preceding section.

i) Distributions of stars in H-R diagram

The significant features of the differences between the H-R diagrams for two populations are: (1) the absence of the main sequence brighter than $M_p = +3.5$ in Population II and the absence of a horizontal giant branch in Population I, (2) the difference in the red giant sequences between two Populations: namely the giants of Population II differ from those in Population I by being generally 2 or 3 magnitude brighter than the latter. It is obvious from our assumption for Population II that there exist no bright main sequence.

Bright main-sequence stars of Population II have evolved to their collapsing stage in the early stage of the universe. In this connection, it is interesting to note that the horizontal branch is, by consideration of the evolutionary history of the M 3 stars, believed to be composed of stars of about 1.2 solar mass. The available evidence also suggests that stars settle down as stars in regions near the left hand side of the horizontal branch in the H-R diagram after reaching collapsing stars—the novae and supernovae in particular. As to the magnitude difference between the giant sequences of two populations, Hoyle and Schwarzschild²⁷⁾ have shown that it is due to surface conditions that differ because of the different contents of metals in Populations I and II. So in this connection we need only to discuss the difference in chemical abundances.

ii) Difference in chemical abundances

It is generally believed that a low abundance of metals in Population II stars is res-

possible for the spectroscopic peculiarities of high-velocity stars. The metallic lines are again much too weak in comparison with hydrogen lines. These spectroscopic peculiarities could be interpreted as a large deficiency in the abundance of metals, the deficiency factor being of the order of 10. According to our evolutionary scheme, heavy elements are generated and scattered into space by stars of Population II and old Population I stars of large masses which have already reached their collapsing stage. The enrichment in this way of the heavy element component of interstellar gas could mean that, even if the metallic content of the original main-sequence stars were negligibly small, subsequent galactic cloud from which stars of Population I had condensed would have to contain a considerable amount of heavy elements ejected from the previous massive stars.

iii) *Space distributions and velocities*

The stars of Population I have low random velocities relative to the Sun and move in nearly circular orbits about the galactic centre and concentrated toward the galactic plane forming a disk-shaped flat system. The situation is believed to be typical of all spiral nebulae. While the stars of Population II, including globular clusters, which are much less concentrated towards the galactic plane and form an extensive spherical halo of stars around the galaxy have generally highly elliptical orbits about the galactic centre passing near or even within the nuclear region of the galaxy.

According to our evolutionary scheme, the systems of Population II stars which were formed in the very early stage of the galaxy are believed to retain the roughly spherical shapes of the proto-galaxies. A small fraction of the primordial clouds, within which the original main-sequence stars are condensed, now retain their original shapes as globular clusters and the remainders now exist as the spherically shaped halo of Population II stars including high-velocity stars and cluster type variables. In the course of evolution, a considerable fraction of these original stars are of course captured into the galactic centre forming a relatively dense galactic nucleus of Population II stars.

Of these original stars of Population II, massive stars have collapsed in the early stage of the galaxy ejecting most of their material as described in the preceding section. The ejected material is believed to be trapped into the dense galactic plane together with the remaining original interstellar material which had not condensed into original main-sequence stars within clusters. The galactic rotation prevented these ejected material to be captured into the galactic nucleus. Stars of Population I have been condensed within this mixed material and thus formed a disk-shaped flat system of stars. Low random velocities and nearly circular orbits about the galactic centre of Population I stars are obviously understood from our evolutionary scheme described above.

It is also evident why Population I stars are associated with dust, while dust is scarce or absent in globular clusters. Recently, von Weizsäcker have discussed the origin of galactic rotation, the evolution of globular clusters and some related problems from the consideration of encounters of turbulent eddies in the proto-galaxy⁽¹¹⁾. Our scheme of evolution is in general agreement with his theory in this phase.

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Theory of Electron Multiplication in Silicon

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In the previous paper the conductivity of non-polar crystals in the strong electrostatic field was computed by the kinetic-statistical method. The impact ionization process, however, was neglected there. In the present paper the effect of the impact ionization process is also considered and the theory is extended to the case, where the distribution function depends on the velocity and the space-coordinate of the particles. The theory is applied to "the electron multiplication" in the p - n junction of Si, which was observed by McKay and McAfee.

§ 1. Introduction

In the previous paper,¹⁾ subsequently referred to as part I, the conductivity of non-polar crystals in the strong electrostatic field was computed by the kinetic-statistical method. Although the simplified model of the spherical energy surface was used, the results were in qualitative agreement with the observation of Ryder and Shockley. In the previous treatment, however, the impact ionization process was completely neglected, because the applied field was not strong enough to bring forth the above effect appreciably. Recently McKay and McAfee²⁾ have found the "Electron Multiplication" phenomenon in the p - n junction of Si and Ge crystals. The theoretical explanation of the phenomenon was also given by McKay³⁾ and Wolff⁴⁾ by applying the method developed in the field of the gas-discharge. Therefore, we shall extend our previous theory with the intention of treating the electron multiplication process along the general line of the quantum theory of solids. Since the rigorous treatment of the elementary processes in solids is very complicated, we must at present content ourselves with rather crude approximations. Thus, our theoretical results are only qualitative.

§ 2. Bloch equation without space-coordinates

We shall concern ourselves with the problem of the behaviour of electrons as they move in the pure non-polar insulator under very strong electrostatic field. The conduction electrons are assumed to obey the Maxwell-Boltzmann statistics, because their density is very small. We also assume for simplicity that an energy contour forms a sphere in the wave number space.

$$E = \hbar^2 K^2 / 2m \quad (1)$$

As is well known, this assumption is not valid for Si and Ge. but we found in part

I that this assumption led to good results as far as we concern with qualitative affairs.

Now, we shall write down the Bloch's kinetic-statistical equation for the electron distribution function, which is at first assumed to be independent of space coordinates. Later on we shall consider the distribution function which depends upon space-coordinates. In this section we also neglect the motion of positive holes. The equation has to be considered in two regions in the energy space, the region A below the ionization energy E_0 and the region B above it.

(1) We shall begin with the region A. We expand the distribution function of conduction electrons in a series of Legendre polynomials:

$$f(\mathbf{k}) = f_0(E) + k_z g(E) + \dots \quad (2)$$

The stationary condition is given by;

$$[\partial f / \partial t] = [\partial f / \partial t]_F + [\partial f / \partial t]_C = 0. \quad (3)$$

In part I we have derived the equations;

$$(E + P k_0 T) d^2 f_0 / dE^2 + (E / k_0 T + 2 + P k_0 T / E) df_0 / dE + (2 / k_0 T) f_0 = 0 \quad (4)$$

and

$$g = \{ \hbar l e F / (1 + R) \sqrt{2mE} \} df_0 / dE, \quad (5)$$

where P is defined by $P = p / (1 + R)$. The small p is defined by

$$p = P(1 + R) = (e F l)^2 / 6 m u_0^2 k_0 T. \quad (6)$$

Other notations should be understood as:

- k : the wave number vector of conduction electrons,
- K : the magnitude of the vector k ,
- K_z : its component in the field direction,
- l : mean free path of conduction electrons in weak field,
- m : mass of the conduction electron,
- u_0 : velocity of the sound wave,
- ω_0 : frequency of the optical vibration,
- C : an energy parameter describing the coupling between the electron and acoustical lattice vibration,
- D : an energy parameter describing the coupling between the electron and optical lattice vibration,
- ρ : the first non-vanishing reciprocal vector of the lattice,
- F : intensity of the external field, (average value at the p - n junction)
- T : temperature,
- k_0 : Boltzmann's constant,
- e : the absolute value of the electronic charge

and

$$R = (D^2 / 2C^2) (2m u_0^2 / k_0 T) (\hbar^2 \rho^2 / 2m) (1 / \hbar \omega_0). \quad (7)$$

With these definitions $[\partial f/\partial t]$ is found to be ;

$$[\partial f/\partial t] = \{2 (2m)^{1/2} u_0^2 / l \sqrt{E}\} d/dE [E^2 (df_0/dE + f_0/k_0 T) + P k_0 T E \cdot df_0/dE]. \quad (8)^*$$

Multiplying both sides of this equation by $(4\pi m V/h^3) (2mE)^{1/2} dE$ and integrating over the momentum space we obtain :

$$V/h^3 \{[\partial f/\partial t] dp_x dp_y dp_z\} = (16\pi m^2 u_0^2 V/lh^3) \{d/dE [E^2 (df_0/dE + f_0/k_0 T) + P k_0 T E \cdot df_0/dE] dE\}. \quad (9)$$

Since the left side of (9) must vanish in the equilibrium state, it becomes ;

$$(16\pi m^2 u_0^2 V/lh^3) [E^2 (df_0/dE + f_0/k_0 T) + P k_0 T E \cdot df_0/dE] = -S_0, \quad (10)$$

where S_0 is an integral constant. The physical meaning of S_0 is the average number density of conduction electrons passing per unit time into the ionization region B. If we neglect the ionization process, we may take S_0 as zero, and then we obtain the distribution function $f_0(E)$ as in part I ;

$$f_0(E) = N_0 (E + P k_0 T)^P \exp(-E/k_0 T), \quad (11)$$

which is reduced to

$$f_0(E) = N \exp\{-E^2/2P(k_0 T)^2\}, \quad (12)$$

in the case of the very strong field. In the present case, where the ionization process is our main interest, the evaluation of S_0 will be our main task.

Let us now seek for the equation for $f_0(E)$, when the ionization process is predominant. We shall denote the number density of conduction electrons by n and the number density of electrons ionized per unit time into the energy interval between E and $E+dE$ by $nQ(E)dE$. Besides $Q(E)$ some valence electrons are ionized thermally into the conduction band and at the same time some conduction electrons recombine with holes. However, we do not consider them explicitly, because these processes are not very important within the p - n junction under the experimental condition of McKay and McAfee. Taking account of $Q(E)$ we obtain the following equations instead of (10).

$$\frac{16\pi m^2 u_0^2 V}{h^3 l} \left[(E^2 + P k_0 T E) \frac{df_0}{dE} + \frac{E^2}{k_0 T} f_0 \right] + \frac{4\pi m (2m)^{1/2} V}{h^3} \int_0^E Q(x) x^{1/2} dx = 0, \quad (13)$$

and

$$\frac{V}{h^3} \int_0^{E_0} Q(E) dp_x dp_y dp_z = \frac{4\pi m (2m)^{1/2} V}{h^3} \int_0^{E_0} Q(x) x^{1/2} dx = 2S_0 \quad (14)$$

The factor 2 in (14) means that two slow conduction electrons appear after an ionization process. Here we must notice that these equations do not satisfy the conservation law of the S -current, because the absorption process (recombination of electrons and holes) is

*) This expression is only allowable, when $Q(E)$, the electron distribution created by the impact ionization, has the spherical symmetry in the k -space. The validity of this assumption is rather doubtful in the strong electric field.

neglected. In the following section, however, we shall introduce the distribution function which depends upon both energy and space coordinates of electrons, and we shall see there that the conservation law is fulfilled by the existence of the diffusion current. Now, the solution of (13) is given by ;

$$f_0^A(E) = \exp \{ -E^2/2P(k_0T)^2 \} [C - l / \{ (2m)^{1/2} 2u_0^2 k_0TP \} \\ \times \int_0^k (1/\gamma) \exp \{ \gamma^2/2P(k_0T)^2 \} d\gamma \int_0^y Q(x) x^{1/2} dx], \quad (15)$$

where C is an integral constant. If we denote the distribution function of the B-region by $f_0^B(E)$, the constant C is determined by the condition $f_0^A(E_0) = f_0^B(E_0)$ as follows ;

$$C = f_0^B(E_0) \exp \{ E_0^2/2P(k_0T)^2 \} + l / \{ (2m)^{1/2} 2u_0^2 k_0TP \} \\ \times \int_0^{k_0} (1/\gamma) \exp \{ \gamma^2/2P(k_0T)^2 \} d\gamma \int_0^y Q(x) x^{1/2} dx. \quad (16)$$

From (14) we have ;

$$\int_0^k Q(x) x^{1/2} dx = [2S_0 b^3 / 4\pi m (2m)^{-1/2} V] - \int_k^{k_0} Q(x) x^{1/2} dx. \quad (17)$$

Inserting (16) and (17) into (15) we obtain ;

$$f_0^A(E) = \exp \{ -E^2/2P(k_0T)^2 \} [f_0^B(E_0) \exp \{ E_0^2/2P(k_0T)^2 \} \\ + (b^3 l S_0 / 8\pi m^2 u_0^2 k_0 T P V) \int_k^{k_0} (1/\gamma) \exp \{ \gamma^2/2P(k_0T)^2 \} d\gamma \\ \times \left(1 - [4\pi m (2m)^{1/2} V / 2S_0 b^3] \int_y^{k_0} Q(x) x^{1/2} dx \right)]. \quad (18)$$

(2) Next we shall proceed to the region B.

Heller¹⁰⁾ has computed the distribution function of electrons in this region with some assumptions about the cross-section of the impact ionization. He has found that the distribution function decreases very rapidly with increasing energy owing to the large ionization loss. Unfortunately, we have no reliable knowledge about the ionization process in crystals, especially under the strong electrostatic field, (we hope that we shall develop a more rigorous treatment in future), but if we estimate the cross-section of the impact ionization from the data of the secondary electron emission, we obtain a large cross-section. Thus we assume here that the probability of the impact ionization is larger than that of the lattice scattering, and the behaviours of the distribution function are mainly determined by the ionization loss (see appendix). Under these assumptions the approximate distribution function may be obtained by an elementary method. Now, we shall assume tentatively that the distribution function has the following form ;

$$f_0^B(E) = \mathfrak{N} (E_0/E)^{1/2} \exp \{ -\alpha (E - E_0) \}. \quad (19)$$

Then the average value of the energy of electrons in the B-region becomes,

$$\bar{E} = \int_{E_0}^{\infty} E f_0^B(E) E^{1/2} dE / \int_{E_0}^{\infty} f_0^B(E) E^{1/2} dE = E_0 + 1/\alpha. \quad (20)$$

If we denote the mean collision time of the impact ionization by $\tau(E)$ and the mean drift velocity by v_d , then the mean drift distance of an electron with energy E_0 is given by

$$\bar{x} = v_d \langle \tau(E) \rangle_{AV} = v_d \tau. **$$

Thus the mean energy of electrons in the B-region is approximately given by ;

$$\bar{E} = E_0 + eF\bar{x} = E_0 + eF v_d \tau. \quad (21)$$

Comparing (20) with (21) we find ;

$$1/\alpha = eF v_d \tau. \quad (22)$$

The quantity S , which is the current density in the wave-number space, has the value S_0 at the energy-sphere E_0 , and in the B-region it decreases to zero with extreme rapidity owing to the ionization loss. Using $f_0^B(E)$ and τ , S_0 is given by ;

$$\begin{aligned} S_0 &= (V/h^3) \int_{E_0}^{\infty} \{ f_0^B(E) / \tau(E) \} dp_x dp_y dp_z \\ &= (4\pi m (2m)^{1/2} V \mathfrak{A} / h^3 \tau) \int_{E_0}^{\infty} (E_0/E)^{1/2} \exp \{ -\alpha(E-E_0) \} E^{1/2} dE. \end{aligned}$$

Thus, using (22) \mathfrak{A} is determined as follows ;

$$\mathfrak{A} = (h^3 S_0 / 4\pi m (2m)^{1/2} V) (1/v_d e F E_0^{1/2}). \quad (23)$$

(3) The number of disappeared electrons between E and $E+dE$ per unit time is given by $(n/\tau(E)) \int_{E_0}^{\infty} f_0^B(E) E^{1/2} dE$.

If we adopt a further crude approximation that two electrons have approximately the same energy after the ionization collision, then the number of electrons which appear at the energy interval between $(E-E_0)/2$ and $(E+dE-E_0)/2$ is given by ;

$$\begin{aligned} &\{ 2n/\tau(E) \} \int_{E_0}^{\infty} f_0^B(E) E^{1/2} dE \\ &= \{ 2n S_0 / \tau(E) v_d e F \} \exp \{ -\alpha(E-E_0) \} dE. \end{aligned}$$

Changing the variable from E to $E' = (E-E_0)/2$ and considering the definition of $Q(E)$ we find ;

$$\begin{aligned} n \int_0^E Q(x) x^{1/2} dx &= (h^3 / 4\pi m (2m)^{1/2} V) (4n S_0 / v_d e F \tau) \int_0^E \exp \{ -2\alpha E' \} dE' \\ &= (2n S_0 h^3 / 4\pi m (2m)^{1/2} V) (1 - \exp \{ -2\alpha E \}). \end{aligned} \quad (24)$$

Inserting (19), (23) and (24) into (18) $f_0^A(E)$ becomes ;

**) Strictly speaking, this assumption is not valid, because the average drift velocity of electrons in B-region is not equal to v_d , which is the mean drift velocity of the total electrons.

$$f_0^A(E) = (h^3 S_0 / 4\pi m (2m)^{1/2} V) \exp \{-E^2 / 2P(k_0 T)^2\} \\ \times [(1/v_d e F E_0^{1/2}) \exp \{E_0^2 / 2P(k_0 T)^2\} \\ + (l / (2m)^{1/2} v_0^2 k_0 T P) \int_{E_0}^{E_0} (1/E) \exp \{E^2 / 2P(k_0 T)^2\} (1 - \exp \{-2\alpha E\}) dE], \quad (25)$$

while $f_0^B(E)$ is rewritten as

$$f_0^B(E) = (h^3 / 4\pi m (2m)^{1/2} V) (S_0 / v_d e F E^{1/2}) \exp \{-\alpha(E - E_0)\}. \quad (26)$$

Finally S_0 is determined by the normalization condition ;

$$(4\pi m (2m)^{1/2} V / h^3) \int_0^\infty f_0(E) E^{1/2} dE = 1. \quad (27)$$

§ 3. Bloch-equation with space-coordinates

It has hitherto been assumed that the crystal has an infinite dimension. This assumption may lead to reasonable results, when phenomena do not depend on the crystal size as in the case of the ordinary ohmic conduction. The phenomenon of the current multiplication in the p - n junction, however, depends explicitly on the dimension of the junction region. In order to treat such a problem we must use the distribution function $f(E, x)$, which depends both upon energy and a space coordinate x in the field direction. In this case Boltzmann's equation becomes ;

$$[\partial f / \partial t]_{\text{drift}} + [\partial f / \partial t]_{\text{coll}} = 0, \quad (28)$$

where

$$-[\partial f / \partial t]_{\text{drift}} = (\hbar/m) k_x \cdot \partial f / \partial x - (eF/\hbar) \cdot \partial f / \partial k_x \quad (29)$$

and

$$f(k) = f_0(E, x) + k_x g(E, x). \quad (30)$$

In the classical treatment the generalization is attained only through the following replacement⁽⁵⁾ ;

$$eF \left(g + \frac{2}{3} E \frac{dg}{dE} \right) \rightarrow eF \left(g + \frac{2}{3} E \frac{\partial g}{\partial E} - \frac{1}{eF} \frac{2}{3} E \frac{\partial g}{\partial x} \right), \\ eF v_x (df_0/dE) \rightarrow eF v_x \left(\frac{\partial f_0}{\partial E} - \frac{1}{eF} \frac{\partial f_0}{\partial x} \right).$$

Thus we shall follow these procedures formally. Strictly speaking some questions remain unsolved about this point. In the very strong field the wave packet of electrons must be confined to the very small dimension in order to keep the field as nearly constant in the packet. On the other hand, the energy of such a packet spreads out considerably.⁽⁶⁾

If we neglect the ionization process, we obtain the following equations using eqs. (33) and (34) in part I.

$$E(\partial^2 f_0 / \partial E^2) + (2 + E/k_0 T) (\partial f_0 / \partial E) + 2/k_0 T \cdot f_0 \\ = \{ l e F / 2 \hbar u_0^2 (2mE)^{1/2} \} (g + 2E/3 \cdot \partial g / \partial E - 2E/3 eF \cdot \partial g / \partial x), \quad (31)$$

and

$$g(E, x) = - \{ \hbar e F l / [1] (2mE)^{1/2} \} (\partial f_0 / \partial E - 1/eF \cdot \partial f_0 / \partial x), \quad (32)$$

where $[1]$ means simply $[1] = 1$.

Eliminating $g(E)$ we have ;

$$(E^2 + p k_0 T E) (\partial^2 f_0 / \partial E^2) + (2E + E^2/k_0 T + p k_0 T) (\partial f_0 / \partial E) \\ + 2E/k_0 T \cdot f_0 + p k_0 T [E/(eF)^2 \cdot (\partial^2 f_0 / \partial x^2) - 2E/eF (\partial^2 f_0 / \partial E \cdot \partial x) \\ - 1/eF (\partial f_0 / \partial x)] = 0, \quad (33)$$

or

$$\frac{\partial}{\partial E} \left[(E^2 + p k_0 T E) \frac{\partial f_0}{\partial E} + \frac{E^2}{k_0 T} f_0 \right] = p k_0 T \left[- \frac{E}{(eF)^2} \frac{\partial^2 f_0}{\partial x^2} + \frac{2E}{eF} \frac{\partial^2 f_0}{\partial x \partial E} + \frac{1}{eF} \frac{\partial f_0}{\partial x} \right] \quad (34)$$

Thus, we obtain the following equation instead of (10).

$$(E^2 + p k_0 T E) (\partial f_0 / \partial E) + E^2/k_0 T \cdot f_0 \\ + p k_0 T \int [E/(eF)^2 (\partial^2 f_0 / \partial x^2) - 2E/eF (\partial^2 f_0 / \partial x \partial E) - (1/eF) (\partial f_0 / \partial x)] dE \\ = - (\hbar^3 l / 16 \pi m^2 u_0^2 V) S_0(x). \quad (35)$$

Next, we shall proceed to the more practical case, where the ionization process takes place, and both electron- and hole-currents exist. In the following we shall use the suffix e for electrons and h for holes, and newly define the function $\gamma(E, x)$ by

$$\gamma(E, x) = n(x) f_0(E, x).$$

Then our fundamental equation (10) may be generalized to the following simultaneous equations ;

$$(16 \pi m_e^2 u_0^2 / \hbar^3 l_e) [(E^2 + p_e k_0 T E) (\partial \gamma_e / \partial E) + (E^2/k_0 T) \gamma_e] \\ = (16 \pi m_e^2 u_0^2 / \hbar^3 l_e) (p_e k_0 T / eF) (E \partial \gamma_e / \partial x \\ + \int_0^E (\partial^2 \gamma_e / \partial x \partial E) dE - (1/eF) \int_0^E (\partial^2 \gamma_e / \partial x^2) dE) \\ - (4 \pi m_e (2m_e)^{1/2} / \hbar^3) n_e(x) \int_0^E Q_e(Z) Z^{1/2} dZ \\ - (4 \pi m_h (2m_h)^{1/2} / 2 \hbar^3) n_h(x) \int_0^E Q_h(Z) Z^{1/2} dZ, \quad (37) \\ (16 \pi m_h^2 u_0^2 / \hbar^3 l_h) [(E^2 + p_h k_0 T E) (\partial \gamma_h / \partial E) + (E^2/k_0 T) \gamma_h]$$

$$\begin{aligned}
&= - (16\pi m_h^2 v_0^2 / h^3 l_h) (p_h k_0 T / eF) (E \partial \gamma_h / \partial x \\
&\quad + \int_0^E (\partial^2 \gamma_h / \partial x \partial E) dE + (1/eF) \int_0^E E (\partial^2 \gamma_h / \partial x^2) dE) \\
&\quad - (4\pi m_h (2m_h)^{1/2} / h^3) \int_0^E Q_h(Z) Z^{1/2} dZ \\
&\quad - (4\pi m_e (2m_e)^{1/2} / 2h^3) n_e(x) \int_0^E Q_e(Z) Z^{1/2} dZ.
\end{aligned} \tag{38}$$

The last term in the round bracket of the right side of (37) does not depend on the field strength explicitly and it means the pure diffusion in x -space. Other terms in the bracket come from the coupling effects between the electric field and the electronic diffusion. The second term in the right side means the current density, which enters into the conduction band per unit time as a result of collisions between fast electrons and valence electrons. The last term means the current density, which enters into the conduction band as a result of collisions between fast holes and valence electrons. The physical meaning of the terms in (38) is to be interpreted in the similar way. If we extend the integration-limits to infinity, we have two conditions which express the conservation of current in the wave-number space both for electrons and holes respectively. Here we have used eqs. (10), (14) and (6).

$$\begin{aligned}
n_e(x) S_{e0}(x) &= - \{8\pi m_e V e F l_e / 3h^3 [1]\} \{[E \partial \gamma_e / \partial x]_{E=\infty} \\
&\quad + \int_0^\infty E (\partial^2 \gamma_e / \partial x \partial E) dE - (1/eF) \int_0^\infty E (\partial^2 \gamma_e / \partial x^2) dE\} \\
&\quad + 2n_e(x) S_{e1}(x) + n_h(x) S_{h0}(x),
\end{aligned} \tag{39}$$

$$\begin{aligned}
n_h(x) S_{h0}(x) &= \{8\pi m_h V e F l_h / 3h^3 [1]\} \{[E \partial \gamma_h / \partial x]_{E=\infty} \\
&\quad + \int_0^\infty E (\partial^2 \gamma_h / \partial x \partial E) dE - (1/eF) \int_0^\infty E (\partial^2 \gamma_h / \partial x^2) dE\} \\
&\quad + 2n_h(x) S_{h1}(x) + n_e(x) S_{e1}(x).
\end{aligned} \tag{40}$$

We must notice here that the electric field F in (39) or (40) is not always equal to the external applied field F_{ext} , when the density of electrons and holes in the p - n junction reaches a considerable amount. If we denote the electric field by

$$F = F_{ext} + F_{int} = F_{ext} - \partial \mathfrak{S} / \partial x. \tag{41}$$

The potential \mathfrak{S} is determined by

$$\partial^2 \mathfrak{S} / \partial x^2 = -4\pi e \{n_h(x) - n_e(x)\}. \tag{42}$$

In general we must solve the equations (37), (38) and (42) with suitable boundary conditions. Thus, we have found that "the electron multiplication process in the p - n junction" is in general a very complicated problem. However, we shall see that the main character of McKay's experiment is explained by our theory, even when we introduce rather

crude approximations. If the density of electrons and holes is not very large and effects of the diffusion terms are rather small, we may be allowed then to use the appropriate approximations. In the following we shall separate the variables in equations and neglect some diffusion terms. As a result of these approximations we are able to derive some conclusions which may be compared with observations.

Hitherto we neglected the interaction between electrons and the optical lattice vibrations for simplicity. However, the generalization of the formulas to include the contribution from the optical modes is a simple matter as far as we adopt our previously mentioned approximations. That is, we only need to write P instead of p and $(1+R)$ instead of $[1]$ in the equations from (32) to (40).

Now, we separate the variables E and x as follows;

$$\gamma(E, x) = n(x)f_0(E). \quad (43)$$

Then eq. (37) becomes;

$$\begin{aligned} & n_e(x) \left[(E^2 + P_e k_0 T E) (df_0/dE) + E^2/k_0 T \cdot f_0 \right] \\ &= - \{l_e/2u_0^2 (2m_e)^{1/2}\} n_e(x) \int_0^E Q_e(Z) Z^{1/2} dZ \\ &\quad - \{l_e m_h^{3/2}/4 \cdot 2^{1/2} u_0^2 m_e^2\} n_h(x) \int_0^E Q_h(Z) Z^{1/2} dZ \\ &\quad + (P_e k_0 T/eF) \{ (dn_e/dx) E f_0 + (dn_e/dx) \int_0^E (df_0/dE) dE \\ &\quad - (1/eF) (d^2 n_e/dx^2) \int_0^E E f_0(E) dE \}. \end{aligned} \quad (44)$$

As long as we can neglect the last diffusion terms, this equation is essentially equivalent to eq. (13). Thus we are able to obtain the approximate solution of (44). In this case $S_0(E, x)$ is considered to be a function of E only. Further, if we want to see the effects of the diffusion terms, we may use the successive approximations.

Using (43), eqs. (39) and (40) become;

$$\begin{aligned} & \{ (8\pi m_e l_e V/3h^3) \int_0^\infty E f_0 dE \} (d^2 n_e/dx^2) + \{ (8\pi m_e V e F l_e/3h^3 (1+R)) \\ & \times \int_0^\infty E (df_0/dE) dE \} (dn_e/dx) + S_{e0} n_e(x) + S_{h0} n_h(x) = 0, \end{aligned} \quad (45)$$

and

$$\begin{aligned} & \{ (8\pi m_h l_h V/3h^3) \int_0^\infty E f_0 dE \} (d^2 n_h/dx^2) - \{ (8\pi m_h V e F l_h/3h^3 (1+R)) \\ & \times \int_0^\infty E (df_0/dE) dE \} (dn_h/dx) + S_{e0} n_e(x) + S_{h0} n_h(x) = 0. \end{aligned} \quad (46)$$

If the condition

$$\left(\int E f_0 dE \right) (d^2 n / dx^2) \ll (eF) \int E (df_0 / dE) dE \times (dn / dx)$$

is satisfied, these equations are reduced to ;

$$\begin{aligned} & \{ (8\pi m_e V e F l_e / 3 h^3 (1+R)) \int_0^\infty E (df_0 / dE) dE \} (dn_e / dx) \\ & + S_{e0} n_e(x) + S_{h0} n_h(x) = 0, \end{aligned} \quad (47)$$

and

$$\begin{aligned} & - \{ (8\pi m_h V e F l_h / 3 h^3 (1+R)) \int_0^\infty E (df_0 / dE) dE \} (dn_h / dx) \\ & + S_{e0} n_e(x) + S_{h0} n_h(x) = 0. \end{aligned} \quad (48)$$

Here we consider the electric current or the drift velocity v_d , which is given by :

$$\begin{aligned} v_d &= \int v_x k_x g(E) dk_x dk_y dk_z dV \\ &= -V \int v_x k_x \{ \hbar l e F / (2mE)^{1/2} (1+R) \} dk_x dk_y dk_z \\ &= - \{ 8\pi m V e F l / 3 h^3 (1+R) \} \int_0^\infty E (df_0 / dE) dE. \end{aligned} \quad (49)$$

Thus we find that eqs. (47) and (48) can be written in a simple way.

$$\begin{cases} -v_{ed} dn_e / dx + S_{e0} n_e + S_{h0} n_h = 0, \end{cases} \quad (50)$$

$$\begin{cases} v_{hd} dn_h / dx + S_{e0} n_e + S_{h0} n_h = 0. \end{cases} \quad (51)$$

Here we set the boundary conditions as follows ;

$$\begin{cases} x=0 ; & n_e = n_0 \\ x=L ; & n_h = n'_0. \end{cases} \quad (52)$$

Under the condition $F_{\text{ext}} \gg F_{\text{int}}$, we can easily solve these equations :

$$\begin{aligned} n_e(x) &= \{ S_{h0} (n v_d) / v_{ed} v_{hd} \} x \\ &+ n_0 \exp \{ (S_{e0} / v_{ed} - S_{h0} / v_{hd}) x \}, \end{aligned} \quad (53)$$

and

$$\begin{aligned} n_h(x) &= (1 - S_{h0} x / v_{hd}) (n v_d) / v_{hd} \\ &- (v_{ed} / v_{hd}) n_0 \exp \{ (S_{e0} / v_{ed} - S_{h0} / v_{hd}) x \}, \end{aligned} \quad (54)$$

where

$$v_{ed} n_e(x) + v_{hd} n_h(x) = \text{const.} \equiv v_d n$$

and

$$v_d n = v_{hd} (1 - S_{h0} L / v_{hd})^{-1} \times [n'_0 + n_0 (v_{ed} / v_{hd}) \exp \{ (S_{e0} / v_{ed} - S_{h0} / v_{hd}) L \}]. \quad (55)$$

As a special case, if we put $v_{hd} \sim 0$ and $S_{h0} \sim 0$, it becomes:

$$n_e(x) = n_0 \exp \{ (S_{e0} / v_{ed}) x \}; \quad n_h(x) \approx 0. \quad (56)$$

And if we put $S_{h0} = S_{e0} = S_0$ and $v_{ed} = v_{hd} = v_d$, then it becomes:

$$\begin{cases} n_e(x) = n_0 + (S_0 / v_d) n x, \\ n_h(x) = -n_0 + n (1 - S_0 x / v_d) \end{cases} \quad (57)$$

$$\begin{cases} n_e(x) + n_h(x) = \text{const.} \equiv n, \\ n = (n_0 + n'_0) / (1 - S_0 L / v_d) \end{cases} \quad (58)$$

§ 4. Results and discussions

In this section we shall consider the physical meaning of the formulas developed in § 2 and § 3. Since we have introduced many approximations, we can not draw any quantitative conclusions from them. However, we shall show that these formulas can explain the main features of the electron multiplication process in a qualitative sense. Now, we examine eq. (25). When the field is not very strong, that is $E_0^2 \gg 2P(k_0 T)^2$, the first term in [] is much larger than the second and S_0 is given approximately by:

$$S_0 = v_d e F \exp \{ -E_0^2 / 2P(k_0 T)^2 \} / \int_0^\infty \exp \{ -E^2 / 2P(k_0 T)^2 \} dE, \quad (59)$$

where we have used eq. (27). Thus S_0 is very small and the effect of the impact ionization is negligible, when $E_0^2 \gg 2P(k_0 T)^2$. From eq. (59) we see that S_0 increases very rapidly with increasing field intensity. Moreover, the second term of (25) becomes large and it becomes much larger than the first, when the field becomes very strong. The fact that S_0 increases very rapidly with increasing field is a very important result of our theory and the observation of McKay and McAfee confirmed this interesting field-current characteristics.

As is well known the electrical breakdown of alkali-halide crystals is attributed to the electron avalanche caused by the series of the impact ionizations of electrons. We suppose that the behaviours of the avalanche may be represented by S_0 . In the case of polar crystals the interaction between conduction electrons and lattice vibrations is strong and the usual perturbation method is not applicable.⁷⁾

However, if we compute S_0 by the usual perturbation method, we find that the behaviour of S_0 is qualitatively the same as in the case of non-polar crystals. Such a computation has been carried out by Franz⁸⁾ also. Now, we find also that the drift velocity of electrons changes slowly in the strong field region (see appendix). Thus the quantity (S_0 / v_d) , which means the ionization rate, changes very rapidly with increasing field. In

the appendix we estimate the order of magnitude of the ionization rate of Si crystals. The computed (S_0/v_d) versus F curve is shown in Fig. 1 with a dashed line and McKay's data are also shown in a full line. We find that the agreement between theory and experiment is rather good. In fact, the expression of S_0 contains two adjustable parameters. As mentioned in the appendix, we determine them so as to get the best fit between the computed current density and the observation of Ryder in the field region where the ionization process is not important. Thus, we find that our theory gives apparently the correct current-field characteristics in all field intensities.

Now, we proceed to the next problem, that is, the examination of eqs. (56) and (58). The equation (56) says that the number of electrons increases according to the exponential law, when the current is carried by electrons only. The alkali-halide crystals seem to be such examples, but we have not yet reliable data about this point. In Si and Ge crystals, however, the current is carried by both electrons and holes and then we must apply (57) and (58) for these crystals. (In this case we must remember that we have used the simplifying assumptions $S_{h0}=S_{e0}=S$ and $v_{h0}=v_{e0}=v_d$.) Then we find that the electron current or the hole current increases linearly with the distance and the sum of the two current does not depend on x .

We see that eq. (58) has the singularity, when the field intensity reaches the value, where the condition

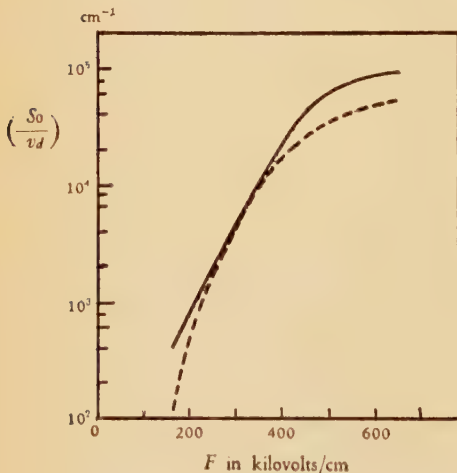


Fig. 1 (S_0/v_d) versus Field Intensity
 — McKay's experiment
 Our Computation

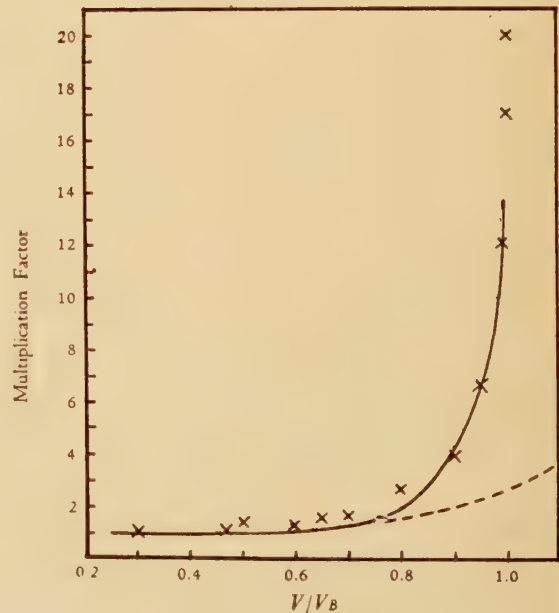


Fig. 2 Current Multiplication
 — (58)
 (56)
 × McKay's data

$S_0 L = v_d$ is satisfied. It expresses the effect of the so-called positive feedback, which is a very important character of the avalanche breakdown of Si and Ge crystals. In Fig. 2 we show multiplication-curve of Si according to (58). The numerical values are tabulated

in appendix. We find that the curve (58) shows the main features of the experiment of McKay. In Fig. 2 we also show the curve according to (56). This is clearly different from the experiment. Thus, we see that the positive feedback effect is the essential character of the p - n junction of Si and Ge, where positive holes and electrons contribute to the current. Finally we must consider F_{int} . Till now we assumed the condition $F_{\text{ext}} > F_{\text{int}}$.

This condition is surely satisfied in the pre-breakdown region ($S_0 L / v_d < 1$). However, when the number density of charge carriers becomes very large as a result of the breakdown, the condition breaks down also. Since the direction of F_{int} is inverse, F_{int} must reduce the effect of F_{ext} , when F_{int} reaches the appreciable intensity, and then the current-multiplication process stops automatically at a certain point by the effect of F_{int} , which is generated by the electron multiplication process itself. We must notice, however, that our theory can not be applied to such a phenomenon.

Finally the author wishes to express his sincere thanks to Prof. T. Muto for the helpful discussions.

Appendix

Since our theory has introduced many simplifying assumptions, it may be meaningless to draw quantitative conclusions from it. However, the estimation of the order of magnitude of some quantities is still necessary for the qualitative understanding of "the electron multiplication process" by our theory. In the following we shall give such an estimation.

(1) cross-section of the impact ionization

The impact ionization process is a very important elementary process in solids and the exact theoretical treatment of it is highly desirable. But we do not touch such a problem here. We shall estimate the order of magnitude of the cross-section only by phenomenological ways.

If we denote the transition probability of ionization process by $W(E)$, and the cross-section by $\sigma(E)$, it becomes

$$W(E) = Nv\sigma(E) = N(2E/m)^{1/2} \cdot \sigma(E)$$

where N is the number of scattering centres in the unit volume. For $N \sim 5.2 \times 10^{22}$ (Si) and $E \sim 1.1$ ev. the average value of $W(E)$ becomes $\overline{W(E)} \sim 3 \times 10^{30} \sigma(E)$. Thus τ becomes ;

$$1/\tau \equiv \overline{W(E)} = 3 \times 10^{30} \overline{\sigma(E)}.$$

Although we do not know the magnitude of $\overline{\sigma(E)}$, it may be $10^{-16} \sim 10^{-17} \text{ cm}^2$. Then the magnitude of τ may be estimated as $3 \times 10^{-14} \sim 3 \times 10^{-15}$ sec.

The secondary electron emission phenomenon of semi-conductors is also useful for the estimation of τ of primary electrons with energy of several hundred ev. There is some evidence which indicates that the mean drift distance between ionization collisions may be at most 10^{-7} cm and the mean time τ is less than 10^{-14} sec. Thus, if we put $\tau \sim 3 \times$

10^{-11} sec and the drift velocity at the field intensity 10^5 volts/cm 10^7 cm/sec, (which is estimated from the experiment of Ryder¹¹), then the mean distance \bar{x} becomes :

$$\bar{x} = v_d \tau \sim 3 \times 10^{-7} \text{ cm.}$$

Thus we see that the mean drift distance of electrons in B-region is much shorter than the usual mean free path of electrons, which is determined mainly by the lattice scattering. (Its magnitude is about $10^{-5} \sim 10^{-6}$ cm at the room temperature.) Thus, $f_0^B(E)$ may be a rapidly decreasing function of E . If we represent it by (19), α is still very large even in the very strong field; for example, putting $F = 3 \times 10^5$ volts/cm $f_0^B(E)$ becomes :

$$f_0^B(E) \propto \exp\{-\alpha(E-E_0)\} = \exp\{-10(E-E_0)\},$$

where the energy-unit is ev. If the above estimation of α is reasonable, the factor $\{1 - \exp(-2\alpha E)\}$ in $f_0^A(E)$ is nearly equal to one except for the region of very small energy. Since the small energy-region contributes little to the S_0 -integral owing to the volume factor, the exact value of α is not important for computing the value of S_0 .

(2) Field intensity

The relation between p and F is given by (6), but this equation is useless for quantitative purposes. Unfortunately, we found in part I that our theory gave too low electric field for the deflection point from ohmic current. The reason for this discrepancy is believed to be in the wrong assumption of the spherical energy surface. However, we know also that, if we re-interpret the meaning of p and F as adjustable parameters and give them suitable values, our previous theory can explain fairly well the observed behaviours of the non-ohmic current in Si and Ge throughout the energy region, where the ionization process is not important. The suitable values of p and F are: (for n -type Si); (1) $R=3$ and (2) the value of $p=1$ corresponds to the field intensity $F=7500$ volts/cm, while eq. (6) gives the value $F=840$ volts/cm.

Hereafter we shall use these values by the field region, where the ionization process is important. Strictly speaking such a treatment is meaningless, because we do not take into account the structure of the energy surface properly. However, we think our phenomenological procedure would be a natural extension, because our theory is led to the previous theory of part I, when the ionization process is neglected.

(3) computation of S_0 and v_d

Now, we eliminate l from (25) by using (6) and insert it into (27), and obtain the following equation for S_0 ;

$$\begin{aligned} 1/S_0 = & (1/eF) \int_0^{E_0} \exp\{-E^2/2P(k_0T)^2\} [(1/v_d E^{1/2}) \exp\{E_0^2/2P(k_0T)^2\} \\ & + (3/(1+R) u_0^2 k_0 T P)^{1/2} \int_E^{E_0} (1/E) \exp\{E^2/2P(k_0T)^2\} (1 - e^{-2\alpha E}) dE] E^{1/2} dE, \quad (\text{A-1}) \end{aligned}$$

where we neglect the contribution from $f_0^B(E)$.

The drift velocity is also written as follows;

$$\begin{aligned}
 v_d &= - (8\pi/3) (l e F m / \hbar^3) \int_0^\infty E (df_0/dE) dE \\
 &= (8\pi m / 3 \hbar^3) \{6mu_0^2 k_0 T (1+R) P\}^{1/2} \int_0^\infty f_0(E) dE.
 \end{aligned}
 \tag{A-2}$$

The results of computations are tabulated in Table I.

The value of S_0 changes rapidly with field intensity, while v_d changes slowly and its value is about 10^7 cm/sec in the computed field region.

Table I.

p	F volts/cm	S_0/v_d
512	1.7×10^5	72
800	2.1	700
1100	2.5	2.0×10^3
1600	3.0	5.0×10^3
2180	3.5	9.2×10^3
2840	4.0	1.3×10^4
3600	4.5	1.8×10^4
7200	6.36	4.3×10^4

According to McKay the linear dimension of the p - n junction of his samples is about $0.4 \sim 2.0 \times 10^{-4}$ cm. Here we take $L = 0.77 \times 10^{-4}$ cm for example, and compute $\exp[S_0 L / v_d]$

Table II.

F	S_0/v_d	$S_0 L / v_d$	$\exp[S_0 L / v_d]$	$1/[1 - S_0 L / v_d]$
2.0×10^5	0.41×10^3	0.03	1.03	1.03
2.4	1.6×10^3	0.12	1.12	1.1
3.0	5.0×10^3	0.38	1.46	1.6
3.2	6.7×10^3	0.51	1.66	2.0
3.6	1.0×10^4	0.77	2.16	4.3
3.8	1.1×10^4	0.85	2.34	6.6
4.0	1.3×10^4	1.0	2.72	∞

in (55) and $1/[1 - S_0 L / v_d]$ in (57) respectively for some values of the field intensity. The necessary values of (S_0 / v_d) are computed by the interpolation method from Table I. The results are also tabulated in Table. II.

(4) effect of diffusion terms

Finally we shall examine the effect of the diffusion terms in (44). In the previous treatment we neglected them and assumed tentatively that the solution of (44) was approximately given by (25). The approximation may not be bad, if the following condition is satisfied:

$$(n_e(x) E^2 / k_0 T) f_0(E) \gg (P k_0 T / e F) (E dn_e / dx) f_0(E). \tag{A-3}$$

Using (57) it becomes :

$$(E/k_0 T) (eFx/k_0 T) (n_0 + S_0 nx/v_d) \gg PS_0 nx/v_d. \quad (\text{A-4})$$

Taking the average value of it we have ;

$$\langle E/k_0 T \rangle_{Av} (eFL/2k_0 T) (n_0 + nS_0 L/2v_d) \gg PnS_0 L/2v_d. \quad (\text{A-5})$$

When the electric field is near the breakdown strength, the number density $f_v(E)E^{1/2}$ has a broad maximum about 0.2~0.4 ev. Thus we may put $\langle E/k_0 T \rangle$ as 10 and the value of $(eFL/2)$ is estimated as 5 ev. by the McKay's experiment. Then, in the breakdown field region we have ;

$$nS_0 L/2v_d \gg n_0$$

and

$$\langle E/k_0 T \rangle (eFL/2k_0 T) \approx 2000 ; P \approx 1000.$$

Thus we see that the effect of the diffusion is not very small and our approximation is not very good in the breakdown field region. However, outside of the breakdown region $S_0 Ln/2v$ is much smaller than n_0 and the condition (A-5) is well satisfied.

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Supernova Origin of Cosmic Rays^{*)}

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The supernova origin of cosmic rays is proposed in connection with the stellar evolution and the building up of heavy elements in the core of stars as they evolve. Nearly equal abundances of heavy and medium nuclei in primary cosmic rays suggest that the sources may be such stars, in which the thermonuclear reactions of building up heavy elements take place and which eventually explode as supernovae. The light elements, Li, Be and B, in galactic matter are considered to be generated by the nuclear bombardment of heavier nuclei at the explosion. The number of cosmic ray particles injected there is estimated in reference to the radio emission of supernova remnants. If the number injected is of the order of 10^{51} per supernova and the explosion is more frequent than now observed, the supernova origin is shown to be not inconsistent with cosmic ray as well as astronomical evidences. The galactic model suitable to our interpretation is a sphere with magnetic clouds. The life time of cosmic rays in this galaxy is estimated to be of the order of 10^8 years, corresponding to the fact that the mean thickness of interstellar matter traversed by cosmic rays is about 1 g cm^{-2} , that is estimated from the abundance of the light nuclei in the primary cosmic rays.

§ 1. Introduction

It has been disputed whether in the primary cosmic radiation the light nuclei, Li, Be and B, are absent or not^{1,2)}. However, recent observations seem to have established the existence of the light nuclei nearly as one half abundant as the medium nuclei, C, N, O and F^{3,4)}. Furthermore, the abundance of the heavy nuclei, $Z \geq 10$, has been found to be as great as that of the medium nuclei^{4,5)}.

Such relative abundances of primary cosmic ray nuclei have important bearing on the origin of cosmic rays, if they are compared with the relative abundances of cosmic elements. The cosmic abundance of the light nuclei is more than six order smaller than that of the medium nuclei and the latter is about one order larger than the abundance of the heavy nuclei, based on the relative abundances in stellar and interstellar matter. The differences in the relative abundances between cosmic rays and galactic matter are considered to be due both to the fragmentation of cosmic ray nuclei by the collisions with interstellar matter and to the peculiarity of cosmic ray sources.

The abundance of the primary light nuclei given above can be accounted for, if the thickness of interstellar matter traversed by cosmic rays is about 1 g cm^{-2} , provided that

*) The essential content of this paper was read at the International Conference on Cosmic Rays at Guanajuato, Mexico (1955). A similar idea was expressed by Peters at the same Conference.

no light nuclei are injected at sources. This thickness is so small, on one hand, that the relative abundances at the top of the earth atmosphere and at sources are not essentially different, except for the light nuclei that are produced with large fragmentation probabilities. The thickness is so large, on the other hand, that the heavy nuclei of energies below about 100 MeV can hardly be accelerated, because of their large ionization loss. Therefore, the thickness of 1 g cm^{-2} is considered as such that traversed by fast nuclei, most of which will have relativistic energies. These nuclei may have traversed further thickness of matter while they have been of low energies. At energies below 100 MeV the spallation of nuclei produces almost exclusively nucleons and α -particles, but an appreciable fraction of the medium nuclei may be transformed into the light nuclei which are left after the evaporation of nucleons and α -particles. However, the relative abundance of the light nuclei will remain small, as will be discussed in § 2 and § 3.

The nuclear transformations which cosmic ray particles suffer in the course of entire acceleration have shifted the abundance distribution of nuclei to considerable extent. The heavy nuclei at the very beginning must, therefore, have been most abundant⁶. Such a high abundance of heavy nuclei can be expected only at the core of heavy stars, where the thermo-nuclear reactions are believed to build up heavy elements⁷. The helium capturing process is estimated to start at about $10^8 \text{ }^\circ\text{K}$, and to terminate by the explosion which seems to be observed as a supernova. The temperature at the explosion may be as high as $10^9 \text{ }^\circ\text{K}$ and even iron atoms can be stripped. This is in accordance with the observation that all primary nuclei are completely stripped.

A possibility of the supernova origin of cosmic rays is thus suggested by the evidence on the primary cosmic radiation. Other supports may be obtained from the solar production of cosmic rays and the close relation between cosmic rays and radio wave emission. The solar activity that gives rise to the production of cosmic rays is regarded as the disturbance of very small scale, provided that an essential part of cosmic rays are produced by stellar activities. The explosion of a supernova is a disturbance of enormous scale and has been presumed as a possible source of cosmic rays.⁹ The scale of disturbance may be correlated with the intensity of non-thermal radio emission, its energy out-put being estimated to be of the same order of magnitude as the energy out-put for cosmic rays.¹⁰ Ginsburg and Sklovskij¹¹ have developed the idea of the supernova origin based on the interpretation of the radio emission in terms of the synchrotron radiation of electrons in the magnetic field produced in the vicinity of supernovae.¹² The strong radio emission from Crab nebula, a supernova remnant, and a rather regular magnetic field there deduced from the polarization of light¹³ strongly support this view.

Since electrons are accelerated up to relativistic energies, nuclear particles should also be accelerated at supernovae. If the number of protons accelerated, a part of which form nuclei, is equal to that of electrons, the supernova origin is found to account for a fraction of percent of cosmic rays now observed, provided that supernovae explode in every three hundred years and the cosmic rays thus produced fill up our galaxy during their lifetime. We do not think the discrepancy of factor thousand as serious, but suspect the existence of unobservable objects that behave more or less like supernovae. This may not be too

speculative, because similar discrepancies are found in the intensity of non-thermal radio waves and the amount of heavy nuclei, if they are due mainly to supernovae. As such objects, some of which are observable, magnetic variable stars are suggested to be sources of cosmic rays⁽¹⁴⁾⁽¹⁶⁾, though their behaviour is a little different.

Another evidence for the local origin will be the generation of such elements that are hardly built up by thermo-nuclear reactions. The collisions of cosmic ray particles with interstellar nuclei can produce an appreciable amount of the light nuclei, Li, Be and B, but their amount is not large enough to account for their observable abundance.⁽¹⁵⁾ The generation of such elements may be more efficient at local sources where the intensity of energetic particles is stronger. Heavy elements may be generated by the capture of the neutrons that are produced by the collisions of energetic particles injected at the surface of some active stars.⁽¹⁶⁾ If these generation processes take place in the stars that produce cosmic rays and eventually explode as novae or supernovae, scattering those elements in the interstellar space, the origins of cosmic rays and elements are closely connected with each other. Hence the abnormal abundance of some elements in a star could be indication of the cosmic ray production.

Thus the injection of cosmic rays at supernovae and similar objects seems highly probable. The injected particles will undergo the gradual acceleration as suggested by Fermi⁽¹⁷⁾⁽¹⁸⁾ in the interstellar space. The power energy spectrum and the near isotropy holding up to 10^{18} eV should result from the acceleration and the diffusion in the galaxy. If the flat galactic model is adopted, those two facts are found hard to come out. The cosmic ray evidence seems to share the preference to the spherical galactic model with radio astronomical observations.

§ 2. Thickness of interstellar matter traversed by cosmic rays

The composition of cosmic ray nuclei observed at the top of the atmosphere provides us a clue to estimate the thickness of interstellar matter traversed by cosmic rays. For this purpose we classify the cosmic ray nuclei into heavy ($Z > 10$), medium ($10 \geq Z \geq 6$) and light ($5 \geq Z \geq 3$) species and also into helium ($Z=2$) and hydrogen ($Z=1$), according to Noon and Kaplon.⁽⁵⁾ These species are designated by indices h , m , l , α and p , respectively. For example, the notations of the collision mean free paths of these species with hydrogen are given in Table 1, together with their values adopted.

Table 1. Mean free paths of various species of nuclei

species	heavy	medium	light	helium	hydrogen
notation	λ_h	λ_m	λ_l	λ_α	λ_p
collision mean free path (g cm ⁻²)	4.0	6.0	8.7	15.0	36.7

As representatives of respective species, we take those nuclei which have charges 15, 7, 4, 2 and 1 and mass numbers corresponding to these charges. The collision cross section is

assumed to be $\pi r_0^2 A^{2/3}$ with $r_0 = 1.2 \times 10^{-12}$ cm, on account of cosmic ray as well as Cosmotron experiments.^{5,10)}

The collision of these cosmic ray nuclei with interstellar matter, mostly consisting of hydrogen, gives rise to the nuclear transformation that produces lighter nuclei. The yield of secondary nuclei belonging to species j from a primary nucleus belonging to i is expressed by P_{ij} , which is the so-called fragmentation probability. Noon and Kaplon⁵⁾ deduced the values of P_{ij} from their photographic experiments, but, unfortunately, statistics is very poor. Hence we refer also to Cosmotron experiments, in which a proton beam of energy 2.2 GeV is used. The bombardment of Cu, for which the reaction cross section is about 700 mb, is found to give various nuclides with mass numbers down to thirty or smaller²⁰⁾. A striking feature is the yield of ${}^7\text{Be}$, ~ 10 mb. Another experiment²¹⁾ shows the yields of Li and Be as about 10 mb respectively. We may, therefore, conclude the yield of whole light nuclei to be at least 20 mb and not to exceed 100 mb, even if all nuclides belonging to the light species are included. It is noted that the yield of medium nuclei is small, while that of other heavy ones is considerable. The bombardment of Al by 2.2 GeV protons²²⁾, the reaction cross section of about 400 mb, yields ${}^7\text{Be}$ and ${}^{11}\text{C}$ with cross sections of 16 mb and 5.1 mb respectively. Thus the relative yield of light nuclei is greater than in the case of Cu target. The yields of medium and heavy nuclei seem to be nearly equal. The cross section for ${}^{12}\text{C}(p, pn){}^{11}\text{C}$ reaction is observed as about 20 mb in an energy region of interest²³⁾. The yields of ${}^3\text{H}$ from N, O and Fe are measured also to be about 10% of geometrical cross sections.

These are direct sources, as far as we know, and are not sufficient to deduce the fragmentation probability P_{ij} . Nevertheless, we may be allowed to set the upper and lower limits of P_{ij} , as in Table 2. It is hoped that the figures in Table 2 will be made accurate in near future by means of Cosmotron and Bevatron experiments as well as of cosmic ray experiments.

Table 2. Fragmentation probabilities

secondary primary	h	m	l	α	p
h	0.2–0.5	0.3–0.7	0.3–0.8	1.0–2.0	3.0–5.0
m		0.1–0.2	0.4–0.9	1.0–2.0	1.0
l			0	~ 1.5	~ 1

With quantities given in Table 1 and Table 2, the intensities of cosmic ray nuclei belonging to respective species heavier than He can be obtained as⁹⁾

$$N_h(x) = N_h(0) \exp(-x/\lambda'_h), \quad (2 \cdot 1a)$$

$$N_m(x) = N_m(0) \exp(-x/\lambda'_m) + (P_{hm} \lambda_{hm}/\lambda_h) [N_h(0) \exp(-x/\lambda'_h) - N_h(x)] \quad (2 \cdot 1b)$$

$$N_l(x) = N_l(0) \exp(-x/\lambda'_l) + (P_{ml} \lambda_{ml}/\lambda_m) [N_m(0) \exp(-x/\lambda'_m) - N_m(x)]$$

$$+ (\lambda_{hl}/\lambda_h) (P_{hl} + P_{hm} P_{ml} \lambda_{ml}/\lambda_m) [N_h(0) \exp(-x/\lambda'_l) - N_h(x)], \quad (2.1c)$$

where

$$1/\lambda_{ij} = 1/\lambda'_i - 1/\lambda'_j > 0, \quad \lambda'_j < \lambda'_i$$

and

$$\lambda'_i = \lambda_i / (1 - P_{ii}).$$

Here unknown quantities are three initial intensities, $N_i(0)$, and the thickness traversed, x . If $N_i(0)$ is assumed to be zero, which seems plausible on the astrophysical ground, a set of equations (2.1) can be solved.

On account of that λ_{ij} is probably much larger than x , we are able to obtain the approximate relation among N_h , N_m and N_l as

$$N_l(x) \simeq [(P_{hl}/\lambda_h) N_h(x) + (P_{ml}/\lambda_m) N_m(x)] x. \quad (2.2)$$

It must be noticed that other uncertain parameters vanish in this approximation. If we take the upper and the lower limits of P_{ij} , the lower and the upper limits of x are obtained respectively, in reference to the observed values of $N_m(x)/N_h(x) \simeq 1^{(4)}$ and $N_m(x)/N_l(x) \simeq 2^{(3,4)}$. Thus the thickness traversed by cosmic rays is obtained as

$$1 \text{ gcm}^{-2} < x < 3 \text{ gcm}^{-2}. \quad (2.3)$$

On account of the ionization loss which is neglected in the above estimate and of the possible presence of a small amount of light nuclei at sources, we prefer to choose a little smaller value of x . For later use we fix

$$x = 1 \text{ gcm}^{-2} \quad (2.3')$$

The magnitude of x is small enough to permit the approximation leading to (2.2). The smallness of x results chiefly from the large fragmentation probability of giving light nuclei.

Since the value of x has been fixed, we can now evaluate the initial values, $N_h(0)$ and $N_m(0)$. The ratio of them is given by

$$\begin{aligned} N_m(0)/N_h(0) &= [N_m(x)/N_h(x)] \exp(-x/\lambda_{hm}) \\ &\quad - (P_{hm} \lambda_{hm}/\lambda_h) [1 - \exp(-x/\lambda_{hm})] \\ &\simeq [N_m(x)/N_h(x)] (1 - x/\lambda_{hm}) - P_{hm} x/\lambda_h. \end{aligned} \quad (2.4)$$

With numerical values given in Tables 1 and 2 we have

$$0.3 < N_m(0)/N_h(0) < 1. \quad (2.5)$$

We may, therefore, conclude that the initial intensities of heavy and medium nuclei are approximately equal. The possibility of vanishing $N_m(0)$ seems to be ruled out, unless the values of the parameters adopted differ greatly from their true values.

Next we discuss how large parts of relativistic helium and hydrogen nuclei are contributed from heavier nuclei. In order to simplify our calculations, the medium nuclei are taken as representatives of nuclei heavier than He. Hence (2.1) are usable also in this

case by changing suffices as $h \rightarrow m$, $m \rightarrow \alpha$ and $l \rightarrow p$. Taking appropriate averages, we assume $P_{mm} = 0.4 - 0.9$, $P_{m\alpha} = 1.0 - 2.0$, $P_{mp} = 3.0 - 4.0$, $P_{\alpha\alpha} = 0$ and $P_{ap} = 4$. Here the fact is taken into consideration that neutrons are eventually turned into protons. Hence $\lambda'_m = 10 - 60 \text{ g cm}^{-2}$; consequently $\lambda_{m\alpha} = 30 - 20 \text{ g cm}^{-2}$.

On account of $|\lambda_{m\alpha}| \gg x$, the initial value of N_α is obtained as

$$N_\alpha(0) \cong \exp(x/\lambda_\alpha) [N_\alpha(x) - P_{m\alpha} N_m(x) (x/\lambda_\alpha)]. \quad (2.6)$$

As observations tell us $N_\alpha(x)/N_m(x) = 8 - 10$, $N_\alpha(0)$ can never be zero, but is several times $N_m(0)$. Analogously, only a part of hydrogen nuclei can be supplied from other nuclei. Therefore, we are led to conclude the presence of relativistic protons and α -particles after injection and the relative abundances of injected relativistic nuclei are only slightly shifted from those observed in the primary cosmic rays.

In concluding this section, we call one's attention to that most of results obtained here are essentially based on the copious presence of the light nuclei in the primary cosmic rays. If the abundance of the light nuclei were as small as in early works by Bradt and Peters¹⁾, our conclusions would have to be modified great deal.

3. Acceleration of cosmic rays in the galaxy

The thickness of traversed matter deduced in the preceding section gives us the mean lifetime of cosmic rays in the galaxy, provided that the average density of interstellar matter is known.*¹⁾ The latter is estimated as about $10^{-24} \text{ g cm}^{-3}$ within the galactic disk in which a substantial part of galactic matter is contained. Hence the lifetime is about $3 \times 10^{13} \text{ sec}$ (one million years), in essential agreement with that deduced by Morrison et al.²⁴⁾ With the disk model of the galaxy arises a difficulty, however, if the isotropy and the energy spectrum of primary cosmic rays are considered on the basis of Fermi's fundamental idea^{17,18)} on the galactic origin.

In Fermi's idea the transport mean free path is introduced as a unit of length for the acceleration and the diffusion of cosmic rays in the magnetic field. The transport mean free path has a meaning, only if it is longer than the radius of curvature of a cosmic ray particle in the magnetic field. As we observe a particle of energy as high as 10^{18} eV ²³⁾ and the galactic magnetic field may not be stronger than 10^{-5} gauss , the transport mean free path is subject to the lower limit

$$l > 3 \times 10^{20} \text{ cm}. \quad (3.1)$$

According to Morrison et al.²⁴⁾ this results in the anisotropy of one percent or larger, in disagreement with observations.²⁶⁾ Another difficulty is too high a rate of energy gain per transport mean free path. In order that the power energy spectrum continues up to 10^{18} eV , the rate of energy gain should be as high as 10^{-2} , which is highly improbable.

¹⁾ The thickness of 1 g cm^{-2} corresponds to the straight passage of cosmic rays in the universe. However, we seriously take the evidence for the galactic magnetic field that keeps cosmic rays longer or shorter in our galaxy.

These difficulties seem to disappear, if one considers the galactic model that has a diffuse envelope of spherical shape surrounding the optical galaxy.²⁷⁾ In this envelope the matter is of low density, $\sim 10^{-26}$ g cm⁻³, but of high velocity, $\sim 10^7$ cm sec⁻¹, so that the magnetic field of strength as large as 10^{-5} gauss is expected. If cosmic rays are kept in this spherical galaxy, the anisotropy is smaller than 0.1% and the rate of energy gain may not be larger than 10^{-5} *). The mean lifetime of cosmic rays is now of the order of a hundred million years and the problem of the energy regeneration in our galaxy²⁸⁾ may become less serious.

Almost independent of detailed models of the galaxy, we can estimate the injection energy. With the ionization loss β per g cm⁻², the energy loss within a transport mean free path, l , where the density of matter is ρ , is obtained as

$$\Delta E = \beta l \rho. \quad (3.2)$$

The rate of energy gain, on the other hand, is related to the number of scattering by magnetic clouds, N , as

$$\alpha = 1/1.5 N, \quad (3.3)$$

where 1.5 is the exponent of the power energy spectrum. N is nothing but the total path length, L , for cosmic rays to pass through divided by the transport mean free path:

$$N = L/l. \quad (3.4)$$

Hence the condition for acceleration

$$JE < \alpha E$$

requires the lower limit of injection energies, on account of (3.2-4), as

$$E > 1.5 \beta L \rho = 1.5 \beta x, \quad (3.5)$$

where x is the thickness of traversed matter estimated in § 2.

For iron, $Z=26$ and $A=56$, the injection energy is estimated as higher than 20 MeV per nucleon. At about 100 MeV nuclear collisions give rise mainly to protons, neutrons that eventually decay into protons and α -particles. A collision of a heavy nucleus with a hydrogen nucleus can, on the average, produce four nucleons and one half α -particle and the heavy nucleus loses, on the average, seven nucleons. If several such collisions would occur in this energy range, the relative abundances of cosmic ray nuclei might be explained⁷⁾. It is an open question, however, whether or not such collision processes can supply sufficient protons and α -particles, if only heavy nuclei are injected.

§ 4. Acceleration in the vicinity of supernovae

It has occasionally been suggested that there may be a close correlation between the

*) This is a little too large to be accounted for by the first Fermi mechanism¹⁷⁾, because $(v/c)^2 \sim 10^{-7}$ where v is the velocity of magnetic clouds. A mechanism proposed by Thompson²⁸⁾ may explain this rate in which the rate of energy gain is given by $(v/c) (\delta H/H)^2$, where δH is the variation of magnetic field strength.

production of cosmic rays and the radio emission.^{10,11)} In fact, the energy emitted from the sun as disturbed radio waves is about 10^{33} eV sec⁻¹, while the energy emitted as high energy nuclear particles is estimated as the same order of magnitude.³⁰⁾ A similar relation is found to hold for a supernova, whose remnant, Crab nebula for example, is observed to be a strong radio source.

The energy emitted from Crab nebula as radio waves is observed as 3×10^{46} eV sec⁻¹ and the emission is believed to continue as long as 6×10^{11} sec, so that the total energy emitted as radio waves is estimated as 2×10^{58} eV. Assuming that only a fraction of the energy of electrons is fed into radio waves, on account of the escape of electrons, the total energy carried by electrons is estimated as 2×10^{59} eV. If supernovae are the main source of cosmic rays, on the other hand, the number density of cosmic ray particles, 10^{-10} cm⁻³, should be accounted for in terms of the explosions of supernovae which occur every three hundred years. Since the particles produced thereby fill the galaxy of volume about 10^{69} cm³ and their average lifetime is of the order of 3×10^{15} sec, the number of particles produced per explosion should be of the order of 3×10^{53} *). If each particle gets an energy of 10^8 eV, the total energy output for cosmic rays is about 3×10^{61} eV per supernova. This is three order larger than the energy output for radio waves. In this connection we call our attention to the fact that the energy output for whole galactic radio waves is approximately three order higher than that due to supernovae. We may, therefore be allowed to suspect that some things like supernovae, though unobservable optically, take part in the production of cosmic rays as well as radio waves. It must be noticed that the discrepancy of order three also appears, if one regards whole interstellar matter as due to the splash of supernovae.**)

Now we discuss the process of producing cosmic rays in reference to the emission mechanism of radio waves, which has been interpreted by Ginsburg¹²⁾ in terms of the bremsstrahlung of electrons in a magnetic field. Ginsburg was able to express the intensity of the radio waves, J_ν , and the frequency at the maximum intensity, ν_m , in terms of the magnetic field strength, H , the density of energetic electrons, n , and the energy of electrons, E ;

$$J_\nu \simeq 1.7 \times 10^{-23} H n D \Delta\Omega, \text{ erg/cm}^2 \text{ sec cyc/sec}, \quad (4.1)$$

$$\nu_m \simeq 1.4 \times 10^6 H (E/mc^2)^2 \text{ cyc/sec}. \quad (4.2)$$

Taking the diameter of a source, D , and the visual solid angle occupied by a source, $\Delta\Omega$,

*) Disintegrations of heavy nuclei do not matter in this estimate, if the number of particles is understood by the number of nucleons.

**) The energy of radio waves generated in the whole galaxy is of the order of 10^{49} eV sec⁻¹, while that due to supernovae is of the order of 2×10^{46} eV sec⁻¹ (The author is indebted to Professor Hatanaka for telling him these figures). There are roughly 10^{68} nucleons in our galaxy and about one third of them form helium and heavier nuclei. Thus far 2×10^7 explosions of supernovae have occurred, provided that their frequency is assumed as once in every three hundred years. If the average mass of a supernova is ten times solar mass and one tenth of its mass is converted into helium and heavier nuclei, the total amount of them thus generated is estimated to contain roughly 2×10^{64} nucleons. These two indicate a discrepancy of order three. This might be regarded as an evidence against our idea.

as 10^{20} cm and 3×10^{-6} respectively, (4.1) gives us

$$Hn \simeq 10^{-11} \text{ gauss cm}^{-3}, \quad (4.3)$$

in reference to the observed intensity at frequencies around 10^8 cyc/sec. Assuming this frequency as ν_m , (4.2) leads us to

$$H(E/mc^2)^2 \simeq 10^2 \text{ gauss}. \quad (4.4)$$

The energy density of electrons is estimated from the total energy of 2×10^{59} eV and the volume 5×10^{59} cm³ corresponding to $D \simeq 10^{20}$ cm as

$$En \simeq 4 \times 10^{-1} \text{ eV cm}^{-3}. \quad (4.5)$$

(4.3), (4.4) and (4.5) allow us to obtain three quantities E , H and n :

$$E \simeq 10^8 \text{ ev}, H \simeq 2 \times 10^{-3} \text{ gauss}, n \simeq 4 \times 10^{-9} \text{ cm}^{-3}. \quad (4.6)$$

The magnetic field of strength given in (4.6) is assumed as due to the magneto-hydrodynamic wave. Since the velocity of exploding matter is supposed to be about $v \simeq 3 \times 10^8$ cm sec⁻¹, we are able to estimate the density of matter as

$$\rho = H^2 / 4\pi v^2 \simeq 3 \times 10^{-21} \text{ cm}^{-3}. \quad (4.7)$$

This is consistent with the value expected in the envelope of an exploding supernova.

Since the nuclear particles may be accelerated up to relativistic energies, say rigidity up to 1 GV, the distance between magnetic irregularities is required to be

$$l \gtrsim 3 \times 10^6 / H \simeq 6 \times 10^9 \text{ cm}. \quad (4.8)$$

The thickness corresponding to this is $\rho l \gtrsim 10^{-11}$ g cm⁻², far smaller than the collision mean free path of nuclear particles.

The magnitude of l may be estimated on account of the fact that the electrons radiating radio waves have spread out over dimension of $R \simeq 5 \times 10^{19}$ cm within about $t \simeq 100$ years, for example, in the case of Crab nebula. Hence

$$l \simeq (3/2) R^2 / ct \simeq 4 \times 10^{18} \text{ cm}. \quad (4.9)$$

If the latter figure is adopted, the number of collisions with such magnetic irregularities is given by

$$N \simeq ct / l \simeq 2 \times 10^2. \quad (4.10)$$

Consequently the rate of energy gain, α , is of the order of 10^{-2} , which seems too high to realize the gradual acceleration.

It seems to us more natural to assume some instantaneous injection like that may occur for the cosmic ray production associated with intensive solar eruptions. We suppose that an instantaneous mechanism is operative to accelerate nuclear particles together with electrons from thermal energies, say 0.1 MeV, to cosmic ray energies, above 100 MeV per nucleon. This may be followed by the acceleration due to the varying magnetic field in the envelope of supernovae.

§ 5. Generation of light elements at local sources

In a previous note¹⁵⁾ the present author has suggested that the light elements, Li, Be and B, are possibly generated at local sources of cosmic rays due to the fragmentation of heavier nuclei caused by energetic particles which will eventually turn into cosmic rays. Fowler et al.¹⁶⁾ have pointed out that the energetic particles may be an important cause of building up such heavy elements that are observed as abnormally abundant at the surface of some peculiar stars. One can thus see that the cosmic radiation takes an important part in the origin of elements.

In estimating the abundance of the light elements, we refer to the hypothesis that heavier elements, which we call metal, are built up in the core of massive stars which eventually explode by scattering the heavy elements into the interstellar space⁷⁾. If all heavy elements are to be accounted for in this way, 10^{10} explosions like supernovae, as discussed in the preceding section, should give the present mass abundance that is about one hundredth of the total mass of galactic matter. On account of that the mass of a supernova is about ten times of that of an average star, i.e., 10^{31} g, there are 10^{11} stars in the galaxy, the mass abundance of metal in a supernova is about one tenth of its mass.*¹⁾ This corresponds to 2×10^{56} metallic nuclei.

On the other hand, there are about 10^{51} particles to be accelerated in a supernova. Only a fraction of them, say, one tenth, are allowed to suffer nuclear collisions, because otherwise the acceleration would be impossible. Therefore, about one hundredth of such particles, namely, 10^{49} particles may collide with metallic nuclei. In each collision the probability of emitting a light nucleus is approximately one half. Thus we obtain that the abundance of the light elements amounts 10^{-7} times that of heavier elements, in rough agreement with astronomical observations.

The estimate performed above is very rough and merely shows that our idea is not unpaluable. The definite conclusion must be reserved for detailed works along this line. It is also required that the generation of the elements heavier than iron can be accounted for in a similar way and in reference to the idea proposed by Fowler et al.¹⁶⁾

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Pion Theory of Nuclear Forces

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Recently a general and formal method to construct nuclear potentials has been proposed by Fukuda, Sawada and Taketani. Applying the method to the pion theory and adopting fixed and extended sources, we have evaluated nuclear potentials. The result seems to be in favour of that of Taketani, Machida and Onuma rather than that of Erueckner and Watson. As a cut-off function we assumed a Gaussian error function with $k_{max}=6\mu c$. In the case of the effective coupling constant $g^2/4\pi=0.08$ it is found that the effect of the extended source becomes very strong in the region for $r<0.7 \text{ } \hbar/\mu c$. If we use these potentials for $r>0.7 \text{ } \hbar/\mu c$ and phenomenological potentials inside this range, satisfactory results may be obtained to explain the properties of the two nucleon system in the low energy region.

§ 1. Introduction

Since Taketani, Nakamura and Sasaki proposed a method how to attack the meson theory by comparing it with the results of experiments in 1951¹⁾, a series of works on nuclear forces have been done extensively in Japan.²⁾⁻⁵⁾ As Taketani et al. predicted early in the paper, the applicability of the pion theory to the nuclear force problem is established in the present paper with remarks on the region of validity.

However, it may be interesting to bring us to mind the historical background of this problem. After Yukawa predicted the existence of the meson to explain the nuclear force in 1935, so many difficulties appeared in the meson theory as well as in the property of the meson itself. Most meson physicists had given up a hope to explain the nuclear force, and tried to analyse the nuclear force only in a phenomenological way. The main reasons why they inclined to think so seemed to be due to the discrepancies between the meson potential and experimental data on the nucleon-nucleon scattering at that time. Furthermore, the mathematical difficulties, e.g., $1/r^3$ singularity, were considered as very serious hindrances.

In such a situation Taketani et al. analysed the problem of nuclear forces with much care and arrived at the convince that a concept of the potential produced by pions should be fairly useful and valid if a distance between two nucleons was not so small. When two nucleons approach closely to one another, the effects of the higher order perturbations, non-static corrections, relativistic effects, and recoil effects may not be negligible. They are by no means mathematical difficulties, but physical ones. In fact, the existence of many heavy mesons should be taken into account in such a region.

In order to justify the convince it is very necessary to perform the calculation of the meson-theoretical potential first of all. With respect to the complicated effects mentioned above Taketani et al. pointed out that they appeared only when two nucleons approached very nearly and proposed to replace them by a phenomenological potential.*

Concerning the derivation of the potential from the meson theory several authors have attempted to apply the Tamm-Dancoff approximation method besides the usual perturbation technique.⁶⁾⁻⁸⁾ One of them is Brueckner and Watson's treatment. In the treatment the Fock space is used, and amplitudes of two nucleon plus more than zero pion substates are eliminated algebraically from the coupled integral equations. Unfortunately they did not take into account the normalization of the Fock subspace amplitudes. Therefore, the Schroedinger-like equation derived from such procedures can not be regarded as a genuine Schroedinger equation for the nonrelativistic two nucleon system. In other words the no-pion subspace was not decoupled completely from other subspaces which include one or more pion.

Overcoming the difficulty, Fukuda, Sawada and Taketani have proposed a more general and formal method to construct potentials with the field theory (henceforth cited as FST).⁹⁾ In the new method the normalization of the Fock-subspace amplitudes is considered properly. The expression obtained for the potential involves two parts. One of them is a normal part and the other is a probability part. The latter means a probability of finding the two nucleons bare.

Although usual divergence difficulties appear during the evaluation of the expression, the divergent terms must be treated so that the probability part may have a physical meaning. At the present stage we have to adopt some suitable cut-off procedure. It may be plausible to assume the same cut-off as that applied to other processes involving a single

*) After Taketani, Machida and Onuma's paper had been published²⁾, J. Leite Lopes and R. P. Feynman presented a paper entitled "On the Pseudoscalar Meson Theory of the Deuteron" at "Symposium on New Research Techniques in Physics", July 15-29, 1952, Rio de Janeiro, Brasil. As Lopes and Feynman cited correctly the previous works by Taketani, Nakamura and Sasaki and also by Taketani, Machida and Onuma in their paper, their idea and method are almost the same as ours. Unfortunately, since they could not get the "good argument establishing its validity", they concluded the different opinion from ours: "it is unlikely that any calculation by a perturbation expansion has value for nucleon- π -meson problems." We are grateful to Dr. Lopes for sending us a reprint of the paper which arrived just after we had written the present paper.

On the other hand, after Taketani, Machida and Onuma had calculated the pion potential to the fourth-order term,²⁾ Lévy did the same kind of work.⁶⁾ However, since Lévy did not consider that the region of the applicability of the pion potential should be divided into two parts as Taketani, Nakamura and Sasaki had pointed, he could not reach the important results obtained by Japanese workers, which is described as follows:

Outside the region $x \geq 1.5 \times$ (pion Compton wave length) we obtain the same potential as the second order potential with the renormalized coupling constant independently on any approximation^(a). Furthermore, the deuteron problems and low energy phenomena can be explained completely by this outside part of the potential. The theory is in excellent agreement with experiments, while the usual phenomenological potentials contradict seriously to, for example, the experiment on the P -wave proton-proton scattering in the low energy region⁷⁾. Details will be discussed in the Supplement No. 3 of Progress of Theoretical Physics which will soon appear.

nucleon, e.g., P -wave pion-nucleon scattering, photo-pion effect, and anomalous magnetic moments of proton and neutron, because their high frequency components of virtual pion field have to be cut off. In fact the same cut-off procedure is found to give reasonable qualitative agreements with experiments.¹⁰⁾⁻¹⁴⁾ Furthermore, when one wants to describe the two nucleon system by the nonrelativistic Schroedinger picture, the following condition should be kept in mind: The coupling of the nucleon with high frequency components of virtual pion field must be weak and the energy of the system must be small in the nonrelativistic Schroedinger picture.¹⁵⁾

In § 2 the construction of the potential is presented, in § 3 the functional form to describe the nucleon as an extended source is discussed, in § 4 the explicit forms of the potentials are given in our approximation, in § 5 our approximation is examined, and finally in § 6 conclusions and remarks are given.

§ 2. Formulation

A. Approximated nonrelativistic coupling

It has been shown from the experiments of the pion-nucleon scattering and the prediction from pion theory in the low energy region that the P -wave part is rather predominant than the S -wave part in the interaction of pion with nucleon. Furthermore, in the nuclear force problem, it has been already pointed out that the part of the nuclear potential which resulted from the S -wave part of virtual pions is rather weak in the outside region.⁷⁾

The coupling between nucleons and pions is approximated by the following interaction Hamiltonian:

$$H = g/\mu \sum_{i=1}^2 \sigma^{(i)} \cdot F \int dV \rho(\mathbf{r} - \mathbf{r}_i) \boldsymbol{\tau}^{(i)} \cdot \boldsymbol{\phi}(\mathbf{r}), \quad (1)$$

where $\boldsymbol{\phi}(\mathbf{r})$ is the pion field operator, g is the unrenormalized coupling constant and μ is a pion mass, $\sigma^{(i)}$, $\boldsymbol{\tau}^{(i)}$, and \mathbf{r}_i are the spin operator, the isospin operator and the coordinate of the i -th nucleon respectively. We shall assume the nucleons at rest and characterize a nucleon by a spherically symmetrical source function $\rho(\mathbf{r}) = \rho(|\mathbf{r}|)$, which is normalized as follows:

$$\int \rho(\mathbf{r}) dV = 1. \quad (2)$$

This interaction can be interpreted as a coupling between the P -wave pions and fixed nucleons only.

To obtain the approximated form of the interaction Hamiltonian (1) from the charge independent pseudoscalar meson theory, we need to assume the nucleon fixed and the interaction linear with respect to the pion field. Such an approximation means to restrict the number of pions emitted or absorbed in an elementary process to only one.

B. Formal derivation

Recently the most reasonable nonrelativistic nuclear potential has been given by FST.⁽⁹⁾ It is expressed as follows :

$$V = P_0 (J^+ J)^{-1/2} P_0 (J^+ (H_0 + H) J) P_0 (J^+ J)^{-1/2} P_0 - P_0 H_0 P_0, \quad (3)$$

where P_0 is a projection operator to the no pion subspace, H_0 is the free Hamiltonian of the nucleon and the pion field, H is the interaction Hamiltonian between them, and J_0 is the same as in FST's paper (FST, Eq. (2.9)). If we remember that the interaction (1) is linear with respect to the pion variable, we can expand (3) in the static limit as follows :

$$V = [(H^{(-)} (-H_0)^{-1} H^{(+)}) + (H^{(-)} (-H_0)^{-1} H^{(-)} (-H_0)^{-1} H^{(+)} (-H_0)^{-1} H^{(+)} + \dots] \times \\ \times [1 + (H^{(-)} (-H_0)^{-2} H^{(+)}) + \dots]^{-1}, \quad (4)$$

where H is divided into two parts, $H^{(+)}$ and $H^{(-)}$, referring to the emission and absorption of a pion respectively.

In order to make evaluation of the expression (4), we have to make some approximation. We shall expand both the denominator, which means the inverse of the probability of finding the nucleons at bare states in the static limit, and the numerator of the right hand side of (4), taking up to the second order terms in the denominator and up to the fourth-order terms in the numerator. The reason for this procedure may come from the following consideration. In the case of the hypothetical neutral scalar meson theory, if we take the probability part up to the $2n$ -th order and the normal part up to $2(n+1)$ -th order terms, we obtain always a well-known result :⁽⁹⁾

$$V(N.S.) = -\frac{g^2}{4\pi} \frac{e^{-\mu r}}{r} + 2\mathcal{A}, \quad (5)$$

where \mathcal{A} means the self-energy of a single nucleon. It seems worth-while to note that this prescription corresponds to the well-known fact that, in order to calculate the $2n$ -th order perturbation effect, we need knowledge of the $2(n-1)$ -th order perturbed wave function.

Concerning the self-interaction terms, they have been always discarded in the usual treatments. However, it is quite obvious that we should not discard the self-interaction terms in our case. Because the second factor in the right hand side of (4) means the probability as mentioned above, the following condition should always hold :

$$0 < [1 + (H^{(-)} (-H_0)^{-2} H^{(+)}) + \dots]^{-1} < 1. \quad (6)$$

Therefore, we can not take the conventional procedure where the self-interaction terms are omitted. On the contrary we are forced to introduce some suitable cut-off hypothesis.

There are other two kinds of approximation which we have made. One is related to the renormalization process of the coupling constant and the other is concerned with the constant shift of the potential.

Although any divergence does not occur in our case, the renormalization procedure is nevertheless very sensitive. If we follow Chew's method,⁽¹⁶⁾⁽¹⁷⁾ the total contribution of the self-interaction up to the second-order to the coupling constant is expressed by

$$g = g_r Z_1 Z_2^{-1} \\ \cong g_r (1 - (1/6) \Delta_2') (1 - (3/2) \Delta_2')^{-1}, \quad (7)$$

where

$$\Delta_2' = \left(\frac{g_r}{\mu} \right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) k^2}{\omega^2} dk, \quad (8)$$

$\omega = (k^2 + \mu^2)^{1/2}$, g_r is the renormalized coupling constant and $C(k)$ is the cut-off function defined by Eq. (10) in § 3. After we substitute the unrenormalized coupling constant g into the expression (4) and expand it in the power series of the renormalized coupling constant, we take the denominator up to the second-order terms and the numerator up to the fourth-order terms.

As a consequence we obtain the static potential. However, the static potential should be defined by

$$V(r) - V(\infty), \quad (9)$$

where $V(\infty)$ is a constant value of $V(r)$ for $r = \infty$. Therefore, to obtain the static potential, we should subtract $V(\infty)$ from the expression (4). Unfortunately, with the different choice of n in the expression (4), we have different results for $r \gg \hbar/\mu c$. Since we started with the approximation taking the finite orders, it is impossible to remove entirely $V(\infty)$ by such a subtraction procedure. However it is possible to find in the numerator the diagrams which contribute to $V(\infty)$ and to remove them before we start the calculation. Such a treatment may be allowed, because, if we apply it to the case of the hypothetical neutral meson theory, it gives the well-known result, which has no term of $V(\infty)$.

§ 3. Nucleon source function

Since we cannot derive the form of the nucleon source function from the fundamental field theory at present, it may be determined entirely phenomenologically. Therefore the form of the source function has a rather ambiguous character. The only way to avoid this difficulty is to pick up the problems where the precise form of the source function may not be so important. Unfortunately, the nuclear force problem is not the case. It will be shown that the behaviour of the source function gives appreciable effects to the nuclear potential, even if a distance between two nucleons is larger than the nucleon Compton wave length.

After some calculation the nuclear potential can be expressed in terms of integrals over virtual pion momenta. These integrals involve the factor $C(k)$ which is the Fourier transform of the source function:

$$C(k) = \int dV e^{ik \cdot r} \rho(\mathbf{r}). \quad (10)$$

From Eq. (2) it is obvious that $C(0) = 1$. At present we can require $C(k)$ only to be

roughly equal to unity up to $k \simeq k_{max}$ and then to fall rapidly to zero.

Let us recall analyses of the low energy phenomena involving a single nucleon. One used to approximate $C(k)$ by a square cut-off function for the sake of simplicity :

$$C(k) = 1 \text{ for } k < k_{max}$$

$$C(k) = 0 \text{ for } k > k_{max}$$

This crude approximation is useful in the cases where the final results to be compared with experiments are roughly approximated in terms of the following integrals over virtual pion momenta :

$$I_1 = \int_0^\infty C^2(k) dk, \quad (11)$$

$$I_2 = 2 \int_0^\infty C^2(k) k dk. \quad (12)$$

In such cases, for example, as in the P -wave pion-nucleon scattering, the precise functional form of $C(k)$ is not so important and the results are characterized mainly by the cut-off momentum k_{max} .¹⁸⁾

However, the situation is different in the nuclear force. In order to see the functional dependence of the potential on $C(k)$ clearly, we shall now examine the central part of the second-order potential,

$$V_c^{(2)} = -\frac{g^2}{4\pi} \frac{1}{\mu^2} \frac{2}{\pi} \frac{\tau^{(1)} \cdot \tau^{(2)}}{3} \sigma^{(1)} \cdot \sigma^{(2)} \int_0^\infty \frac{C^2(k)}{\omega^2} \frac{\sin kr}{r} k^3 dk. \quad (13)$$

If we substitute the square cut-off function into (13), it gives rise to an oscillation of the potential for large values of r and its behaviour depends appreciably on the cut-off momentum k_{max} . On the physical standpoint, however, it seems reasonable to expect that the effect of the extended sources is probably large at small distances between two nucleons rather than at large ones.

Secondly, let us assume a smooth function

$$C(k) = k^2 (k^2 + k_{max}^2)^{-1}. \quad (14)$$

Then we can evaluate analytically the integral (13) and obtain the following result :

$$V_c^{(2)} = \frac{\tau^{(1)} \cdot \tau^{(2)}}{3} \sigma^{(1)} \cdot \sigma^{(2)} \left[\frac{g'^2}{4\pi} \frac{e^{-\mu r}}{r} - \frac{g^2}{4\pi} F(r) e^{-k_{max} r} \right] \quad (15)$$

where

$$g' = g [1 - (\mu/k_{max})^2]^{-1},$$

and

$$F(r) = \mu/2 \cdot [(k_{max}/\mu)^3 + (k_{max}/\mu)] + (1/r) [1 - (\mu/k_{max})^2]^{-2}.$$

As concerns the cut-off momentum k_{max} it may be plausible to assume the same value as the one used in the theory of the pion-nucleon scattering, i.e., $k_{max} = 6\mu c$. Then, in the

outside region of the pion Compton wave length the expression (15) is nearly equal to the result for the case of the point sources, that is, the Yukawa potential. Therefore, as expected above, a smooth function $C(k)$ gives rather satisfactory results for the outside region of the potential.

On the other hand it has been pointed out that the experimental data on the P -wave pion-nucleon scattering can be explained fairly well if the following relation is satisfied :

$$I_1 \simeq \sqrt{I_2}. \quad (16)$$

The relation is exactly satisfied when we use the square shape function for $C(k)$. Therefore, unless we assume a cut-off function close to the square shape function, the relation (16) may be destroyed.

From the above consideration we shall now assume a Gaussian error-function for $C(k)$:

$$C(k) = \exp(-k^2/2k_{max}^2). \quad (17)$$

§ 4. Numerical results

Let us now evaluate the potential V from the expression (4). In order to illustrate the processes which contribute to the potential, some of the diagrams are given in Fig. 1. Applying the prescription of the renormalized coupling constant to the evaluation we obtain the explicit form of the potential in our approximation as follows :

$$\begin{aligned} V = & - (g/\mu)^2 \cdot 1/(2\pi)^3 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int \frac{C^2(k) d\mathbf{k}}{\omega^2} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} - \\ & \left\{ (g/\mu)^4 \frac{1}{(2\pi)^6} \int \frac{C^2(k_1) C^2(k_2) d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1^2 \omega_2 (\omega_1 + \omega_2)} \left[2 + \frac{1}{\omega^2} \right] e^{i(k_1 + k_2) \cdot \mathbf{r}} \times \right. \\ & \times [2(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 + 3\boldsymbol{\sigma}^{(1)} \cdot [\mathbf{k}_1 \times \mathbf{k}_2] \boldsymbol{\sigma}^{(2)} \cdot [\mathbf{k}_1 \times \mathbf{k}_2]] \Big\} \times \\ & \times \left\{ 1 + 3J'_2 + (g/\mu)^2 \frac{1}{(2\pi)^3} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int \frac{C^2(k) d\mathbf{k}}{\omega^3} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} \right\}^{-1}, \quad (18) \end{aligned}$$

where g is the renormalized coupling constant. The suffix r of g_r is dropped henceforth as we shall use only the renormalized one. The integration of the angular parts can be done in a straightforward way in virtue of the spherical symmetry of the source function. With respect to the radial parts we have to perform the numerical calculation of the following expression :

$$\begin{aligned} V = & \frac{1}{3} \frac{g^2}{4\pi} \frac{1}{\mu^2} \frac{2}{\pi} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int_0^\infty \frac{C^2(k_1) k_1 dk_1}{\omega^2} [(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) (2A_1 + B_1) - \\ & - S_{12} (A_1 - B_1)] - \left\{ (g^2/4\pi)^2 \frac{1}{\mu^4} (2/\pi)^2 \int_0^\infty \int_0^\infty \frac{C^2(k_1) C^2(k_2) k_1 k_2 dk_1 dk_2}{\omega_1^2 \omega_2 (\omega_1 + \omega_2)} \times \right. \\ & \times \left[2 + \frac{1}{\omega^2} \right] [2 \cdot (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (B_1 B_2 + 2A_1 A_2) + \end{aligned}$$

$$\begin{aligned}
 & + 2 (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) (A_1 B_2 + B_1 A_2 + A_1 A_2) - S_{12} (A_1 B_2 + B_1 A_2 - 2 A_1 A_2) \Big\} \times \\
 & \times \left\{ 1 + 3 A_2' - \frac{1}{3} \frac{g^2}{4\pi} \frac{1}{\mu^2} \frac{2}{\pi} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int_0^\infty \frac{C^2(k_1) k_1 dk_1}{\omega^3} \times \right. \\
 & \times \left. [(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) (2 A_1 + B_1) - S_{12} (A_1 - B_1)] \right\}^{-1}
 \end{aligned}$$

where

(19)

$$A_i = 1/r^3 \cdot [k_i r \cos k_i r - \sin k_i r],$$

$$B_i = 1/r^3 \cdot [2 \sin k_i r - 2 k_i r \cos k_i r - k_i^2 r^2 \sin k_i r],$$

and

$$S_{12} = 3 (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}) r^{-2} - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}).$$

The details of the approximation which we have made during the numerical calculation are given in the appendices. We have performed the numerical calculation assuming the Gaussian cut-off momentum $k_{mac} = 6\mu c$ and the effective coupling constant $g^2/4\pi = 0.08$. The results for even states and odd states are plotted in Fig. 2 and Fig. 3 respectively.

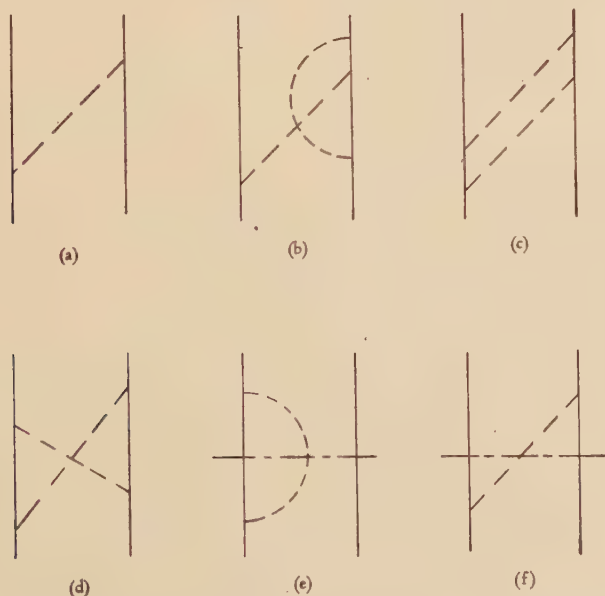


Fig. 1. Sample diagrams which contribute to the normal part and probability part, (a, b, c, d) and (e, f) are given respectively.

For even states (Fig. 2) the potentials show the following shapes. The triplet central potential is very weak and attractive outside the pion Compton wave length $\hbar/\mu c$ and

becomes repulsive inside the region. On the other hand the triplet tensor potential is strong and attractive even inside the region. The singlet potential is attractive in the outside region and changes its sign at $r \cong 0.6\hbar/\mu c$.

For odd states (Fig. 3) the potentials show the following shapes. The triplet central potential is very weak and repulsive for $r \gtrsim 0.8\hbar/\mu c$. The triplet tensor potential is repulsive, contrary to that in the even states. Although it is not shown in the figure, the triplet tensor potential changes its sign near the origin. The singlet potential is repulsive for $r \gtrsim 0.7\hbar/\mu c$ and attractive inside the range.

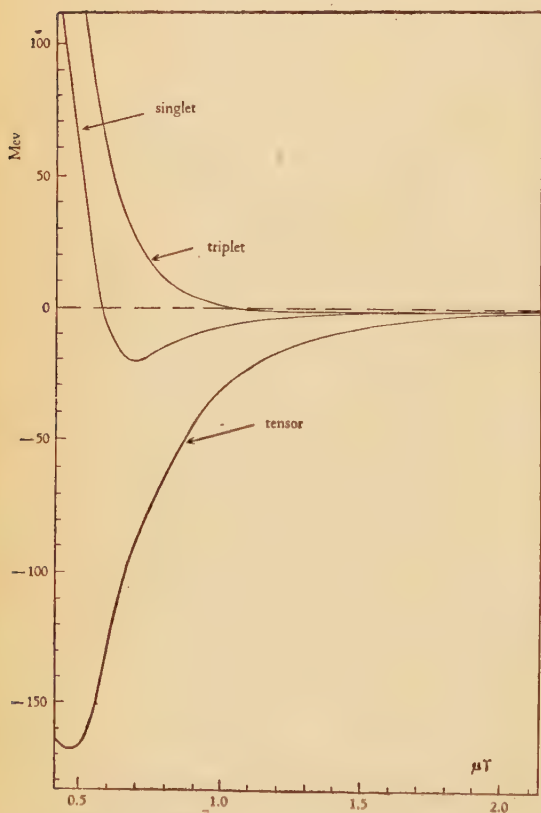


Fig. 2. Potentials in even state. The cut-off momentum $k_{max} = 6\mu c$, where μ is the pion rest mass. The ordinate is always measured in Mev except in Fig. 7.

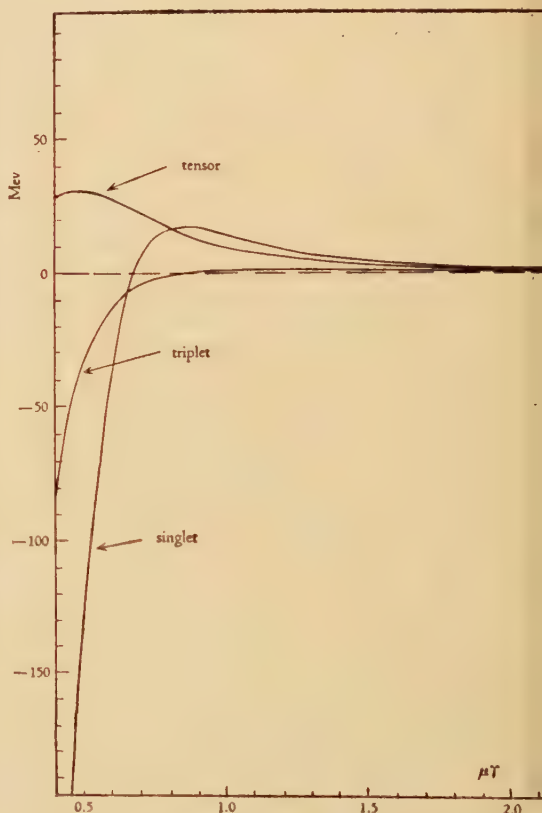


Fig. 3. Potentials in odd state. The cut-off momentum $k_{max} = 6\mu c$.

Concerning the so-called "contact interaction" we would like to make a few remarks. If we assume a point source for each nucleon, the contact interaction does not give any contribution to the potential except at the origin. For instance, the second order central force contains the following contact interaction:

$$-1/3 \cdot (\sigma^{(1)} \cdot \sigma^{(2)}) (\tau^{(1)} \cdot \tau^{(2)}) (g/\mu)^2 \delta(\mathbf{r}). \quad (20)$$

However, in our case it is written due to the cut-off procedure as follows :

$$-1/3 \cdot (\sigma^{(1)} \cdot \sigma^{(2)}) (\tau^{(1)} \cdot \tau^{(2)}) (g/\mu)^2 (16\pi^{3/2})^{-1} k_{max}^3 e^{-k_{max}^2 r^2/4}. \quad (21)$$

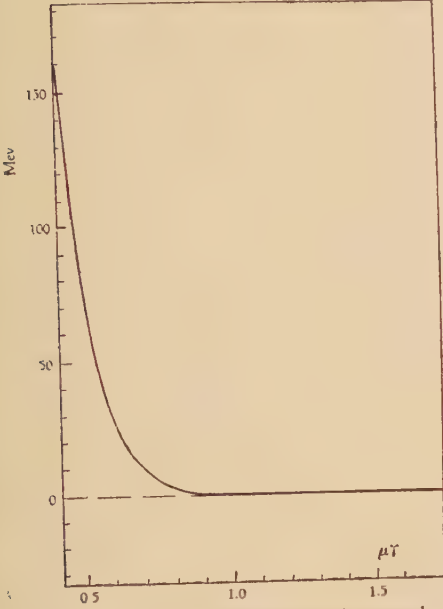


Fig. 4. A contact interaction in the second-order central force of the even state. The cut-off momentum $k_{max} = 6\mu c$.

It is seen from Fig. 4 that it becomes important in the region $r \lesssim 0.7\hbar/\mu c$ in the case of $k_{max} \simeq 6\mu c$ and $g^2/4\pi = 0.08$. It is also found that the contact interaction of the fourth-order potential is effective only at very small distances.

§ 5. Validity of approximation

The potentials which we have obtained in the present paper seem to be valid for $r \gtrsim 0.7\hbar/\mu c$, because of the following reasons. As it is well known, there are many problems in the short internucleon distances, e.g., the effect of the higher order terms of the expansion, the contribution from *S*-wave pions, nonadiabatic corrections, the effect of isobaric states of nucleons, etc. Adding to these difficulties we should expect that the cut-off procedure may modify seriously the potential in the region. The cut-off function has been chosen so that the potential may not be different at

the large distances from the case without cut-off. However, as we mentioned above, the cut-off procedure has very intricate relationship to our evaluation of the potential. Therefore it is difficult to discuss the effects of the cut-off procedure on the final results. While the role of the cut-off procedure is mainly to make possible to estimate the probability part, it results in the appreciable modification of the contact term.

For the region $r \lesssim 0.7\hbar/\mu c$ the contact term is predominant. On the other hand for the region $r \gtrsim 1.5\hbar/\mu c$ the cut-off procedure has nearly no effect on the potential. However if we assume $k_{max} \simeq 4\mu c$, the effect of the contact term spreads to about $\hbar/\mu c$. Therefore, for the region $r \lesssim 0.7\hbar/\mu c$, we shall follow Taketani, Nakamura and Sasaki's method¹⁾ and adopt suitable phenomenological potentials. They may be square wells or extrapolations of the outside potentials to the inside region plus, if necessary, a narrow repulsive core at a very short distance. Although the potentials for the region $0.7\hbar/\mu c < r < 1.5\hbar/\mu c$ are sensitively affected by the cut-off momentum, they are almost the same in the outside region as the usual second-order potentials without the cut-off procedure, i.e.,

$$\frac{g^2}{4\pi} \frac{(\tau^{(1)} \cdot \tau^{(2)})}{3} \left[(\sigma^{(1)} \cdot \sigma^{(2)}) + S_{12} \left(\frac{3}{\mu^2 r^2} + \frac{3}{\mu r} + 1 \right) \right] \frac{e^{-\mu r}}{r}. \quad (22)$$

For comparison between TMO method and BW method we plot in Figs. 5 and 6 the

potentials for even states which are calculated by respective methods, using the same cut-off procedure.

On the other hand, we have dropped the higher terms than the fourth-order one in the numerator and the higher terms than the second-order one in the denominator of the expression (4) as mentioned above. Therefore we have to examine the validity of this

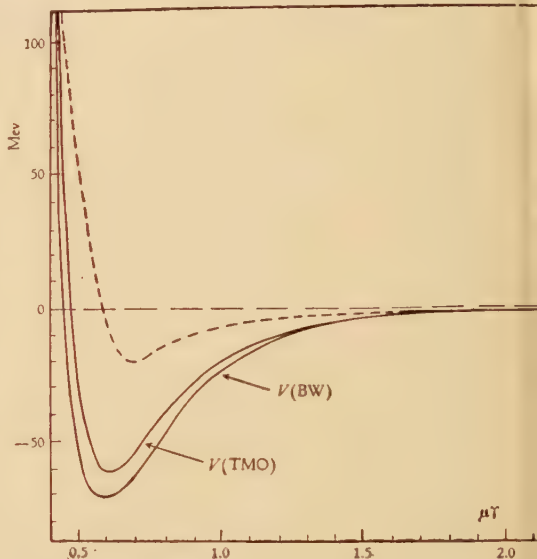
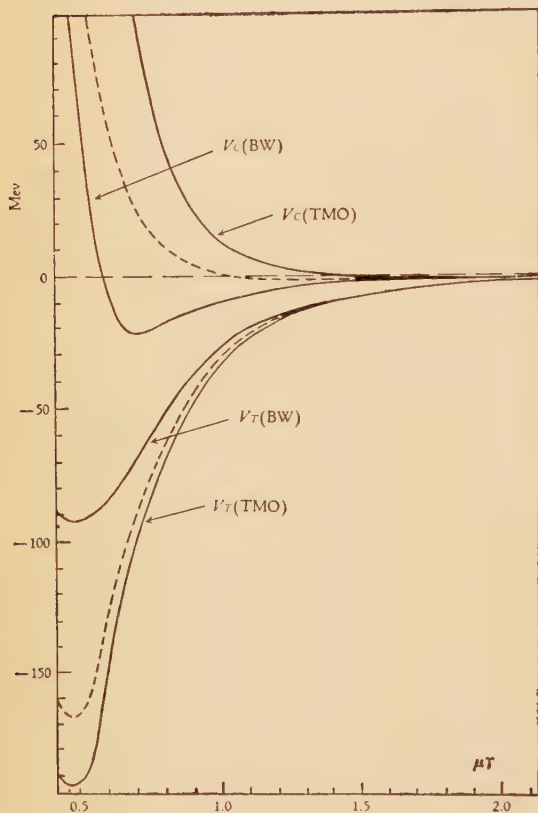


Fig. 5. Comparison between TMO and BW potentials in triplet even state in the case of the same cut-off procedure. The dotted curves are the same ones as in Fig. 2. The cut-off momentum $k_{max} = 6\mu c$.

Fig. 6. Comparison between TMO and BW potentials in singlet even state in the case of the same cut-off procedure. The dotted curve is the same one as in Fig. 2. The cut-off momentum $k_{max} = 6\mu c$.

approximation. At first, let us discuss the expansion of the numerator of (4). Machida and Senba have already investigated the higher order corrections to the second plus fourth-order adiabatic potentials.⁴⁾ Brueckner and Watson discussed also the same problem using the multiple scattering method.⁸⁾ According to their estimates the second plus fourth-order adiabatic potential is not altered seriously by the sixth-order potential and multiple scattering effects for the region $r > 0.6 \sim 0.7\hbar/\mu c$. Therefore it may be plausible to take up to the fourth-order term and drop the higher terms in the numerator of (4).

Secondly we shall examine the validity of the expansion in the denominator. Although there are many difficult problems, we shall estimate numerically the contribution of the fourth-order terms, assuming that the probability part should have a meaningful value at very large internucleon distances (See Appendix B). In Fig. 7 we show a comparison of

the probability part in the triplet even state between two cases. One includes only the second-order term, and the other includes the second-order term plus the fourth-order term. Since the triplet even potential is strongly affected by the probability part, we give only the case.

It is seen from Fig. 7 that the potentials are not so much modified qualitatively for $r \lesssim 0.7\hbar/\mu c$, while inside the range the influences become very strong. As far as the probability part concerns, the power series expansion breaks down completely in the region $r \lesssim 0.7\hbar/\mu c$. Consequently we can conclude that the static potential obtained from FST's theory (4) using our approximations may be meaningful in the outside region ($r > 0.7\hbar/\mu c$).

§ 6. Concluding remarks

As FST pointed out, the so-called nonadiabatic or velocity dependent correction, which gives the difference between the potentials obtained by TMO and ones by BW, is by no means nonadiabatic effect, but is related to the renormalization of the amplitude which can be understood in the static limit as the probability for the absence of pions. One of our conclusions is that the correction has appreciable influences on the behaviour of the potentials inside the pion Compton wave length as shown in § 5.

Concerning the magnitude of the so-called nonadiabatic correction to the second-order potential, Brueckner and Watson estimated the expectation value using the "bound state wave function" of the deuteron. Then they concluded that the correction is less than one percent of the 20 Mev expectation value for the static potential. However, 20 Mev is not the expectation value of the potential, but an appropriate average of the potential inside the well when we represent the property of the nuclear force in the small internucleon distances by the square well. If one would compare the effects inside the pion Compton wave length by estimating the expectation value, one should obviously obtain no appreciable change of the potentials due to the "bound state wave function" of the deuteron. It is worth while to notice that the potential near the origin plays a very essential role for

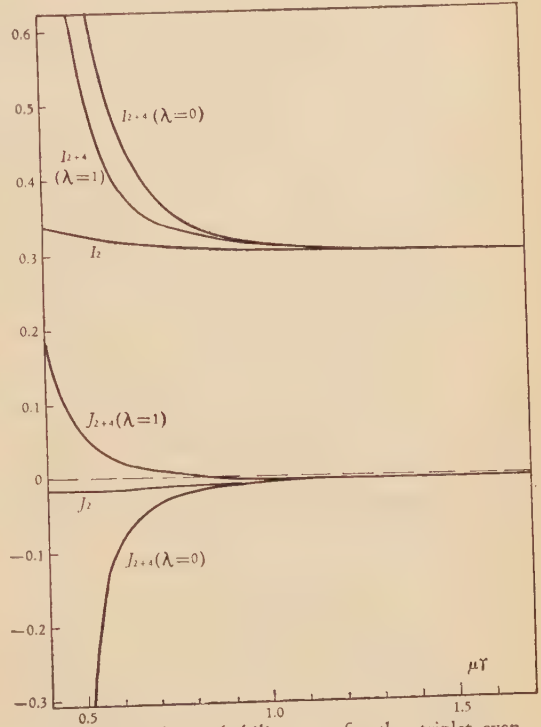


Fig. 7. The probability part for the triplet even state. I_2 and I_{2+4} are the second-order and the second plus fourth-order central force parts respectively. J_2 and J_{2+4} are the second-order and the second plus fourth-order tensor force parts respectively. The parameter λ expresses the effects of the higher order corrections which arise from the self-interactions (See Appendix B). The cut-off momentum $k_{max} = 6\mu c$.

the binding of the deuteron and the detailed form of the wave function is determined mainly by the potential outside the pion Compton wave length.⁵⁾ Therefore the correction should be treated more carefully.

We have also found that it is possible to derive the reasonable potentials from the P -wave pion interaction working with the cut-off procedure and Chew's renormalization method, which gives satisfactory results in the low energy P -wave pion phenomena with a single nucleon.¹⁶⁾¹⁷⁾ However, we have to make a remark that the static potentials obtained in the present paper should not be taken seriously when r is less than $0.7\hbar/\mu c$, because the effect of the contact interactions which arise from the extended sources becomes very appreciable inside the region. Therefore it seems very necessary to attack this region from rather different sides; new treatments or considerations are strongly desirable.

Finally, we would like to mention that our results are in good agreement qualitatively with the previous works of Taketani's group.¹⁾⁻⁵⁾ With respect to the excellent comparison of the pion theory with experiments, especially, the detailed analysis of the low energy phenomena of the two nucleon system by Otsuki and Tamagaki should be quoted. An amazing agreement of the pion theoretical potentials and the P -wave nucleon-nucleon scattering data is clearly shown in their paper.⁵⁾ We find also that the potentials obtained in the present paper have such properties as they suggested to be satisfied by nuclear forces.

One of the authors (K.I.) is indebted to the Yomiuri Yukawa fellowship for the financial aid.

Appendix A

We illustrate here the approximation which we have made for the integration in the fourth-order terms in eq. (19). Since the integrand involves Gaussian function, we should perform a tedious numerical calculation unless we introduce some approximation method. If we make the following replacement in the integrand, the integrations become separable and very easy for the calculation:

$$\frac{1}{\omega_1 + \omega_2} \left[\frac{2}{\omega_1} + \frac{1}{\omega_2} \right] \rightarrow \frac{2}{3} \frac{1}{\omega_1 \omega_2} \left[1 + \frac{\omega_0}{\omega_1 + \omega_0} \right] \left[2 - \frac{\omega_0}{\omega_2 + \omega_0} \right], \quad (\text{A1})$$

where ω_0 is an adjustable parameter. The error due to this approximation does not exceed about several percent. It may be sufficient for our purpose.

Appendix B

Assuming the fixed sources we can write down the fourth-order terms in the probability part as follows:

$$\begin{aligned} (P_0 J^+ J P_0)^{(4)} = & (H^{(-)} (-H_0)^{-2} H^{(-)} (-H_0)^{-1} H^{(+)} (-H_0)^{-1} H^{(+)}) + \\ & + (H^{(-)} (-H_0)^{-1} H^{(-)} (-H_0)^{-2} H^{(+)} (-H_0)^{-1} H^{(+)}) + \\ & + (H^{(-)} (-H_0)^{-1} H^{(-)} (-H_0)^{-1} H^{(+)} (-H_0)^{-2} H^{(+)}) - \end{aligned}$$

$$\begin{aligned}
 & - (H^{(-)} (-H_0)^{-3} H^{(+)}) (H^{(-)} (-H_0)^{-1} H^{(+)}) - \\
 & - (H^{(-)} (-H_0)^{-1} H^{(+)}) (H^{(-)} (-H_0)^{-3} H^{(+)}).
 \end{aligned} \tag{B1}$$

Then we obtain directly

$$\begin{aligned}
 (P_0 J^+ J P_0)^{(4)} = & \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int \frac{C^2(k_1) C^2(k_2) dk_1 dk_2}{\omega_1^4 \omega_2^2} e^{i(k_1 + k_2) \cdot r} \times \\
 & \times \left\{ \left[\frac{1}{2} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})^2 \frac{\omega_1}{\omega_2} - \frac{9\omega_1^2 + 24\omega_1\omega_2 + 12\omega_2^2}{(\omega_1 + \omega_2)^2} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_1) \times \right. \right. \\
 & \times (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_2) + (3 + 2\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \frac{9\omega_1^2 + 24\omega_1\omega_2 + 12\omega_2^2}{(\omega_1 + \omega_2)^2} \Big\} + \\
 & + \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int \frac{C^2(k_1) C^2(k_2) dk_1 dk_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \cdot r} k_2^2 (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_1) \times \\
 & \times \left\{ \frac{10/3}{\omega_1^2 \omega_2^2} - \frac{6}{\omega_1^3 \omega_2} - \frac{6}{\omega_1 \omega_2^3} + \frac{1/3}{\omega_1^2 (\omega_1 + \omega_2)} \left[\frac{2}{\omega_1} + \frac{1}{\omega_2} \right] + \right. \\
 & + \frac{3}{\omega_2^2 (\omega_1 + \omega_2)} \left[\frac{1}{\omega_1} + \frac{2}{\omega_2} \right] \Big\} - \left(\frac{g}{\mu} \right)^4 \frac{1}{(2\pi)^6} \int \frac{C^2(k_1) C^2(k_2) dk_1 dk_2}{(\omega_1 + \omega_2)^2} \times \\
 & \times k_1^2 k_2^2 \left[\frac{3}{\omega_1^2 \omega_2^2} - \frac{1}{\omega_1^2 \omega_2} \right].
 \end{aligned} \tag{B2}$$

Here we make the following approximation adding to the replacement (A1):

$$\frac{1}{(\omega_1 + \omega_2)^2} \left[\frac{4}{\omega_1^2} + \frac{8}{\omega_1 \omega_2} + \frac{3}{\omega_2^2} \right] \rightarrow \frac{15}{4} \frac{1}{\omega_1^2 \omega_2^2}. \tag{B3}$$

If we use these approximations, we can perform the integrations in (B2). Introducing notations for these integrals we rewrite (B2) as follows:

$$\begin{aligned}
 (P_0 J^+ J P_0)^{(4)} \cong & (\text{constant}) + \frac{1}{2} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})^2 L^2 - \frac{45}{4} YN + \frac{15}{4} (3 + 2\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) I + \\
 & + \lambda (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \left[\frac{10}{3} \mathcal{A}''_2 L + \frac{2}{9} \mathcal{A}_3 N' + 2 \mathcal{A}_3'' Y' - 6 \mathcal{A}_2 N - 6 \mathcal{A}''_2 Y \right].
 \end{aligned} \tag{B4}$$

λ represents the effects of the higher order corrections and is expected probably to be small. Notations for integrals are given as follows:

$$Y = \left(\frac{g}{\mu} \right) \frac{1}{(2\pi)^3} \int \frac{C^2(k) dk}{\omega^2} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \tag{B5^1}$$

$$Y' = \left(\frac{g}{\mu} \right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) dk}{\omega^2} \left[2 - \frac{\omega_0}{\omega + \omega_0} \right] (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \tag{B5^2}$$

$$L = \left(\frac{g}{\mu} \right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) dk}{\omega^3} (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \tag{B5^3}$$

$$N = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) dk}{\omega^4} (\sigma^{(1)} \cdot \mathbf{k}) (\sigma^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (\text{B5}^4)$$

$$N' = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) dk}{\omega^4} \left[1 + \frac{\omega_0}{\omega + \omega_0}\right] (\sigma^{(1)} \cdot \mathbf{k}) (\sigma^{(2)} \cdot \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (\text{B5}^5)$$

$$I = \left(\frac{g}{\mu}\right)^4 \frac{1}{(2\pi)^6} \int \frac{C^2(k_1) C^2(k_2) dk_1 dk_2}{\omega_1^4 \omega_2^2} (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 e^{i(k_1 + k_2) \cdot \mathbf{r}}, \quad (\text{B5}^6)$$

$$J_2 = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) k^2 dk}{\omega^2}, \quad (\text{B6}^1)$$

$$J_2'' = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) k^2 dk}{\omega^4}, \quad (\text{B6}^2)$$

$$J_3 = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) k^2 dk}{\omega^2} \left[2 - \frac{\omega_0}{\omega + \omega_0}\right], \quad (\text{B6}^3)$$

$$J_3'' = \left(\frac{g}{\mu}\right)^2 \frac{1}{(2\pi)^3} \int \frac{C^2(k) k^2 dk}{\omega^4} \left[1 + \frac{\omega_0}{\omega + \omega_0}\right]. \quad (\text{B6}^4)$$

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Note added in proof. Using the extended source, Gartenhauss calculated Brueckner-Watson's potential to the two-meson exchange term. However, he assumed that the potential can be used near $r=0$ (Phys. Rev. **100** (1955), 900). As we discussed in the present paper, such a consideration has no physical meaning, even if the result agrees accidentally with experiments. Detailed discussions will be given in the Supplement No. 3 of Progress of Theoretical Physics which will soon appear.

Intermediate Coupling Meson Theory of Nuclear Forces, II

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In the Part I of the same title a method has been proposed to construct the nuclear potential by means of the intermediate coupling theory, where the charged scalar theory has been taken as an example. In this paper, the method is applied to the realistic case, symmetrical pseudo-scalar theory with pseudo-vector coupling, and the static nuclear potential is obtained up to the order of e^{-2x} , where $x = \mu r$.

The results are found to depend on the cut-off procedure which is inevitably introduced to avoid the divergence troubles. A cut-off factor should be chosen so that effects of the so-called contact interaction do not extend onto the exterior region $x \gtrsim 1.0$. We take a cut-off factor of gaussian type as a suitable one. Our potential does not essentially differ from the FST-potential and the BW-potential with cut-off. After the renormalization of the coupling constant, the potential of the order of e^{-x} coincides with the conventional second order perturbation potential in the exterior region, irrespectively of an assumed "structure" of the clothed nucleon. The potential of the order of e^{-2x} , however, depends essentially on the cut-off factor.

§ 1. Introduction

In the interaction of nucleons with the meson field, the inertia of the self field is large and its reactive effects play an important role. It is expected that these circumstances are well exhibited by the intermediate coupling theory. In this theory, meson configurations are classified into bound and unbound ones. The mesons in the bound configuration, the so-called zero-mesons, are strongly bound to a nucleon. The mesons in the unbound configuration are called s-mesons. For the one nucleon problem, the intermediate coupling theory has been applied extensively and we are now well acquainted with this, as far as the one nucleon problem is concerned. However, its generalization to the two nucleon problem is somewhat meandering, since it is not an easy task to solve the eigenvalue problem for the cloud around two nucleons in a straightforward way.

Recently, one of us (H.H.) has proposed a method of construction of the nuclear potential by means of the intermediate coupling theory taking the charged scalar theory as an example¹⁾. The essential point of this method is the transformation (I.3.1), in virtue of which the two nucleon problem turns to be solved exploiting the knowledge about the one nucleon problem. The single clothed nucleon states are taken as a standard, and the potential is obtained as the effects of change induced on the cloud in a course of approach of two nucleons from infinitely separated positions.

In this paper the method is applied to the realistic case where the nucleons interact with the pseudo-scalar meson field through the pseudo-vector coupling. Effects of nucleon recoil are neglected. Moreover the following simplifying assumptions are employed: (i) effects of the excited or isobar states of the single clothed nucleon other than four states: $(I, J) = (1/2, 1/2)_{1st\ excited}, (1/2, 3/2), (3/2, 1/2), (3/2, 3/2)$ can be neglected, and (ii) possibility of a transition of a nucleon from the ground state to any one of the excited states accompanied by emission of the s -meson can be neglected. These two assumptions are called "one level approximation" hereafter. Restricting the number of mesons exchanged between two nucleons we evaluate the potential up to the order of e^{-2x} .

In actual evaluations, the divergence difficulties arise. Therefore, at the present stage, we have to employ a cut-off procedure. A form of the cut-off factor $F(k)$ should be chosen so that the so-called contact interaction terms do not give appreciable effects on the potential in the exterior region $x \gtrsim 1.0$. We use the cut-off factor of gaussian type

$$F(k) = \exp(-k^2/2a^2) \quad (1.1)$$

with an appropriate cut-off momentum a .

In § 2 a formal derivation of the potential is presented. In § 3 renormalization of the coupling constant and the cut-off procedure are discussed. In § 4 the results are summarized. Appendix contains brief reviews on the single clothed nucleon and details of the calculations.

§ 2. Derivation of the potential

(1) The Hamiltonian

The positions of two nucleons are denoted by x_1 and x_2 , respectively. Then the interaction Hamiltonian is

$$H_{int} = \frac{-i}{\sqrt{2\pi^3}} \frac{g}{\mu} \sum_i \tau_i^{(n)} \sigma_i^{(n)} \int dk \frac{F(k)}{\sqrt{2K}} k_l \phi_\alpha(k) e^{ikx_i} + c.c. \quad (2.1)$$

Here, the suffix $i (= 1, 2)$ refers to the nucleons, the suffices $\alpha (= 1, 2, 3)$ and $l (= 1, 2, 3)$ refer to the meson charge and the spatial component, respectively. The dummy suffices α and l are to be automatically summed up. The direction of $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$ is chosen as the third coordinate axis. For other notations, refer to Appendix A. The wave functions of zero-mesons are taken to have the following form:

$$\begin{aligned} \varphi_i^s(k) &= \frac{1}{V_i^s} \frac{i}{\sqrt{2\pi^3}} \frac{g}{\mu} \frac{F(k)}{\sqrt{2K}} \frac{k_l e^{-ikx_1} + e^{-ikx_2}}{\sqrt{2}}, \\ \varphi_i^a(k) &= \frac{1}{V_i^a} \frac{i}{\sqrt{2\pi^3}} \frac{g}{\mu} \frac{F(k)}{\sqrt{2K}} \frac{k_l e^{-ikx_1} - e^{-ikx_2}}{\sqrt{2}}. \end{aligned} \quad (2.2)$$

($l = 1, 2, 3$)

The normalization factors V_i^s and V_i^a are functions of internucleon distance r . For very large r , both are nearly equal to V , the normalization factor in the case of one nucleon

problem [cf. (A·4)]. While, when the two nucleons approach closely, V_l^s and V_l^a tend to $\sqrt{2} V$ and zero, respectively.

We expand the meson field $\phi_\alpha(\mathbf{k})$ into the complete orthonormal set involving $\varphi_i^s(\mathbf{k})$ and $\varphi_i^a(\mathbf{k})$ as its six members :

$$\begin{aligned}\phi_\alpha(\mathbf{k}) &= A_{\alpha l}^s \varphi_l^s(\mathbf{k}) + A_{\alpha l}^a \varphi_l^a(\mathbf{k}) + a_\alpha(\mathbf{k}), \\ \phi_\alpha^*(\mathbf{k}) &= A_{\alpha l}^{s*} \varphi_l^{s*}(\mathbf{k}) + A_{\alpha l}^{a*} \varphi_l^{a*}(\mathbf{k}) + a_\alpha^*(\mathbf{k}).\end{aligned}\quad (2.3)$$

$A_{\alpha l}^{s*}$ and $A_{\alpha l}^{a*}$ or $A_{\alpha l}^s$ and $A_{\alpha l}^a$ are the creation or the annihilation operators of the symmetric and the antisymmetric zero-mesons. The commutation relations are

$$\begin{aligned}[A_{\alpha l}^s, A_{\alpha' l'}^{s*}] &= [A_{\alpha l}^a, A_{\alpha' l'}^{a*}] = \delta_{\alpha\alpha'} \delta_{ll'}, \\ \text{others} &= 0.\end{aligned}\quad (2.4)$$

$a_\alpha^*(\mathbf{k})$ and $a_\alpha(\mathbf{k})$ are the operators for s-mesons.

In order to express all the quantities in the language of each single nucleon, we perform the canonical transformation (I·3·1) :

$$\begin{aligned}A_{\alpha l}^{(1)} &= (A_{\alpha l}^s + A_{\alpha l}^a) / \sqrt{2}, \quad A_{\alpha l}^{(2)} = (A_{\alpha l}^s - A_{\alpha l}^a) / \sqrt{2}, \\ A_{\alpha l}^{(1)*} &= (A_{\alpha l}^{s*} + A_{\alpha l}^{a*}) / \sqrt{2}, \quad A_{\alpha l}^{(2)*} = (A_{\alpha l}^{s*} - A_{\alpha l}^{a*}) / \sqrt{2}.\end{aligned}\quad (2.5)$$

$A_{\alpha l}^{(1)*}$ and $A_{\alpha l}^{(1)}$ are the creation and the annihilation operators of the zero-mesons which extend around the nucleon 1, and so on.

The total Hamiltonian H is rearranged as follows :

$$H = H_0 + H_{0s}^{(1)} + H_{1s}^{(1)} + H_{0s}' + H_s, \quad (2.6)$$

$$\begin{aligned}H_0 &= (K_{00} + U_1) \sum_i Q_i^{(1)} + U_{1l} \sum_i Q_i^{(1)} \\ &+ U_2 \sum_i A_{\alpha l}^{(1)*} A_{\alpha l}^{(1)} + U_{2l} \sum_i (3A_{\alpha 3}^{(1)*} A_{\alpha 3}^{(1)} - A_{\alpha l}^{(1)*} A_{\alpha l}^{(1)}) \\ &+ W \sum_{i \neq j} A_{\alpha l}^{(i)*} A_{\alpha l}^{(j)} + W_l \sum_{i \neq j} (3A_{\alpha 3}^{(i)*} A_{\alpha 3}^{(j)} - A_{\alpha l}^{(i)*} A_{\alpha l}^{(j)}) \\ &+ \frac{GK_{00}}{6\pi V} \sum_{i \neq j} [v_s (A_{\alpha l}^{(i)*} \tau_\alpha^{(j)} \sigma_l^{(j)} + \text{c.c.}) + v_t \{ (3A_{\alpha 3}^{(i)*} \tau_\alpha^{(j)} \sigma_3^{(j)} - A_{\alpha l}^{(i)*} \tau_\alpha^{(j)} \sigma_l^{(j)}) + \text{c.c.} \}] \\ &+ \frac{G\mu}{3} (\tau_\alpha^{(1)} \tau_\alpha^{(2)}) [u_s (\sigma_l^{(1)} \sigma_l^{(2)}) + u_t (3\sigma_3^{(1)} \sigma_3^{(2)} - \sigma_l^{(1)} \sigma_l^{(2)})],\end{aligned}\quad (2.7)$$

$$H_{0s}^{(1)} = \int dk K \varphi_l^*(\mathbf{k}) a_\alpha(\mathbf{k}) A_{\alpha l}^{(1)*} e^{ikx_l} + \text{c.c.}, \quad (2.8)$$

$$H_{0s}' = \frac{GK_{00}}{6\pi V} \sum_{i \neq j} \int dk K [v_s \varphi_l^*(\mathbf{k}) \sigma_l^{(i)} + v_t (3\varphi_3^*(\mathbf{k}) \sigma_3^{(i)} - \varphi_l^*(\mathbf{k}) \sigma_l^{(i)})] \tau_\alpha^{(i)} a_\alpha(\mathbf{k}) e^{ikx_j} + \text{c.c.}, \quad (2.9)$$

$$H_s = \int dk K a_\alpha^*(\mathbf{k}) a_\alpha(\mathbf{k}), \quad (2.10)$$

with

$$G \equiv g^2/4\pi.$$

Here the conventional notations

$$A_{\alpha l}^{(i)} = A_{\alpha l}^{(i)} - V_{\alpha}^{(i)} \sigma_l^{(i)}, \quad A_{\alpha l}^{(i)*} = A_{\alpha l}^{(i)*} - V_{\alpha}^{(i)*} \sigma_l^{(i)} \quad (2.11)$$

are used, of which the diagonal elements vanish, and

$$Q_i^{(i)} = 3[A_{\alpha 3}^{(i)*} A_{\alpha 3}^{(i)} - V_{\alpha}^{(i)*} \sigma_3^{(i)} (A_{\alpha 3}^{(i)*} + A_{\alpha 3}^{(i)})], \quad (2.12)$$

Coefficients U 's, W 's, u 's and v 's, the functions of $x = \mu r$, are defined by

$$\begin{aligned} u_s &= -(1/2\pi^2) \int dk F(k)^2 \cos k_3 r \cdot k^2/K^2, \\ u_t &= -(1/2\pi^2) \int dk F(k)^2 \cos k_3 r \cdot (3k_3^2 - k^2)/2K^2, \\ v_s &= -(1/4\pi) \int dk F(k)^2 \cos k_3 r \cdot k^2/K^3, \\ v_t &= -(1/4\pi) \int dk F(k)^2 \cos k_3 r \cdot (3k_3^2 - k^2)/2K^3, \end{aligned} \quad (2.13)$$

and

$$\begin{aligned} U_1 &= -\frac{G^2}{36\pi V} \left[\mu(u_s v_s + 2u_t v_t) - \frac{3K_{00}}{2\pi} (v_s^2 + 2v_t^2) \right], \\ U_2 &= -\frac{G^2}{36\pi V} \left[\mu(u_s v_s + 2u_t v_t) - \frac{5K_{00}}{2\pi} (v_s^2 + 2v_t^2) \right]. \end{aligned} \quad (2.14)$$

Other functions are unnecessary for later calculations and are not given here.

u 's etc., of course, depend on the cut-off factor $F(k)$. In the case of no cut-off, we have

$$\begin{aligned} u_s(x) &= \frac{e^{-x}}{x} + 2 \left(\partial'(x) + \frac{2\partial(x)}{x} \right), \\ u_t(x) &= \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{e^{-x}}{x} + 2 \left(\partial'(x) - \frac{\partial(x)}{x} \right), \\ v_s(x) &= K_0(x) - K_1(x)/x, \\ v_t(x) &= K_2(x). \end{aligned} \quad (2.15)$$

$K_n(x)$'s are the modified Bessel functions of the second kind. Terms involving the delta-type functions are the so-called contact interaction. Even in the case of cut-off, u 's and v 's can be approximated by the no cut-off functions (2.15) for large x , and U 's are of the order of e^{-2x} . The expressions (2.7) \sim (2.9) are obtained after the omission of terms higher than e^{-2x} for large x .

(2) The potential

The potential is obtained as the x -dependent eigenvalue of the total Hamiltonian H . Let the potentials of the order of e^{-x} and e^{-2x} for large x be denoted by $V(e^{-x})$ and V' ,

respectively. $V(e^{-x})$ is given by the diagonal element of the last row of (2.7) :

$V(e^{-x}) = \text{diagonal element of}$

$$\frac{\mu}{3} \left(\frac{g^2}{4\pi} \right) (\tau_{\alpha}^{(1)} \tau_{\alpha}^{(2)}) [u_s (\sigma_i^{(1)} \sigma_i^{(2)}) + u_t S_{12}], \quad (2.16)$$

where

$$\begin{aligned} S_{12} &= 3\sigma_{\alpha}^{(1)} \sigma_{\alpha}^{(2)} - \sigma_i^{(1)} \sigma_i^{(2)} \\ &= 3(\sigma^{(1)} \mathbf{r})(\sigma^{(2)} \mathbf{r})/r^2 - (\sigma^{(1)} \sigma^{(2)}). \end{aligned} \quad (2.17)$$

It should be noted that τ and σ are the isospin and the spin operators of a bare nucleon. Similarly to the case of charged scalar theory, we must introduce the isospin operator T and the spin operator S of a clothed nucleon. For instance, $(T^{(1)} T^{(2)})$ is equal to 1 for the charge triplet state of two clothed nucleons, and -3 for the charge singlet state. $(S^{(1)} S^{(2)})$ is equal to 1 for the spin triplet state of two clothed nucleons, and -3 for the spin singlet state. Moreover we introduce the operator

$$\sum_{12} = 3(S^{(1)} \mathbf{r})(S^{(2)} \mathbf{r})/r^2 - (S^{(1)} S^{(2)}), \quad (2.18)$$

corresponding to the operator S_{12} .

Since the diagonal element of $\tau_{\alpha}^{(1)} \tau_{\alpha}^{(2)} \cdot \sigma_i^{(1)} \sigma_i^{(2)}$ is $T_{\alpha}^{(1)} T_{\alpha}^{(2)} \cdot S_i^{(1)} S_i^{(2)} \cdot (P_0 - P_{\alpha} - P_{\tau} + P_{\sigma\tau})^2$, as is shown in Appendix A, (2.16) becomes

$$V(e^{-x}) = \frac{\mu}{3} \left(\frac{g^2}{4\pi} \right) (P_0 - P_{\sigma} - P_{\tau} + P_{\sigma\tau})^2 (T_{\alpha}^{(1)} T_{\alpha}^{(2)}) [u_s (S_i^{(1)} S_i^{(2)}) + u_t \sum_{12}]. \quad (2.19)$$

The factor $(P_0 - P_{\sigma} - P_{\tau} + P_{\sigma\tau})^2$ corresponds to the factor $(1 - P_d)^2$ in the charged scalar theory. $V(e^{-x})$ can be interpreted as the potential due to an exchange of one zero-meson between two nucleons.

V' consists of two parts, V_{dis} and V_{iso} . V_{dis} comes from the effect of distortion of the zero-meson cloud of a nucleon due to the existence of the other nucleon nearby. And we obtain, from the diagonal element of the first and the second row of (2.7) :

$$V_{\text{dis}} = 2Q_0 U_1 + 2n_0 U_2, \quad (2.20)$$

$K_{00} Q_0$ is the energy of the ground state and n_0 the mean number of zero-mesons in the cloud of the ground state nucleon.

V_{iso} is the potential energy due to the effect of the excitation of the clothed nucleons to the isobar levels and the effect of the s -meson emission. The method of its evaluation is the same as in I, and is given in Appendix B. It should be noted that the s -meson exchange plays no role.

§ 3. Coupling constant renormalization and cut-off procedure

The potential up to the order of e^{-2x} is expressed in terms of the functions u_s , u_t , v_s and v_t . The shape of these functions depends strongly on an assumed form of the cut-off factor in contrast to the case of the charged scalar theory. These circumstances

are due to the appearance of the "contact interaction." As is seen from (2.15), if the cut-off procedure is not applied, the contact interaction term becomes zero except an infinitely steep peak at $x=0$, and can be disregarded since it has no physical effects. However, when the cut-off procedure is applied, the peak is spread out. But the effects of the spread of this peak must be suppressed in the region of large x .

For this purpose, the cut-off factor must be a function which smoothly decreases with increasing k . We employ a cut-off factor of gaussian type (1.1) as a suitable one. The cut-off momentum a is now to be determined.

Using (1.1) and (2.13), we have

$$\begin{aligned}
 u_e(x) &= \frac{e^{-x}}{x} \cdot e^{1/a^2} \left\{ 1 + \frac{1}{\sqrt{\pi}} \mathcal{G}\left(\frac{ax}{2} - \frac{1}{a}\right) \right\} \\
 &\quad - \frac{e^x}{x} \cdot e^{1/a^2} \frac{1}{\sqrt{\pi}} \mathcal{G}\left(\frac{ax}{2} + \frac{1}{a}\right) - \frac{a^2}{2} \frac{e^{-(ax/2)^2}}{x}, \\
 u_l(x) &= \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^{-x}}{x} \cdot e^{1/a^2} \left\{ 1 + \frac{1}{\sqrt{\pi}} \mathcal{G}\left(\frac{ax}{2} - \frac{1}{a}\right) \right\} \\
 &\quad - \left(1 - \frac{3}{x} + \frac{3}{x^2}\right) \frac{e^x}{x} \cdot e^{1/a^2} \frac{1}{\sqrt{\pi}} \mathcal{G}\left(\frac{ax}{2} + \frac{1}{a}\right),
 \end{aligned} \tag{3.1}$$

with

$$\mathcal{G}(z) = \int_z^\infty e^{-y^2} dy.$$

Therefore $V(e^{-x})$ is, for large x ,

$$\begin{aligned}
 V(e^{-x}) &= \frac{\mu}{3} \left(\frac{g^2}{4\pi} \right) (P_0 - P_\sigma - P_\tau + P_{\sigma\tau})^2 e^{1/a^2} \\
 &\quad \times (T_\alpha^{(1)} T_\alpha^{(2)}) \left[(S_i^{(1)} S_i^{(2)}) + \left(1 + \frac{3}{x} + \frac{3}{x^2}\right) \sum_{i \neq j} \right] \frac{e^{-x}}{x}.
 \end{aligned} \tag{3.2}$$

In this region, V' is very small compared with $V(e^{-x})$ and the resultant potential is predominately determined by (3.2). According to the analysis of Otsuki and Tamagaki¹⁾, the effective coupling constant

$$\frac{g_{eff}^2}{4\pi} = \frac{g^2}{4\pi} (P_0 - P_\sigma - P_\tau + P_{\sigma\tau})^2 e^{1/a^2} \tag{3.3}$$

should be renormalized to 0.08.

Table 1

V	0.2	0.4	0.6
$(P_0 - \dots)^2$	0.633	0.341	0.224
$g^2/4\pi$	0.114	0.221	0.343
a/μ	3.04	4.11	4.84

The value of V determines the "structure" of the meson cloud and $(P_0 - P_\sigma - P_\tau + P_{\sigma\tau})^2$ depends only on V , but neither on g nor on a . Then noting that there is the relation (A·5) among g , V and a , we can determine the values of g and a for given V . The values of the parameters determined in this manner are listed in Table 1. As to the structure of the meson cloud of the single clothed nucleon, we use Takeda's results^{b)}.

The functions u_s , u_t , v_s and v_t are shown in Figs. 1 and 2. The effects of the contact interaction are spread almost to $x \approx 1.0$. In the interior region $x \lesssim 1.0$, the sign of the central potential of $V(e^{-x})$ changes.

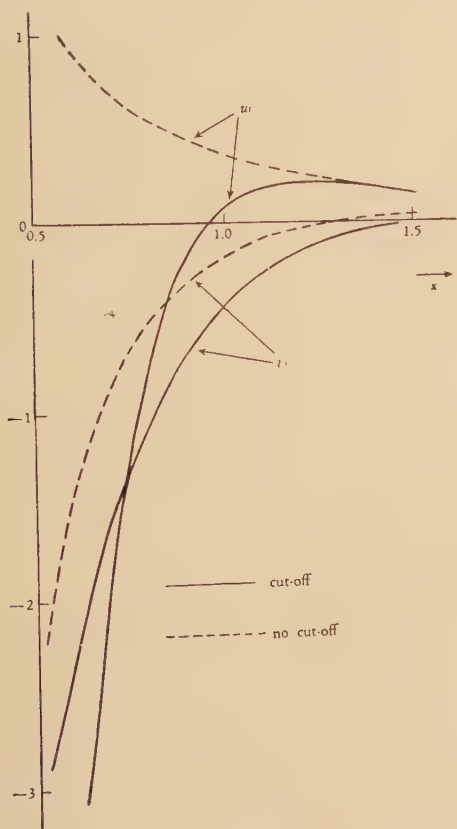


Fig. 1. $u_s(x)$ and $v_s(x)$ with $a=4.11\mu$ and with no cut-off.

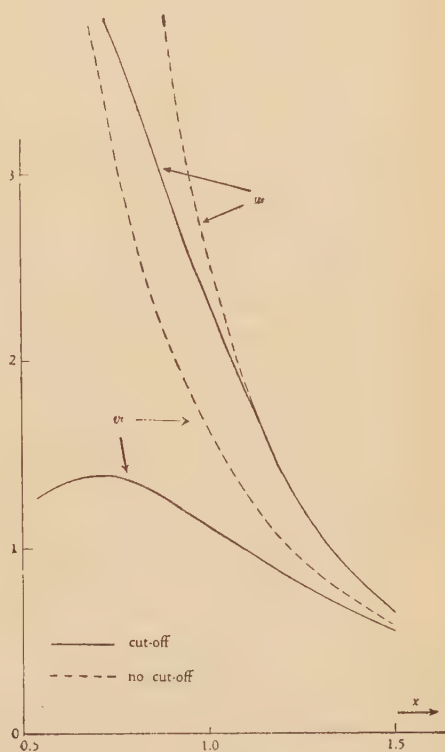
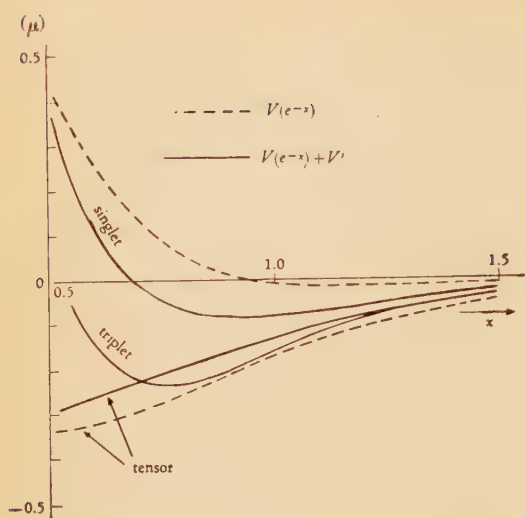
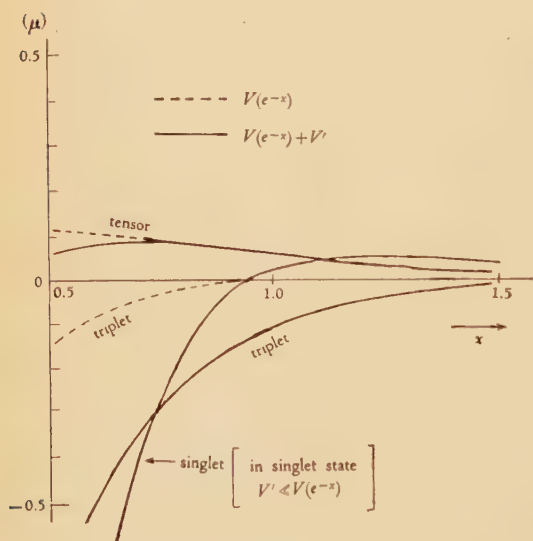


Fig. 2. $u_t(x)$ and $v_t(x)$ with $a=4.11\mu$ and with no cut-off.

§ 4. Results and discussions

The results are shown in Figs. 3 and 4. Qualitative tendency of the potential is tabulated in Table 2. There is no serious discrepancy among our results and the FST-potential⁽²⁾ and the BW-potential⁽³⁾. As to V' , V'_{dis} is repulsive and the central part of V'_{iso} is always attractive, and the central part of the resultant V' is attractive for all states. While the tensor part of V' changes its sign with states and regions.

Fig. 3. The potential in even state. $V=0.4$, $a=4.11\mu$.Fig. 4. The potential in odd state. $V=0.4$, $a=4.11\mu$.

to the change of the energy levels of excited states, etc. For example, K_{00} obtained by the straight cut-off [cf. (A·2)], which is usually employed in the one nucleon problem, is smaller than that obtained by the gaussian cut-off with common cut off momentum. Consequently, the differences between energy levels are small and the effect of the isobaric transitions becomes larger in the case of the straight cut-off than in the case of the gaussian cut-off.

In Fig. 5 the results of the straight cut-off and of the gaussian cut-off are shown for the triplet odd state. Here, of course, if we apply the straight cut-off to (2·13),

Moreover, V_{is0} mainly consists of the effects of the isobaric transitions of nucleons. The effects of s-meson turn to be very small. Therefore the potential up to the order of e^{-2x} , i.e., $V(e^{-x}) + V'$ is almost determined only by the effects of zero-mesons. This fact suggests that the zero-mesons assumed in the intermediate coupling theory give a fairly good picture of the real meson cloud around the nucleons in the problem of the low energy nuclear forces.

Table 2. Tendency of V' .

		$x \lesssim 1.0$	$x \gtrsim 1.0$
even state	singlet	—	—
	triplet central	—	—
	tensor	+	+
odd state	singlet	—	—
	triplet central	—	—
	tensor	—	+

The effects of the cut-off procedure appear in two ways: (i) effect on shape of the functions u 's, etc. and (ii) effect on the coefficients by which the functions u 's etc. are to be multiplied. The effect (i) is estimated from the spread of the contact term. Using the gaussian cut-off with the cut-off momentum $a \gtrsim 4.0\mu$, this effect is suppressed to be small enough in the exterior region where $x \gtrsim 1.0$. The effect (ii) is due

the functions u 's etc. oscillate inconveniently for $x \gtrsim 1.0$. Therefore we have used u 's etc. obtained by the straight cut-off only in the interior region $x \lesssim 1.0$, and connected them smoothly to the functions obtained without cut-off (2.15) in the exterior region.

The effect (i) is characteristic to the two nucleon problem, while the effect (ii) appears also in the one nucleon problem, e.g. meson scattering by nucleon. The effect (ii) on $V(e^{-x})$ disappears after the "coupling constant renormalization". Therefore, if we suppress the effect (i) in the exterior region, $V(e^{-x})$ coincides with the conventional second order perturbation potential, irrespectively of the form of the cut-off factor, and also of the "structure" of the clothed nucleon. (Because the structure of the clothed nucleon is determined by V , and the same effective coupling constant is obtained after renormalization for any value of V .)

Thus the potential $V(e^{-x})$ is well established, as long as we understand that the coupling constant is the renormalized one and the isospin and the spin operators are the operators for the clothed nucleons. However, the total potential $V(e^{-x}) + V'$ can by no means be independent of the cut-off procedure. In other words, V' depends essentially on the structure of the clothed nucleon including the isobar states, through the effect (ii).

The above results have been shown in Figs. only for $V=0.4$. It would be desirable to choose larger cut-off momentum, e.g. $a \sim$ nucleon mass, so that the appearance of the effect (i) is confined in the more interior region. However, for such a large cut-off momentum, V would become ~ 1.0 . Then the one level approximation which simplifies evaluations would not be applicable.

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Appendix A. The single clothed nucleon

As a preparation, we briefly review the static problem of a single clothed nucleon^{5,6)}. The nucleon is considered to be at rest at the origin and interacting with the symmetrical pseudo-scalar meson field through the pseudo-vector coupling.

Then the Hamiltonian is*

* The natural unit ($\hbar=c=1$) is used.

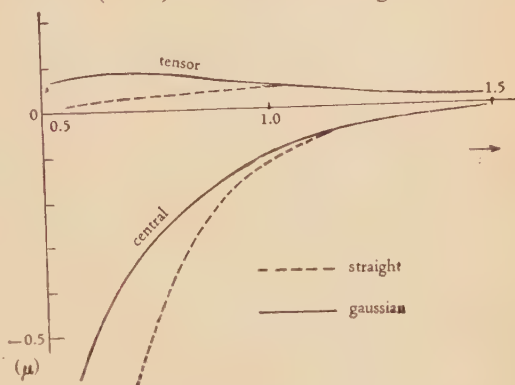


Fig. 5. Comparison of potentials for triplet odd state obtained by gaussian cut-off and by straight cut-off. $V=0.4$, $a=4.11\mu$.

$$H = H_{\text{meson}} + H_{\text{int}},$$

$$H_{\text{meson}} = \int dk K \phi_{\alpha}^*(k) \phi_{\alpha}(k), \quad (\text{A} \cdot 1)$$

$$H_{\text{int}} = \frac{-i}{\sqrt{2\pi^3}} \frac{g}{\mu} \tau_{\alpha} \sigma_l \int dk \frac{F(k)}{\sqrt{2K}} k_l \phi_{\alpha}(k) + \text{c.c.},$$

with

$$K = \sqrt{k^2 + \mu^2}.$$

Here $\phi_{\alpha}^*(k)$ and $\phi_{\alpha}(k)$ are the creation and the annihilation operators of a meson with momentum k . Suffix $\alpha (= 1, 2, 3)$ refers to the charge of the meson, and $l (= 1, 2, 3)$ to the spatial component. τ_{α} and σ_l are the usual isospin and spin operators, respectively. μ is the rest mass of meson, and g is the coupling constant.

$F(k)$ is the cut-off factor. For instance, in the case of the straight cut-off,

$$F(k) = \begin{cases} 1 & \text{for } \begin{cases} k < a \\ k > a \end{cases} \\ 0 & \end{cases} \quad (\text{A} \cdot 2)$$

In the case of no cut-off, $a \rightarrow \infty$.

Now we split the meson field $\phi_{\alpha}(k)$ into bound and unbound parts, by means of

$$\begin{aligned} \phi_{\alpha}(k) &= A_{\alpha l} \varphi_l(k) + a_{\alpha}(k), \\ \phi_{\alpha}^*(k) &= A_{\alpha l}^* \varphi_l^*(k) + a_{\alpha}^*(k). \end{aligned} \quad (\text{A} \cdot 3)$$

The "wave function" of the zero-meson $\varphi_l(k)$ is taken to have the form

$$\varphi_l(k) = \frac{1}{V} \frac{i}{\sqrt{2\pi^3}} \frac{g}{\mu} \frac{F(k) k_l}{\sqrt{2K}}, \quad (l = 1, 2, 3), \quad (\text{A} \cdot 4)$$

where V is the normalization constant:

$$V^2 = \frac{1}{(2\pi)^3} \left(\frac{g}{\mu} \right)^2 \frac{1}{3} \int dk \frac{F(k) k^2}{2K^3}. \quad (\text{A} \cdot 5)$$

$a_{\alpha}(k)$, the so-called s-meson part, is orthogonal to $\varphi_l^*(k)$. $A_{\alpha l}^*$ and $A_{\alpha l}$ are the creation and the annihilation operators of zero-mesons, respectively. The commutation relations are

$$[A_{\alpha l}, A_{\alpha' l'}^*] = \delta_{\alpha \alpha'} \delta_{l l'}, \quad \text{others} = 0. \quad (\text{A} \cdot 6)$$

Using the expansions (A·3), the total Hamiltonian is rewritten in the form

$$\begin{aligned} H &= K_{00} Q + [A_{\alpha l}^* \int dk K \varphi_l^*(k) a_{\alpha}(k) + \text{c.c.}] \\ &\quad + \int dk K a_{\alpha}^*(k) a_{\alpha}(k), \end{aligned} \quad (\text{A} \cdot 7)$$

with

$$\begin{aligned} K_{00} &= \int dk K |\varphi_l(k)|^2, \quad (\text{independent of } l), \\ Q &= A_{\alpha l}^* A_{\alpha l} - V \tau_{\alpha} \sigma_l (A_{\alpha l}^* + A_{\alpha l}), \\ A_{\alpha l} &= A_{\alpha l} - V \tau_{\alpha} \sigma_l, \quad A_{\alpha l}^* = A_{\alpha l}^* - V \tau_{\alpha} \sigma_l. \end{aligned} \quad (\text{A} \cdot 8)$$

$K_{00}Q$ is the Hamiltonian of the system composed of a bare nucleon and the zero-meson cloud. The eigenvalue equation for a clothed nucleon

$$(K_{00}Q - E)\psi = 0 \quad (\text{A} \cdot 9)$$

has been already solved by several authors⁽⁷⁾⁽⁸⁾. The normalization constant V is the coupling constant between the zero-mesons and the bare nucleons, and the structure of the clothed nucleon is determined by the value of V .

There are the following constants of motion: the total angular momentum J , its third component J_z and similar quantities I, I_z in the charge space. For the clothed nucleon in the ground state, $I=J=1/2$. $I_z=1/2$ state is the proton state of a clothed nucleon, and $I_z=-1/2$ state is the neutron state. The probability that the isospin and the spin of a bare nucleon in a clothed nucleon are the same as those of a clothed nucleon is denoted by P_0 ; the probability that the bare nucleon isospin is reversed, by P_z ; the probability that the bare nucleon spin is reversed, by P_σ ; and the probability that the bare nucleon isospin and spin are both reversed, by $P_{\sigma\tau}$. Of course

$$P_0 + P_\sigma + P_z + P_{\sigma\tau} = 1. \quad (\text{A} \cdot 10)$$

Denoting the probability of finding the total orbital angular momentum of the meson cloud to be 1 by P_1 , we have

$$P_0 = 1 - (8/9)P_1, \quad P_\sigma = P_z = (2/9)P_1, \quad P_{\sigma\tau} = (4/9)P_1. \quad (\text{A} \cdot 11)$$

Hereafter the ground state is denoted by suffix 0, and the expectation value of any operator \mathcal{O} for the ground state by $(0|\mathcal{O}|0)$. Now we introduce the isospin operator T and the spin operator S for a clothed nucleon; for instance,

$$\begin{aligned} \frac{1}{2}(T_1 - iT_2)|\text{proton state of a clothed nucleon}) \\ = |\text{neutron state of a clothed nucleon}), \\ T_3|\text{proton (neutron) state of a clothed nucleon}) \\ = 1(-1)|\text{proton (neutron) state of a clothed nucleon}). \end{aligned} \quad (\text{A} \cdot 12)$$

Then the diagonal element of $\tau_\alpha\sigma_l$ is expressed in terms of $T_\alpha S_l$, namely,

$$(0|\tau_\alpha\sigma_l|0) = (P_0 - P_\sigma - P_z + P_{\sigma\tau})(0|T_\alpha S_l|0). \quad (\text{A} \cdot 13)$$

As to the excited or isobar states we consider only the following four states: $(I, J) = (\frac{1}{2}, \frac{1}{2})_{1\text{st excited}}, (\frac{3}{2}, \frac{3}{2})_{1\text{st excited}}, (\frac{3}{2}, \frac{1}{2})_{1\text{st excited}}, (\frac{3}{2}, \frac{3}{2})_{1\text{st excited}}$. The level of the state $(\frac{3}{2}, \frac{3}{2})_{1\text{st excited}}$ coincides with that of $(\frac{3}{2}, \frac{1}{2})_{1\text{st excited}}$. We denote the above states by suffix ν ($\neq 0$), and matrix elements of any operator \mathcal{O} by $(0|\mathcal{O}|\nu)$, $(\nu|\mathcal{O}|\nu')$, etc.

Appendix B. Evaluation of V_{iso}

Here we calculate V_{iso} , the effects of isobaric transitions and emission and absorption of s -mesons.

(1). Matrix elements

First we write down the necessary non-diagonal elements of H_0 , $H_{is}^{(1)}$ and H'_{0s} . Under the assumption of the one level approximation, the third row of H_0 (2.7) is omitted.

Now, there is a well known formula

$$(\nu | A_{\alpha}^{*} | 0) = V \varepsilon_{\nu} (\nu | \tau_{\alpha} \sigma_l | 0) \quad (\text{B} \cdot 1)$$

with

$$\varepsilon_{\nu} = Q_{\nu} / (1 - Q_{\nu})$$

where $K_{00} Q_{\nu}$ is the energy difference between the ν -state and the ground state. Then the non-diagonal element between the ground state and the excited state $|\nu_1 \nu_2\rangle$ is, (two nucleons are in the ν_1 - and ν_2 -states, respectively)

$$\begin{aligned} (\nu_1 \nu_2 | H_0 | 00) &= X_{\nu_1 \nu_2} (\nu_1 \nu_2 | \tau_{\alpha}^{(1)} \tau_{\alpha}^{(2)} \sigma_l^{(1)} \sigma_l^{(2)} | 00) \\ &\quad + Y_{\nu_1 \nu_2} (\nu_1 \nu_2 | \tau_{\alpha}^{(1)} \tau_{\alpha}^{(2)} S_{12} | 00) \end{aligned} \quad (\text{B} \cdot 2)$$

with

$$\begin{aligned} X_{\nu_1 \nu_2} &= \frac{GK_{00}}{6\pi} (\varepsilon_{\nu_1} + \varepsilon_{\nu_2}) v_s + \frac{G_l \mu}{3} u_s, \\ Y_{\nu_1 \nu_2} &= \frac{GK_{00}}{6\pi} (\varepsilon_{\nu_1} + \varepsilon_{\nu_2}) v_t + \frac{G_l \mu}{3} u_t. \end{aligned} \quad (\text{B} \cdot 3)$$

Further, the state is specified by the following quantities; (i_1, j_1) and (i_2, j_2) , the isospins and the spins of nucleons 1 and 2 respectively; i and j , the isospin and the spin of the two nucleon system; m , the third component of j . The state vector is represented by $|\nu_1 \nu_2; i_1, i_2, i; j_1, j_2, j, m\rangle$. Especially the ground state is, using capital letters, represented by $|00; \frac{1}{2}, \frac{1}{2}, I; \frac{1}{2}, \frac{1}{2}, J, M\rangle$ or simply $I; J, M\rangle$. Because of the charge independence, there is no need to specify the third component of i . Then the matrix element (B.2) is expressed as follows:

$$\begin{aligned} &(\nu_1 \nu_2; i_1, i_2, i; j_1, j_2, j, m | H_0 | I; J, M) \\ &= \left[X_{\nu_1 \nu_2} \delta_{jJ} (-1)^{j_1 - J + \frac{1}{2}} W(1/2, 1/2, j_1, j_2; J, 1) \right. \\ &\quad \left. + Y_{\nu_1 \nu_2} \sqrt{6} \sqrt{5(2J+1)} (J, 2; M, 0 | j, M) U \left(\begin{matrix} j_1 & 1/2 & 1 \\ j_2 & 1/2 & 1 \\ j & J & 2 \end{matrix} \right) \right] \\ &\quad \times (-1)^{i_1 - I + \frac{1}{2}} \delta_{iI} \delta_{mM} W(1/2, 1/2, i_1, i_2; I, 1) \\ &\quad \times (\nu_1 | \tau_{\alpha}^{(1)} \sigma_l^{(1)} | 0) (\nu_2 | \tau_{\alpha}^{(2)} \sigma_l^{(2)} | 0). \end{aligned} \quad (\text{B} \cdot 4)$$

Here $(J, 2; M, 0 | j, M)$, $W(\dots)$ and $U(\dots)$ are the Clebsch-Gordan, Racah and generalized Racah coefficients, respectively. $(\nu | \tau_{\alpha} \sigma_l | 0)$ is the so-called physical part of the matrix element⁷⁾.

Next, the system composed of one s-meson and one nucleon is to be considered. We specify the state composed of one nucleon in the ground state and one s-meson, by the isospin i , the spin j and the s-meson momentum k . Then the matrix element of $H_{is}^{(1)}$ is

$$\begin{aligned}
& (0s, \nu_2 | H_{0s}^{(1)} | \nu_1 \nu_2) \\
&= (0\nu_2; \underline{i}_1', \underline{i}_2', i'; \underline{j}_1', \underline{j}_2', j', m'; \mathbf{k} | H_{0s}^{(1)} | \nu_1 \nu_2; i_1, i_2, i; j_1, j_2, j, m) \\
&= \delta_{\underline{i}_1' i_1} \delta_{\underline{i}_2' i_2} \delta_{\underline{j}_1' j_1} \delta_{\underline{j}_2' j_2} \delta_{i' i} \delta_{j' j} \delta_{m' m} \\
&\quad \times V \mathcal{E}_{\nu_1}(\nu_1 | \tau^{(1)} \sigma^{(1)} | 0) (K - K_{00}) \varphi(k) / \sqrt{(2i_1 + 1)(2j_1 + 1)}, \tag{B.5}
\end{aligned}$$

where

$$\varphi(k)^2 = \varphi_i^*(\mathbf{k}) \varphi_i(\mathbf{k}) / 3. \tag{B.6}$$

The matrix element of $H_{0s}^{(2)}$ is similar to (B.5).

Finally, the matrix elements of H_{0s}' e.g. $(0s; \nu_2 | H_{0s}' | 00)$ can be obtained by the following substitution in (B.4).

$$X_{\nu_1 \nu_2} \rightarrow \frac{GK_{00}}{6\pi} v_s, \quad Y_{\nu_1 \nu_2} \rightarrow \frac{GK_{00}}{6\pi} v_s, \tag{B.7}$$

$$(\nu_1 | \tau^{(1)} \sigma^{(1)} | 0) \rightarrow \sqrt{(2i_1 + 1)(2j_1 + 1)} (K - K_{00}) \varphi(k).$$

(2) Effects of isobaric transitions and interactions with s-mesons

Since we are concerned with the potential up to the order of e^{-2x} , the processes containing the exchange of an s-meson between two nucleons are inhibited. As $(0s, \nu | H_{0s}' | 00)$,

$(\nu, 0s | H_{0s}' | 00)$ and $(\nu_1 \nu_2 | H_0 | 00)$ are the quantities of the order of e^{-x} for large x , the potential energy of the order of e^{-2x} is obtained by a sort of perturbation method. There are two types of transition schemes, just as in I, which are shown in Fig. 6. V_{iso} is the sum of the contributions of respective processes.

In the process of the type I, contribution $V_{\nu 0}$ is given by (I.6.9), i.e.

$$V_{\nu 0} = -\frac{(\alpha + \beta \lambda R(0))^2}{K_{00} Q_\nu (1 - J)} + \beta^2 R(0), \tag{B.8}$$

with abbreviations

$$\begin{aligned}
(\nu 0 | H_0 | 00) &= \alpha, \quad (00 | H_{0s}' | 0s, 0) = \beta (K - K_{00}) \varphi(k), \\
(0s, 0 | H_{0s}^{(1)} | \nu 0) &= \lambda (K - K_{00}) \varphi(k)
\end{aligned}$$

and

$$R(X) = \frac{1}{Q(X)} - X - K_{00}, \quad Q(X) = \int dk \frac{\varphi(k)^2}{K + X}, \quad J = -\frac{\lambda^2 R(0)}{K_{00} Q_\nu}. \tag{B.9}$$

In the process of the type II, we obtain contribution $V_{\nu_1 \nu_2}$, using an appropriate approximation, as follows:

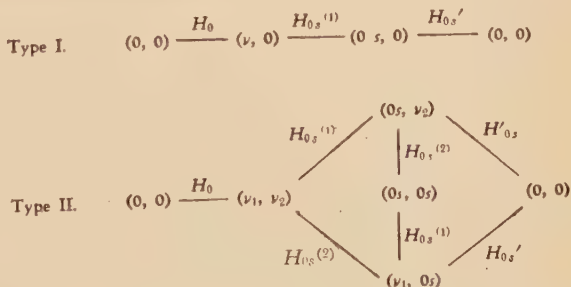


Fig. 6. Transition schemes. $\nu \rightarrow (0s)$ indicates that a nucleon makes transition from ν -state to the ground state emitting one s -meson.

$$V_{\nu_1\nu_2} = -\frac{(\alpha' + \beta_1\lambda_1 R_2 + \beta_2\lambda_2 R_1)^2}{K_{00}(\mathcal{Q}_1 + \mathcal{Q}_2)(1 - A')} + (\beta_1^2 R_2 + \beta_2^2 R_1) \quad (\text{B} \cdot 10)$$

with abbreviations

$$\begin{aligned} (\nu_1\nu_2|H_0|00) &= \alpha', \\ (00|H'_{08}|0s, \nu_2) &= \beta_1(K - K_{00})\varphi(k), \quad (00|H'_{08}|\nu_1, 0s) = \beta_2(K - K_{00})\varphi(k), \\ (0s, \nu_2|H_{08}^{(1)}|\nu_1\nu_2) &= (0s, 0s|H_{08}^{(1)}|\nu_1, 0s) = \lambda_1(K - K_{00})\varphi(k), \\ (\nu_1, 0s|H_{08}^{(2)}|\nu_1\nu_2) &= (0s, 0s|H_{08}^{(2)}|0s_2, \nu_2) = \lambda_2(K - K_{00})\varphi(k), \\ R_1 &= R(K_{00}\mathcal{Q}_1 + \lambda_2^2 R(0)), \quad \mathcal{Q}_1 \equiv \mathcal{Q}_{\nu_1}, \\ A' &= -(\lambda_1^2 R_2 + \lambda_2^2 R_1)/K_{00}(\mathcal{Q}_1 + \mathcal{Q}_2). \end{aligned}$$

It is easily seen that (B·9) is a special case of (B·10).

In V_{ν_0} and $V_{\nu_1\nu_2}$, the terms containing α^2 are due to the isobaric transitions, and the terms containing β^2 to the emission and reabsorption of the s -mesons. The cross terms are due to the interference of these two processes.

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Quantum Statistical Mechanics of Electron-Phonon System

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The problem of the interaction between conduction electrons and the thermal vibration of the lattice ions in the metal is treated by a statistical mechanical method. We adopt the free electron and the phonon models as usual, using the Bloch-Etche type interaction Hamiltonian. By the method of expanding the grand partition function in powers of the interaction parameter, we get a formula for the free energy of the system of conduction electrons in the presence of the interaction. A technique analogous to the renormalization method in the quantum mechanics is utilized in the calculation.

§ 1. Introduction

In 1950, H. Fröhlich¹⁾ put forward a basic idea concerning the important role of the interaction between conduction electrons and thermal vibration of the lattice in the fundamental mechanism of the phenomena of the superconductivity. Since then many discussions and calculations on this problem have been published by various authors.^{2)~7), 9)} But the situations concerning the difficult problems such as the validity of the perturbation procedures that are used by Fröhlich and others, the problem of the renormalization of the sound velocity and the role of the Coulomb interaction between electrons are now still very far from what can be said to be solved. In addition to this, only few papers have appeared in regard to the problem at the temperature above absolute zero.^{8), 12)}

In the present paper,* we want to treat this problem of the interaction between conduction electrons and quantized lattice vibrations (phonons) by a statistical mechanical method and show as clearly as possible the results of the adopted Hamiltonian in the thermal properties of the system of conduction electrons. Our treatment is essentially based on the perturbation procedure in terms of the interaction energy and we are forced to stop at the second order terms for the present. In the course of the calculation we make use of a method which may be considered as a modification of the renormalization method in the quantum theory of fields which has been applied to this problem at 0°K by Fröhlich⁶⁾ and other authors.^{4), 5)} We obtain the formula for the free energy of the system of conduction electrons in the presence of the interparticle interaction which is caused by the electron-phonon interaction. To the order of our approximation it has a very simple form and bears a close resemblance to the formula for the free electron gas.

* The main part of this paper has been published in Japanese in *Busseiron Kenkyu* No. 76 (1954), and here some revisions and additions are made.

§ 2. Grand partition function of electron-phonon system

Following the method in the reference (7), we define the grand partition function Ξ of our system as

$$\Xi = \mathcal{S}p(\lambda^{\sum n_r} e^{-\beta H}) ; \lambda = e^{\beta \mu}, \beta = 1/\kappa T \quad (2.1)$$

where μ is the chemical potential of the electron and κ is the Boltzmann constant. H and $\sum n_r$ are the operators which represent the total Hamiltonian and the total number of the electron of our system respectively. Ξ is determined as the function of the chemical potential μ , the volume V and the temperature T . The chemical potential for the phonon must be zero because these particles can change their total number even under the condition of the closed system.

As our Hamiltonian H we take the following form.*

$$H = H_0 + J(H_1 + H_r) \quad (2.2)$$

$$H_0 = \sum_k \epsilon_k a_k^\dagger a_k + \sum_q \hbar s q b_q^\dagger b_q \quad (2.3)$$

$$\epsilon_k = \hbar^2 k^2 / 2m^*, \quad k = |\mathbf{k}|, \quad q = |\mathbf{q}| \quad (2.3)$$

$$H_1 = -i \Gamma_0 \sum_k \sum_q \sqrt{q} (b_q a_{k+q}^\dagger a_k - b_q^\dagger a_{k-q}^\dagger a_k) \quad (2.4)$$

$$\Gamma_0^2 = 4F_0 \hbar \mu_0 s_0 / 3N_A, \quad F_0 = 3C^2 / 8\mu_0 M s_0^2 \quad (2.4)$$

$$\mu_0 = \hbar^2 / 2m \cdot (3N / 8\pi V)^{2/3}$$

$$H_r = \hbar(s_0 - s) \sum_q q b_q^\dagger b_q + \alpha \sum_k \epsilon_k a_k^\dagger a_k \quad (2.5)$$

$$m^* = m(1 + \alpha)$$

Here a_k^\dagger, a_k are the creation and annihilation operators for the electron field and satisfy the usual anti-commutation relations, and b_q^\dagger, b_q are those for the phonon field which have the canonical commutation relations; \mathbf{k} and \mathbf{q} mean the wave number vectors of electrons and phonons respectively. For the conduction electron we use the free electron model, and the effects of the periodic potential field of the lattice ions and the Coulomb interaction between electrons are taken into account in terms of the effective mass m . In H_0 , this effective mass m is replaced by m^* which is to be determined under the condition that the reaction of the interaction H_1 may be taken into account. For the longitudinal component of the lattice wave we use the phonon model. s means the sound velocity under the influence of H_1 , that is, the renormalized sound velocity, and s_0 is that before renormalization. In our treatment we neglect the dispersion of the sound velocity s or s_0 . H_1 represents the interaction Hamiltonian between conduction electrons and lattice vibrations as usually adopted. In this operator, F_0 is the coupling constant which has been used by Fröhlich; N_A is the number of atoms in the metal of the volume V ; M is the mass of the metal ion; C is the coupling energy which is introduced by Bloch and Bethe; μ_0 is the energy of the Fermi surface of our electron gas at 0°K with the effective mass m and we define the wave number k_0 by the relation $\mu_0 = \hbar^2 k_0^2 / 2m$.

* We omit the zero point energy term $1/2 \cdot \sum_q \hbar s q$ which makes only a constant contribution.

\mathbf{H}_r is the part of the Hamiltonian which has appeared owing to the introduction m^* and s in \mathbf{H}_0 . The parameter J has been introduced for convenience and should be put equal to unity in final results.

In the quantum mechanical treatment of the electron-phonon interaction, two different techniques have been proposed concerning the renormalization of the sound velocity; one is the method adopted by Fröhlich⁽⁶⁾ and the other is the method used by Kitano and Nakano.⁽⁴⁾ In our statistical mechanical version of the renormalization method, we have taken Fröhlich's Hamiltonian as the starting point, chiefly for the sake of simplicity, and obtained the results in § 4 which is equivalent to his formula in the limit of 0°K. The method of Kitano and Nakano which is based on the Bohm-Pines transformation yields a different formula for the renormalized sound velocity which contains terms of higher orders in F_0 even in the first step. But if we stop at the first order term in F_0 , both methods give the same results.*

With the above Hamiltonian we calculate our grand partition function Ξ using the scheme of the representation in which the operator \mathbf{H}_0 takes the diagonal form, that is, the operators $a_k^+ a_k$ and $b_q^+ b_q$ which represent the numbers of the particles take the eigenvalues

$$a_k^+ a_k = n_k, \quad n_k = 0, 1, \\ b_q^+ b_q = N_q, \quad N_q = 0, 1, 2, \dots, \infty.$$

So the calculation of the trace in Ξ is equivalent to the following multiple summation

$$\sum_{n_1=0}^1 \sum_{n_2=0}^1 \dots \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \dots;$$

To carry out this summation, we expand the operator $\exp(-\beta(\mathbf{H}_0 + J(\mathbf{H}_1 + \mathbf{H}_r)))$ in powers of $\beta J(\mathbf{H}_1 + \mathbf{H}_r)$ as in the reference (7);

$$\begin{aligned} \exp(-\beta(\mathbf{H}_0 + J(\mathbf{H}_1 + \mathbf{H}_r))) &= \sum_{k=0}^{\infty} J^k O_k \\ (O_0)_{\nu\nu} &= \exp(-\beta \mathbf{H}_{0\nu}) \\ (O_1)_{\nu\nu} &= -\beta \exp(-\beta \mathbf{H}_{0\nu}) (\mathbf{H}_1 + \mathbf{H}_r)_{\nu\nu} \\ (O_2)_{\nu\nu} &= \beta^2 \left\{ 1/2 \exp(-\beta \mathbf{H}_{0\nu}) ((\mathbf{H}_1 + \mathbf{H}_r)_{\nu\nu})^2 \right. \\ &\quad \left. + \sum_{\sigma(\neq \nu)} \left(\frac{e^{-\beta \mathbf{H}_{0\nu}}}{\beta(\mathbf{H}_{0\sigma} - \mathbf{H}_{0\nu})} + \frac{e^{-\beta \mathbf{H}_{0\sigma}} - e^{-\beta \mathbf{H}_{0\nu}}}{\beta^2(\mathbf{H}_{0\nu} - \mathbf{H}_{0\sigma})^2} \right) (\mathbf{H}_1 + \mathbf{H}_r)_{\nu\sigma} (\mathbf{H}_1 + \mathbf{H}_r)_{\sigma\nu} \right\} \quad (2.6) \end{aligned}$$

where ν or σ stands for a set of numbers $(n_1, n_2, \dots; N_1, N_2, \dots)$ and $\mathbf{H}_{0\nu}$ is the ν -th diagonal element of \mathbf{H}_0 . Then we can write

$$\Xi = \sum_{k=0}^{\infty} J^k \Xi_k, \quad \Xi_k = \text{Sp}(\lambda^{\sum n_r} O_k). \quad (2.7)$$

Setting $\Xi_k = \Xi_0 \xi_k$, we get

$$\Xi = \Xi_0 \left(\sum_{k=0}^{\infty} J^k \xi_k \right). \quad (2.8)$$

* Concerning these points, the author owes very much to kind remarks by Dr. F. Nakano.

We can show* that this power series in J may be cast in a form

$$\Xi = \Xi_0 \exp \left(\sum_{i=1}^{\infty} J^i C_i \right) \quad (2.9)$$

$$C_i = \langle \langle \gamma_i \rangle \rangle$$

$$\gamma_i = \sum_{(\sum l m_l = i)} (-1)^{\sum m_l - 1} (\sum_l m_l - 1)! \prod_l (\hat{\xi}_l)^{m_l} / m_l! \quad (2.10)$$

where the symbol $\langle \rangle$ means to take only the terms which is proportional to N (the number of the conduction electrons in the volume V), the relation between γ_i and $\hat{\xi}_i$ has the same form as the formula which appears in the theory of the imperfect gas, and the summation in this formula means to take all terms which appear under the condition indicated there (the partition of the number i having m_l sets of length l).

The first several terms are

$$\begin{aligned} \gamma_1 &= \hat{\xi}_1, & \gamma_2 &= \hat{\xi}_2 - 1/2 \cdot \hat{\xi}_1^2, \\ \gamma_3 &= \hat{\xi}_3 - \hat{\xi}_1 \hat{\xi}_2 + 1/3 \cdot \hat{\xi}_1^3, & \gamma_4 &= \hat{\xi}_4 - \hat{\xi}_1 \hat{\xi}_3 - 1/2 \cdot \hat{\xi}_1^2 \hat{\xi}_2 + \hat{\xi}_1^2 \hat{\xi}_2 - 1/4 \cdot \hat{\xi}_1^4. \end{aligned} \quad (2.11)$$

With our form (2.9) for the grand partition function Ξ , we use the usual formula

$$N = \lambda \partial \log \Xi / \partial \lambda \quad (2.12)$$

to determine μ in terms of T , N , V and the Helmholtz free energy of our system is given by

$$\Phi = N\mu - \kappa T \log \Xi. \quad (2.13)$$

§ 3. Calculations of Ξ_k

(1) Ξ_0 : From the definition in (2.7), we have

$$\begin{aligned} \Xi_0 &= Sp(\lambda^{\sum n_r} \exp(-\beta H_0)) \\ &= \sum_{n_1=0}^1 \dots \prod_k (\lambda e^{-\beta \epsilon_k})^{n_k} \sum_{N_1=0}^{\infty} \dots \prod_q (e^{-\beta \hbar \omega_q})^{N_q} N_q \\ &= \prod_k (1 + \lambda e^{-\beta \epsilon_k}) \prod_q (1 - e^{-\beta \hbar \omega_q})^{-1}. \end{aligned} \quad (3.1)$$

This is identical with the grand partition function of the system of electrons and phonons without interaction except the fact that m and s_0 are replaced by m^* and s respectively.

(2) Ξ_1 : From (2.7) and (2.6) we have

$$\Xi_1 = -\beta Sp(\lambda^{\sum n_r} e^{-\beta H_0} (H_1 + H_r)). \quad (3.2)$$

Now we introduce the following notation for an operator A .

$$\langle A \rangle = Sp(\lambda^{\sum n_r} e^{-\beta H_0} A) / \Xi_0 \quad (3.3)$$

Then it becomes

$$\Xi_1 = -\beta \Xi_0 \langle H_1 + H_r \rangle$$

and we get

* In Reference (7), the relation between C_k and $\hat{\xi}_k$ was not given in a general form, so we want to supplement it here.

$$\hat{\varepsilon}_1 = -\beta \langle \mathbf{H}_1 + \mathbf{H}_r \rangle. \quad (3.4)$$

As may be seen easily we have $\langle \mathbf{H}_1 \rangle = 0$. Hence it becomes

$$\hat{\varepsilon}_1 = -\beta \langle \mathbf{H}_r \rangle \quad (3.4')$$

and

$$\langle \mathbf{H}_r \rangle = R_0 + R_1 \quad (3.5)$$

$$R_0 = \alpha \sum_k \varepsilon_k \langle a_k^+ a_k \rangle = \alpha \sum_k \varepsilon_k f_k$$

$$f_k = \frac{\lambda e^{-\beta \varepsilon_k}}{1 + \lambda e^{-\beta \varepsilon_k}} \quad (3.6)$$

$$R_1 = \hbar (s_0 - s) \sum_q q \langle b_q^+ b_q \rangle = \hbar (s_0 - s) \sum_q q \bar{N}_q$$

$$\bar{N}_q = \frac{e^{-\beta \hbar s q}}{1 - e^{-\beta \hbar s q}}. \quad (3.7)$$

R_0 and R_1 can be evaluated easily as follows. We can approximate the summation \sum_k and \sum_q by the integral $2V \int d\mathbf{k} / (2\pi)^3$ and $V \int dq / (2\pi)^3$. The factor 2 in the first integral means the spin weight of the electron. Hence it becomes

$$\begin{aligned} R_0 &= \alpha \frac{2V}{(2\pi)^3} \int \frac{\varepsilon_k d\mathbf{k}}{e^{\beta(\varepsilon_k - \mu)} + 1} = \alpha \frac{3}{2} N \left(\frac{1}{\mu_0^*} \right)^{3/2} \int \frac{\varepsilon^{3/2} d\varepsilon}{e^{\beta(\varepsilon - \mu)} + 1} \\ &= \alpha \frac{3}{5} N \left(\frac{\mu}{\mu_0^*} \right)^{3/2} \mu \left(1 + \frac{5}{8} \pi^2 \left(\frac{\kappa T}{\mu} \right)^2 + \dots \right) \end{aligned} \quad (3.8)$$

$$\mu_0^* = \hbar^2 / 2m^* \cdot (3N/8\pi V)^{2/3}.$$

The function f_k has the same form as the Fermi distribution function in the ideal gas, but of course they have lost their meanings as the distribution function in \mathbf{k} , because the mutual interaction between electrons is introduced through the electron-phonon interaction \mathbf{H}_1 . In calculating the integral, we have assumed that μ is very large compared with κT and used the same method of evaluation as in the case of the ideal Fermi gas. Next, we have

$$\begin{aligned} R_1 &= \hbar (s_0 - s) \frac{V}{(2\pi)^3} \int \frac{q dq}{e^{\beta \hbar s q} - 1} = \hbar (s_0 - s) \frac{4\pi V}{(2\pi)^3} \int_0^{q_m} \frac{q^3 dq}{e^{\beta \hbar s q} - 1} \\ &= \hbar (s_0 - s) \frac{4\pi V}{(2\pi)^3} \left(\frac{\kappa T}{\hbar s} \right)^4 \int_0^{\gamma} \frac{x^3 dx}{e^x - 1} \end{aligned} \quad (3.9)$$

$$\gamma = \hbar s q_m / \kappa T, \quad q_m = 2\pi (3N_A / 4\pi V)^{1/3}.$$

At sufficiently low temperatures, we can assume that γ is very large and the approximation

$$\int_0^{\gamma} \frac{x^3 dx}{e^x - 1} \approx \int_0^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

may be used. Hence we obtain

$$R_1 = \hbar (s_0 - s) q_m (\pi^4 / 5) N_A (\kappa T / \hbar s q_m)^4. \quad (3.10)$$

Here q_m means the wave number which corresponds to the Debye maximum of the frequency spectrum of the lattice vibration.

(3) Ξ_2 : Using the notation given in (3.3), we have from (2.6) and (2.7),

$$\Xi_2 = \Xi_2^{(1)} + \Xi_2^{(2)} \quad (3.11)$$

$$\Xi_2^{(1)} = 1/2 \cdot \beta^2 \Xi_0 \langle \mathbf{H}_r^2 \rangle \quad (3.12)$$

$$\left. \begin{aligned} \Xi_2^{(2)} &= \beta^2 \Xi_0 \langle N_2 \rangle \\ (N_2)_{\nu\nu} &= \sum_{\sigma(\neq \nu)} \mathbf{H}_{1\nu\sigma} \mathbf{H}_{1\sigma\nu} / \beta (\mathbf{H}_{0\sigma} - \mathbf{H}_{0\nu}) \end{aligned} \right\} \quad (3.13)$$

Here we have utilized the fact that \mathbf{H}_r has only the diagonal elements, and on the contrary, \mathbf{H}_1 has only the off-diagonal elements.

The contribution from the third term in O_2 turns out to be zero due to its anti-symmetry property concerning σ and ν . For $\Xi_2^{(1)}$, we get the following relation.

$$\begin{aligned} \Xi_2^{(1)} &= 1/2 \cdot \beta^2 \Xi_0 \left\{ \langle (\hbar(s_0 - s) \sum_q q b_q^+ b_q)^2 \rangle + \langle (\alpha \sum_k \epsilon_k a_k^+ a_k)^2 \rangle \right. \\ &\quad \left. + 2 \langle (\hbar(s_0 - s) \sum_q q b_q^+ b_q) (\alpha \sum_k \epsilon_k a_k^+ a_k) \rangle \right\} \\ &= 1/2 \cdot \beta^2 \Xi_0 \left\{ (\hbar(s_0 - s) \sum_q q \bar{N}_q)^2 + (\alpha \sum_k \epsilon_k f_k)^2 \right. \\ &\quad \left. + 2 (\hbar(s_0 - s) \sum_q q \bar{N}_q) (\alpha \sum_k \epsilon_k f_k) \right\} \\ &= 1/2 \cdot \Xi_0 \hat{\epsilon}_1^2 \end{aligned} \quad (3.14)$$

This relation is valid except the terms which give the effects of the order of magnitude $1/N_A$ or $1/N$ in the final results.

For the evaluation of $\langle N_2 \rangle$, we know that we must calculate

$$\begin{aligned} \mathbf{H}_{1\nu\sigma} \mathbf{H}_{1\sigma\nu} &= -\Gamma_0^2 \left\{ \sum_k \sum_q \sqrt{q} (b_q a_{k+q}^+ a_k - b_q^+ a_{k-q}^+ a_k) \right\}_{\nu\sigma} \\ &\quad \times \left\{ \sum_{k'} \sum_{q'} \sqrt{q'} (b_{q'} a_{k'+q'}^+ a_{k'} - b_{q'}^+ a_{k'-q'}^+ a_{k'}) \right\}_{\sigma\nu} \end{aligned}$$

and we use the relations such as

$$\nu = (n_1, n_2, \dots; N_1, N_2, \dots), \quad \sigma = (n'_1, n'_2, \dots; N'_1, N'_2, \dots)$$

$$n'_k = n_k + 1, \quad n'_{k+q} = n_{k+q} - 1, \quad N'_q = N_q + 1$$

for non zero $(b_q a_{k+q}^+ a_k)_{\nu\sigma}$. Then we obtain

$$\begin{aligned} (N_2)_{\nu\nu} &= \Gamma_0^2 \sum_k \sum_q \frac{q}{\beta} \left\{ \frac{N_q n_{k-q} (1 - n_k)}{\hbar^2 / 2m^* \cdot (k^2 - (k-q)^2) - \hbar s q} \right. \\ &\quad \left. + \frac{(N_q + 1) n_{k+q} (1 - n_k)}{\hbar^2 / 2m^* \cdot (k^2 - (k+q)^2) + \hbar s q} \right\}. \end{aligned} \quad (3.15)$$

We can rearrange this as follows. Putting $\sigma = 2m^* s / \hbar$, we get

$$\begin{aligned} (N_2)_{\nu\nu} &= \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_q \left[q \left\{ \frac{(N_q + 1) n_{k+q}}{k^2 - (k+q)^2 + \sigma q} + \frac{N_q n_{k-q}}{k^2 - (k-q)^2 - \sigma q} \right\} \right. \\ &\quad \left. - q \left\{ \frac{(N_q + 1) n_{k+q} n_k}{k^2 - (k+q)^2 + \sigma q} + \frac{N_q n_{k-q} n_k}{k^2 - (k-q)^2 - \sigma q} \right\} \right]. \end{aligned}$$

Since the summation \sum_q is equivalent to \sum_{-q} , we can replace q by $-q$ in the second and the fourth terms. So, putting $k+q=k'$ ($q=k'-k$), we obtain

$$(N_2)_{vv} = \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \left[q \left\{ \frac{N_q + 1}{k^2 - k'^2 + \sigma q} + \frac{N_{-q}}{k^2 - k'^2 - \sigma q} \right\} n_{k'} \right. \\ \left. - q \left\{ \frac{N_q + 1}{k^2 - k'^2 + \sigma q} + \frac{N_{-q}}{k^2 - k'^2 - \sigma q} \right\} n_k n_{k'} \right].$$

In the second waved bracket, we can transform as

$$\frac{N_{-q} n_k n_{k'}}{k^2 - k'^2 - \sigma q} \rightarrow \frac{N_q n_k n_{k'}}{k'^2 - k^2 - \sigma q} = - \frac{N_q n_k n_{k'}}{k^2 - k'^2 + \sigma q}.$$

So we get finally

$$(N_2)_{vv} = \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} q \left\{ \frac{N_q + 1}{k^2 - k'^2 + \sigma q} + \frac{N_{-q}}{k^2 - k'^2 - \sigma q} \right\} n_{k'} \\ - \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q n_k n_{k'}}{k^2 - k'^2 + \sigma q}. \quad (3.16)$$

Hence we obtain*

$$\langle N_2 \rangle = A_0 + A_1 + A_2 \quad (3.17)$$

$$A_0 = \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q}{k'^2 - k^2 + \sigma q} f_k \quad (3.18)$$

$$A_1 = \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} q \left(\frac{1}{k'^2 - k^2 + \sigma q} + \frac{1}{k'^2 - k^2 - \sigma q} \right) \bar{N}_q f_k \\ = \frac{4m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q (k'^2 - k^2)}{(k'^2 - k^2)^2 - \sigma^2 q^2} \bar{N}_q f_k \quad (3.19)$$

$$A_2 = - \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q f_k f_{k'}}{k'^2 - k^2 + \sigma q} \\ = - \frac{2m^* \Gamma_0^2}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q}{2} \left(\frac{1}{k'^2 - k^2 + \sigma q} + \frac{1}{k^2 - k'^2 + \sigma q} \right) f_k f_{k'} \\ = \frac{2m^* \Gamma_0^2 \sigma}{\beta \hbar^2} \sum_k \sum_{k'} \frac{q^2}{(k'^2 - k^2)^2 - \sigma^2 q^2} f_k f_{k'}. \quad (3.20)$$

These quantities can be evaluated in similar ways as in $\langle H_r \rangle$.

A_0 can be written as

$$A_0 = \left(\frac{2m^* \Gamma_0^2}{\beta \hbar^2} \right) \mathcal{Q} \int q dq \int \frac{f_k dk}{k'^2 - k^2 + \sigma q}, \quad \mathcal{Q} = \frac{2V^2}{(2\pi)^6} \\ |k' - k| = q \leq q_m \quad (3.21)$$

where the factor 2 before V^2 in \mathcal{Q} is due to the spin degeneracy of the electron. We can calculate the above integral using the properties of f_k at large $\beta\mu$ and taking the principal values when the denominator $k'^2 - k^2 + \sigma q$ becomes zero. (See Appendix I) It becomes

* Here we interchange k' and k .

$$A_0 = (2m^* \Gamma_0^2 / \beta \hbar^2) I_0 = 4F_0 \sigma_0 \mu_0 / 3N_A \beta \cdot I_0 \quad (3 \cdot 21')$$

$$I_0 = 8V^2 / (2\pi)^4 (1/4\nu)^{4/3} \bar{k} \cdot k_0^4 \cdot P$$

$$\therefore A_0 = \frac{3}{2} N (\mu_0 / \beta) F_0 (\sigma_0 / k_0) (\bar{k} / k_0) (1/4\nu)^{1/3} P \quad (3 \cdot 22)$$

where $\sigma_0 = 2m^* s_0 / \hbar$ and $\nu N_A = N$ (ν is the number of conduction electrons per atom) are used, and P is a dimensionless factor of the order of unity as shown in the Appendix I. \bar{k} is defined by the relation $\mu = \hbar^2 \bar{k}^2 / 2m^*$.

Similarly we obtain

$$A_1 = (4m^* \Gamma_0^2 / \beta \hbar^2) I_1 = \frac{4}{3} \pi^4 N_A (\mu_0 / \beta) F_0 (\sigma_0 \bar{k} / q_m^2) (\kappa T / \hbar s q_m)^4$$

$$I_1 = Q \int q \bar{N}_q dq \int \frac{k'^2 - k^2}{(k'^2 - k^2)^2 - \sigma^2 q^2} f_k dk \quad (3 \cdot 23)$$

(Appendix II), and

$$A_2 = (2m^* \Gamma_0^2 \sigma / \beta \hbar^2) I_2 = (4\mu_0 \sigma F_0 \sigma_0 / 3N_A \beta) I_2$$

$$I_2 = Q \int dk' \int dk \frac{q^2 f_k f_{k'}}{(k'^2 - k^2)^2 - \sigma^2 q^2} \quad (3 \cdot 24)$$

$$q = |k' - k| \leq q_m$$

The evaluation of the integral I_2 is very troublesome, but, by an approximate evaluation in the Appendix IV, we know that this has the form

$$A_2 = -\frac{3}{2} \frac{N}{\beta} \nu F_0 \frac{\sigma \sigma_0}{k_0^2} \mu_0 Q. \quad (3 \cdot 25)$$

Q is dimensionless and is given in the Appendix IV, (A4.5).

Now we get

$$\Xi_2 = \Xi_0 \hat{\Xi}_2 = \Xi_0 \{ \frac{1}{2} \hat{\Xi}_1^2 + \beta^2 \langle N_2 \rangle \} \quad (3 \cdot 26)$$

and then

$$C_2 = \beta^2 \langle N_2 \rangle = \beta^2 (A_0 + A_1 + A_2).$$

§ 4. Free energy of the system of conduction electrons

From the results in the previous section, we know the expression for the grand partition function of our electron-phonon system up to the J^2 term.

$$\begin{aligned} \Xi &= \Xi_0 \exp(-\beta \langle H_r \rangle + \beta^2 \langle N_2 \rangle) \\ &= \Xi_0 \exp(-\beta (R_0 + R_1) + \beta^2 (A_0 + A_1 + A_2)) \end{aligned} \quad (4 \cdot 1)$$

From this Ξ , we derive three equations by which m^* , s and μ are determined.

$$-\beta R_0 + \beta^2 A_0 = 0, \quad (4 \cdot 2)$$

$$-\beta R_1 + \beta^2 A_1 = 0, \quad (4 \cdot 3)$$

$$N = \lambda \frac{\partial}{\partial \lambda} \log \Xi, \quad \Xi = \Xi_0 \exp(\beta^2 A_2). \quad (4 \cdot 4)$$

Although these are simultaneous equations, we can estimate α (or m^*) from (4.2), s from (4.3) and μ from (4.4).

We begin with the equation (4.3). This equation gives the condition that in $-\beta\langle\mathbf{H}_r\rangle + \beta^2\langle\mathbf{N}_q\rangle$ all the terms which contain N_q should reduce to zero and it corresponds to the condition in Fröhlich's theory which states that in the expression of the total energy the sum of the coefficients of each $b_q^+b_q$ other than those in \mathbf{H}_0 reduce to zero. In consequence, the phonon field is separated formally from the electron field and the phonon part in Ξ has the ideal gas form except that s appears instead of s_0 . In other words, the condition (4.3) plays the role of the renormalization condition of the sound velocity which has been used by Fröhlich⁽⁶⁾ and other authors.^{(4),(5)} Although they have established that condition for each normal mode, our condition is established in an averaged form. Substituting (3.10) and (3.23) into (4.3), we get approximately

$$(s_0 - s)/s_0 = 4F_0(\sigma_0\bar{k}/q_m^2)\mu_0/\hbar s_0 q_m = 2\nu F_0(m^*/m)(\bar{k}/k_0). \quad (4.5)$$

Next, the equation (4.2) also states that the sum of terms which contain single f_k should vanish. By this equation the parameter α is determined, that is, the relation between m^* and m is given. From (3.8), (3.22) and (4.2) we get

$$\alpha = \frac{5}{2}(1/4\nu)^{1/3}F_0(\sigma_0/k_0)(\mu_0^*/\mu)^2(1+\alpha)\cdot P. \quad (4.6)$$

As will be shown below, the value of μ is very near to that of μ_0^* . m^* and s may be taken for the observed effective mass and the observed sound velocity. Naturally there will be some doubts on these interpretations, because the elimination of the terms other than $\beta^2 A_2$ in the exponential of the equation (4.1) can be effected in some different ways. But we think that the condition (4.3) and hence (4.2) are the natural extension of the idea of the renormalization method in the quantum theory of fields to the statistical mechanical treatment, and the fact that the order of the magnitude of terms in (4.2) are different from those in (4.3) ($\beta R_0, \beta^2 A_0, \beta R_1, \beta^2 A_1$) may offer some supports for the above proposition.

Then, considering the fact that μ_0^* is the order of several electron volts and the sound velocity is the order of a few thousand meters per second in ordinary metals, we know that

$$\alpha \text{ is the order of } 10^{-2} \sim 10^{-3}$$

and the equation for $s_0 - s/s_0$ is essentially the same as that of Fröhlich,

$$(s_0 - s)/s_0 = 2\nu F_0.$$

So we can introduce a new coupling constant F by the relation

$$sF = s_0 F_0 \quad (4.7)$$

as has been done by Fröhlich.* We use this F in all equations hereafter.

* Following Fröhlich we interpret this F as the really acting coupling constant which one can estimate from the conductivity at the room temperature.

The equation (4.4) becomes

$$N = \sum_k \lambda e^{-\beta \varepsilon_k} / (1 + \lambda e^{-\beta \varepsilon_k}) + \beta^2 \lambda \frac{\partial}{\partial \lambda} A_2.$$

So we get

$$N = N(\mu/\mu_0^*)^{3/2} (1 + \pi^2/8 \cdot (\kappa T/\mu)^2 + \dots) + n(\mu) \quad (4.8)$$

$$n(\mu) = \beta^2 \lambda \frac{\partial}{\partial \lambda} A_2 = (8F\sigma^2/3N_A) \mu_0 \beta J_2 \quad (4.9)$$

$$J_2 = \Omega \int dk \int dk' G(k, k') f_k g_{k'}$$

$$G(k, k') = q^2 / \{ (k'^2 - k^2)^2 - \sigma^2 q^2 \}, \quad q = |k' - k| \leq q_m \quad (4.10)$$

$$g_{k'} = \lambda \frac{\partial}{\partial \lambda} f_{k'} = \frac{1}{(1 + e^{\beta(\varepsilon_{k'} - \mu)}) (1 + e^{-\beta(\varepsilon_{k'} - \mu)})}. \quad (4.11)$$

If $n(\mu)$ is zero, the equation (4.8) yields the ideal gas formula for μ . Here we can assume $n(\mu)/N \ll 1$ and solve this equation by the iteration method. To the first order in F (to the second order in Γ_0), we obtain

$$\mu = \mu_0^* - (\pi^2/12) (\kappa T)^2 / \mu_0^* - a(\mu_0^*) \quad (4.12)^*$$

$$a(\mu_0^*) = \frac{2}{3} \mu_0^* n(\mu_0^*) / N = (16\nu F \sigma^2 / 9N^2) \mu_0 \beta J_2^0 \quad (4.13)$$

where J_2^0 means that in J_2 , μ is replaced by μ_0^* . The evaluation of J_2^0 is given in Appendix III and we obtain, in the approximation of low temperature,

$$\begin{aligned} J_2^0 &= -\Omega (8\pi^2/3) (k_0^4/\beta \mu_0^*) (1/4\nu) S_2 \\ S_2 &= 1 + \frac{3}{4} (1/4\nu)^{1/3} + \frac{3}{8} (1/4\nu)^{2/3} + \dots \end{aligned} \quad (4.14)$$

Then we have

$$a(\mu_0^*) = -\frac{1}{6} F (\sigma/k_0)^2 \mu_0^* S_2. \quad (4.15)$$

After the above preparations, we can calculate the free energy $\Phi^{(e)}$ of the system of the conduction electrons as follows. Because of the elimination of the direct coupling between the electron field and the phonon field, $\Phi^{(e)}$ is given by

$$\begin{aligned} \Phi^{(e)} &= N\mu - \kappa T \log \Xi_0^{(e)} - \beta A_2 \\ \Xi_0^{(e)} &= \prod_k (1 + \lambda e^{-\beta \varepsilon_k}) \end{aligned} \quad (4.16)$$

and for $\kappa T \log \Xi_0^{(e)}$ we use

$$\begin{aligned} \kappa T \log \Xi_0^{(e)} &= \frac{3}{2} \kappa T N (1/\mu_0^*)^{3/2} \int_0^\infty \log(1 + \lambda e^{-\beta \varepsilon}) \sqrt{\varepsilon} d\varepsilon \\ &= \frac{2}{5} N \mu (\mu/\mu_0^*)^{3/2} (1 + (5\pi^2/8) (\kappa T/\mu)^2 + \dots). \end{aligned} \quad (4.17)$$

* In this formula, we have neglected the terms of the order $(\kappa T/\mu_0^*)^2 \cdot (\sigma/k_0)^2$.

Substituting the value of μ in (4.16), we get to the first order in F ,

$$\phi^{(e)}/N = \phi^{(e)} = \frac{3}{5}\mu_0^* - (\pi^2/4)(\kappa T)^2/\mu_0^* + \Delta\phi^{(e)} \quad (4.18)$$

$$\Delta\phi^{(e)} = -\beta A_2^0/N = -\frac{4}{3}\nu F\sigma^2/N^2 \cdot \mu_0 I_2^0 \quad (4.19)$$

where I_2^0 means to put $\mu = \mu_0^*$ in I_2 . We evaluate this integral in Appendix IV, and get

$$\begin{aligned} I_2^0 &= -24\pi^2 k_0^4 Q \\ Q &= \frac{5}{2}\gamma^4 + \gamma^3 - \frac{1}{2}\gamma^2 - \gamma - \log(1-\gamma) + \gamma^4 \log \frac{1-\gamma}{\gamma} \\ &\quad + 2\gamma^4 \log(k_*^2/\sigma q_m) \\ \gamma &= q_m/2k_0 = (1/4\nu)^{1/3}. \end{aligned} \quad (4.20)$$

The largest term in Q is $2\gamma^4 \log(k_*^2/\sigma q_m)$. The above value of I_2^0 is for 0°K and later we will discuss the temperature variation of this quantity.

Then finally, we get

$$\phi^{(e)} = \frac{3}{5}\mu_0^* - (\pi^2/4)(\kappa T)^2/\mu_0^* + \frac{3}{2}\nu F(\sigma/k_0)^2 \mu_0 Q. \quad (4.21)$$

In this formula, the influence of the electron-phonon interaction appears in μ_0^* and the last term, as can be seen easily.

§ 5. Discussions

Since our method of treatment is essentially a perturbation procedure, the convergence of the series $\sum J^i C_i$ is of essential importance. But, unfortunately, we cannot discuss the convergency of this series here. The evaluation of higher order terms is very troublesome and we can only infer that C_3 is zero and C_4 will give terms of the order $(F(\sigma/k_0))^2$ and $(F(\sigma/k_0)^2)^2$ in ordinary cases. But, by our present treatment, one can clearly see the roles of various terms in the second order approximation. In the cases of superconducting metals, the above series will show some divergent characters as was pointed out by Matsubara¹¹⁾.

In the cases where the convergence of our series can reasonably be assumed, the influence of the electron-phonon interaction on the thermal properties of the electron gas may be approximated by our free energy formula (4.21). The last term of this formula is positive and we see that to our approximation the interaction between electrons caused by the electron-phonon interaction is of repulsive nature.

When we calculate the electronic specific heat from (4.21), the temperature dependence of m^* and s must be carefully taken into consideration and we want to discuss it in the next paper.

Fröhlich¹⁾ has shown that the above mentioned repulsive interaction could be converted to the attractive one if we used a shell structure distribution in k -space as a trial function, and obtained a criterion for the appearance of this special state at 0°K . As he used the Hamiltonian in the second order perturbation approximation, the questions on the convergence

of his procedure occurred. But we think that the problem of the convergence of his variational procedure may have some different features compared with the problem of the convergence of our perturbation series in the grand partition function. Concerning the possibilities of deducing a phase transition in our system of conduction electrons by the present treatment, we want to make some trials in another paper.

In conclusion, the author expresses his sincere thanks to Professors M. Toda, R. Kubo, T. Matsubara, T. Yamamoto, Y. Kitano and F. Nakano, and Prof. A. Harasima in this Institute for their valuable discussions and criticisms.

Appendix I. The evaluation of A_0

$$A_0 = (2m^* \Gamma_0^2 / \beta \hbar^2) I_0$$

$$I_0 = \Omega \int dq \int dk \frac{q f_k}{(k'^2 - k^2) + \sigma q}$$

$$q = |\mathbf{k}' - \mathbf{k}| \leq q_m \quad (\text{A1} \cdot 1)$$

Introducing the angle θ as in Fig. 1, we can transform the integral I_0 as

$$\begin{aligned} I_0 &= 2\pi \Omega \int dk f_k \int_0^{q_m} q^2 dq \int_0^\pi \frac{d \cos \theta}{q + \sigma - 2k \cos \theta} \\ &= 4\pi^2 \Omega \int_0^\infty k dk f_k \int_0^{q_m} q^2 \log \left| \frac{q + \sigma + 2k}{q + \sigma - 2k} \right| dq. \end{aligned} \quad (\text{A1} \cdot 2)$$

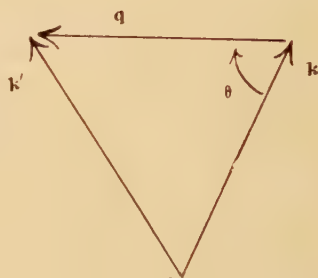


Fig. 1.

Then we use the usual approximal method in the case of large $\beta \mu$ in f_k ,

$$\int_0^\infty \Phi(\varepsilon) f(\varepsilon) d\varepsilon = \int_0^\mu \Phi(\varepsilon) d\varepsilon + \frac{\pi^2}{6} (\kappa T)^2 \left(\frac{\partial \Phi}{\partial \varepsilon} \right)_\mu + \dots$$

and retain only the first term. Because our βA_0 is already small (of the order $F(\sigma/k_0)$ compared with $N\mu_0^*$) and the neglected terms are of the order $F(\sigma/k_0) (\kappa T/\mu_0^*)^2$, they are smaller than the terms of the order $F(\sigma/k_0)^2$ and $(\kappa T/\mu_0^*)^2$ which will be retained in our calculations.

Hence we have

$$I_0 = 4\pi^2 \Omega \int_0^{q_m} q^2 dq \int_0^{\bar{k}} k \log \left| \frac{2k + q + \sigma}{2k - q - \sigma} \right| dk. \quad (\text{A1} \cdot 3)$$

Using the principal values where $2k - q - \sigma = 0$ holds, we get to the order of (σ/k_0) ,

$$\begin{aligned} I_0 &= 4\pi^2 \Omega \int_0^{q_m} \left\{ \left(\frac{\bar{k}^2}{2} - \frac{1}{8} (q + \sigma)^2 \right) \log \left| \frac{2\bar{k} + q + \sigma}{2k - q - \sigma} \right| + \frac{1}{2} \bar{k} (q + \sigma) \right\} q^2 dq \\ &= 4\pi^2 \Omega \int_0^{q_m} dq \left\{ \frac{1}{2} \bar{k} q^3 + \frac{1}{8} ((2\bar{k})^2 q^2 - q^4) \log \left| \frac{q + 2\bar{k}}{q - 2\bar{k}} \right| \right\} \end{aligned} \quad (\text{A1} \cdot 4)$$

$$\begin{aligned}
& + \left(\frac{\sigma}{\bar{k}} \left[-\frac{1}{8} (2\bar{k}) q^3 \log_3 \left| \frac{q+2\bar{k}}{q-2\bar{k}} \right| + ((2\bar{k})^2 q^2 - q^4) \frac{1}{8} \bar{k} \left(\frac{1}{q+2\bar{k}} - \frac{1}{q-2\bar{k}} \right) \right] \right. \\
& \left. + O\left(\left(\frac{\sigma}{\bar{k}}\right)^2\right) \right\} \\
& = 4\pi^2 \Omega \bar{k}^5 \left[\left\{ 2\gamma^4 + \frac{4}{3} \gamma^3 \log \frac{1+\gamma}{1-\gamma} + \frac{4}{3} (\log(1-\gamma^2) + \gamma^2) \right. \right. \\
& \quad \left. \left. - \frac{4}{5} \log \frac{1+\gamma}{1-\gamma} + \frac{4}{5} \left(\log(1-\gamma^2) + \gamma^2 + \frac{1}{2} \gamma^4 \right) \right\} \right. \\
& \quad \left. + \left(\frac{\sigma}{\bar{k}} \right) \left\{ \frac{4}{3} \gamma^2 + 2 \left(\log \frac{1+\gamma}{1-\gamma} - 2\gamma \right) \right. \right. \\
& \quad \left. \left. - \left(\log \frac{1+\gamma}{1-\gamma} - 2\gamma - \frac{2}{3} \gamma^3 \right) - \gamma \log \frac{1+\gamma}{1-\gamma} \right\} \right] \\
& \quad \gamma = q_m / 2\bar{k}.
\end{aligned}$$

Putting

$$I_0 = 4\pi^2 \Omega \bar{k}^5 \cdot 4\gamma^4 \cdot P$$

and expanding in powers of $\gamma = q_m / 2\bar{k} = q_m / 2k_0 \cdot k_0 / k = (1/4\nu)^{1/3} < 1$ we get

$$A_0 = 4\mu_0 F_0 \sigma_0 I_0 / 3N_A \beta = \frac{3}{2} N (\mu_0 / \beta) F_0 (\sigma_0 / k_0) (1/4\nu)^{1/3} (k/k_0) \cdot P, \quad (A1.5)$$

$$P = 1 - \frac{2}{3}\gamma^2 - \frac{1}{30}\gamma^4 - \dots + (\sigma/\bar{k}) \cdot 1/\gamma \cdot \left(\frac{2}{3} - \frac{1}{5}\gamma^2 + \frac{1}{21}\gamma^4 - \dots \right). \quad (A1.6)$$

Appendix II. The evaluation of A_1

$$\begin{aligned}
A_1 &= \frac{8\nu\mu_0 F_0 \sigma_0}{3N_A \beta} I_1 \\
I_1 &= \Omega \int q \bar{N}_q dq \int f_k dk \frac{(k'^2 - k^2)}{(k'^2 - k^2)^2 - \sigma^2 q^2} \\
q &= |\mathbf{k}' - \mathbf{k}| \leq q_m
\end{aligned} \quad (A2.1)$$

Using the relation

$$\bar{N}_q = \frac{e^{-\beta \hbar \epsilon q}}{1 - e^{-\beta \hbar \epsilon q}} = \sum_{n=1}^{\infty} e^{-n\beta \hbar \epsilon q}, \quad (A2.2)$$

we can proceed as in Appendix I.

$$\begin{aligned}
I_1 &= 2\pi^2 \Omega \int_0^{\infty} k f_k dk \sum_{n=1}^{\infty} \int_0^{\infty} q^2 e^{-n\beta \hbar \epsilon q} dq \\
&\quad \times \left\{ \log \left| \frac{q+2k+\sigma}{q-2k-\sigma} \right| + \log \left| \frac{q+2k-\sigma}{q-2k+\sigma} \right| \right\} \\
&\doteq 2\pi^2 \Omega \sum_{n=1}^{\infty} \left[\int_0^{q_m} q^2 e^{-n\beta \hbar \epsilon q} dq \int_0^{\bar{k}} k dk \left\{ \begin{array}{c} \text{ } \end{array} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= 2\pi^2 \Omega \sum_{n=1}^{\infty} \int_0^{q_m} \bar{q}^2 e^{-n\gamma \bar{q} s q} d\bar{q} \\
&\times \left[\frac{1}{8} (4\bar{k}^2 - q^2 - \sigma^2) \log \left| \frac{(2\bar{k} + q + \sigma)(2\bar{k} + q - \sigma)}{(2\bar{k} - q - \sigma)(2\bar{k} - q + \sigma)} \right| \right. \\
&\quad \left. - \frac{1}{4} \sigma q \log \left| \frac{(2\bar{k} + q + \sigma)(2\bar{k} - q + \sigma)}{(2\bar{k} + q - \sigma)(2\bar{k} - q - \sigma)} + q\bar{k} \right| \right] \\
&= 2\pi^2 \Omega \sum_{n=1}^{\infty} \int_0^{q_m} e^{-n\gamma \bar{q} s q} d\bar{q} \left\{ 2\bar{k}q - \frac{1}{6} \frac{q^3}{\bar{k}} - \frac{1}{120} \frac{q^5}{\bar{k}^3} + \cdots + O\left(\frac{\sigma}{\bar{k}}\right) \right\}.
\end{aligned} \tag{A2.3}$$

We retain only the first term in the waved bracket of the last member, so it becomes approximately

$$\begin{aligned}
I_1 &= 2\pi^2 \Omega 6 \left(\sum_1^{\infty} 1/n^4 \right) \bar{k} q_m^6 / q_m^2 (\kappa T / \hbar s q_m)^4 \\
&= 3/10 \cdot \pi^4 N_A^2 (\bar{k} / q_m^2) (\kappa T / \hbar s q_m)^4,
\end{aligned} \tag{A2.4}$$

here the relation $\sum_1^{\infty} 1/n^4 = \pi^4/90$ is used. Hence we obtain the equation (3.23)

$$A_1 = 4/5 \cdot \pi^4 N_A (\mu_0 / \beta) F_0(\sigma_0 \bar{k} / q_m^2) (\kappa T / \hbar s q_m)^4. \tag{A2.5}$$

Appendix III. The evaluation of J_2^0

$$\begin{aligned}
J_2^0 &= \Omega \int d\mathbf{k} \int d\mathbf{k}' G(\mathbf{k}, \mathbf{k}') f_{k'}^0 g_k^0, \\
q &= |\mathbf{k}' - \mathbf{k}| \leq q_m, \\
G(\mathbf{k}, \mathbf{k}') &= \frac{q^2}{(k'^2 - k^2)^2 - \sigma^2 q^2},
\end{aligned} \tag{A3.1}$$

where $f_{k'}^0, g_k^0$ are the functions f_k, g_k in which $\mu = \mu_0^*$ is substituted. Using the angle θ as in Fig. 1, we have

$$\begin{aligned}
J_2^0 &= 8\pi^2 \Omega \int_0^{\infty} k^2 g_k dk \int_0^{q_m} \bar{q}^2 d\bar{q} \int_{-1}^1 \frac{d \cos \theta}{(q - 2k \cos \theta)^2 - \sigma^2} f_{k'} \\
&= 8\pi^2 \Omega \frac{k_*^3}{2\beta \mu_0^*} \int_{-\beta \mu_0^*}^{\infty} \left(1 + \frac{x}{\beta \mu_0^*} \right)^{1/2} g(x) dx \\
&\quad \times \int_0^{q_m} \bar{q}^2 d\bar{q} \int_{-1}^1 \frac{1}{(q - 2kz)^2 - \sigma^2} \frac{1}{1 + \exp(\beta a(k'^2 - k_*^2))} dz \\
g(x) &= 1/(1 + e^x)(1 + e^{-x})
\end{aligned} \tag{A3.2}$$

where $\beta a(k^2 - k_*^2) = x$, $a = \hbar^2/2m^*$, and $\hbar^2 k_*^2/2m^* = \mu_0^*$ are used.

$g(x)$ has the δ -function type property at the origin and $-\beta \mu_0^*$ can be considered as $-\infty$. So we get approximately,

$$\begin{aligned}
J_2^0 &\doteq 8\pi^2 \Omega \frac{k_*^3}{2\beta\mu_0^*} \int_0^{q_m} q^2 dq \int_{-1}^1 dz \frac{1}{(q-2k_*z)^2 - \sigma^2} \frac{1}{1 + \exp(\beta a(k'^2 - k_*^2))} \quad (A3.3) \\
&= 8\pi^2 \Omega \frac{k_*^3}{2\beta\mu_0^*} \frac{1}{4\sigma} \int_0^{q_m} q^2 dq \{ \} , \\
\{ \} &= \left\{ \log \left| \frac{q+2k_*-\sigma}{q+2k_*+\sigma} \right| \frac{1}{1 + \exp(\beta a(q+2k_*)q)} \right. \\
&\quad \left. - \log \left| \frac{q-2k_*-\sigma}{q-2k_*+\sigma} \right| \frac{1}{1 + \exp(\beta a(q-2k_*)q)} \right\} ,
\end{aligned}$$

here the neglected terms are of the order $(\kappa T/\mu_0^*)^2$ compared with the above.

At sufficiently low temperatures, we can use the approximations

$$\begin{aligned}
1/\{1 + \exp(\beta a(q+2k_*)q)\} &\doteq 0, \quad 1/\{1 + \exp(\beta a(q-2k_*)q)\} \doteq 1 \\
(0 < q < q_m < 2k_*)
\end{aligned}$$

So we obtain

$$\begin{aligned}
J_2^0 &\doteq -8\pi^2 \Omega \frac{k_*^2}{4\beta\mu_0^*} \frac{1}{2\sigma} \int_0^{q_m} q^2 \log \left| \frac{q-2k_*-\sigma}{q-2k_*+\sigma} \right| dq \quad (A3.4) \\
&= -8\pi^2 \Omega \frac{k_*^4}{4\beta\mu_0^*} \left\{ 4 \left(\log \frac{2k_*}{2k_*-q_m} - \frac{q_m}{2k_*} + \frac{1}{2} \left(\frac{q_m}{2k_*} \right)^2 \right) \right. \\
&\quad \left. + \left(\frac{\sigma}{k_*} \right)^2 \left(\log \frac{2k_*}{2k_*-q_m} - \frac{q_m}{2k_*-q_m} + \frac{1}{2} \frac{q_m^2}{(2k_*-q_m)^2} \right) \right. \\
&\quad \left. + \dots \right\} ,
\end{aligned}$$

hence

$$\begin{aligned}
J_2^0 &\doteq -8/3 \cdot \pi^2 \Omega (k_0^4/\beta\mu_0^*) (1/4\nu) S_2 , \\
S_2 &= (1 + 3/4 (q_m/2k_*) + 3/5 (q_m/2k_*)^2 + \dots) \\
&\doteq (1 + 3/4 (1/4\nu)^{1/3} + 3/5 (1/4\nu)^{2/3} + \dots) . \quad (A3.5)
\end{aligned}$$

From the course of our calculations we can easily see that the temperature dependence of J_2^0 is of the form

$$J_2^0 = (J_2^0)_0 (1 + C(\kappa T/\mu_0^*)^2 + \dots) . \quad (A3.6)$$

Appendix IV. The evaluation of A_2^0

The evaluation of A_2^0 is very troublesome and we are forced to use its form at 0°K from the beginning.

$$\frac{\beta A_2^0}{N} = \frac{4\nu F \sigma^2}{3N^2} \mu_0 I_2^0 = \frac{4\nu F \sigma^2}{3N^2} \mu_0 (I_2^0)_0$$

$$(I_2^0)_0 = 8\pi^2 Q \int_0^{k_*} k^2 dk \int_0^{q_m} q^2 dq \int_{z_0}^1 \frac{dz}{(q-2kz)^2 - \sigma^2} \quad (\text{A4} \cdot 1)$$

$$k_*^2 = k^2 + q^2 - 2kqz_0.$$

After the z -integration, we perform the k -integration and next the q -integration. Hence we have

$$(I_2^0)_0 = 4\pi^2 Q \frac{1}{2\sigma} \int_0^{k_*} k dk \int_0^{q_m} q^2 dq$$

$$\times \log \left| \frac{(2k-q-\sigma)(k^2-k_*^2+\sigma q)}{(2k-q+\sigma)(k^2-k_*^2-\sigma q)} \right|$$

$$= 4\pi^2 Q \left[\frac{1}{8} k_*^2 q_m^2 + \frac{1}{2} k_*^3 q_m + k_*^4 \log \frac{2k_* - q_m}{2k_*} \right.$$

$$\left. - \frac{5}{32} q_m^4 - \frac{3}{24} k_* q_m^3 - \frac{1}{16} q_m^4 \log \frac{2k_* - q_m}{2k_*} + \frac{1}{8} q_m^4 \log \frac{\sigma q_m}{k_*^2} \right], \quad (\text{A4} \cdot 2)$$

here we have used the principal values where $2k-q \pm \sigma = 0$ and $k^2 - k_*^2 \pm \sigma q = 0$ hold.

Then we have

$$(I_2^0)_0 = 4\pi^2 Q k_*^4 Q \quad (\text{A4} \cdot 3)$$

$$\beta A_2^0 N = \frac{3}{2} \nu F (\sigma/k_0)^2 \mu_0 Q \quad (\text{A4} \cdot 4)$$

$$Q = \left\{ \frac{5}{2} \gamma^4 + \gamma^3 - \frac{1}{2} \gamma^2 - \gamma - \log(1-\gamma) \right.$$

$$\left. + \gamma^4 \log((1-\gamma)/\gamma) + 2\gamma^4 \log(k_*^2/\sigma q_m) \right\} \quad (\text{A4} \cdot 5)$$

$$\gamma = q_m/2k_* = q_m/2k_0 = (1/4\nu)^{1/3}.$$

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On a Regular Formulation of Quantum Field Theory, I

— Non Relativistic Theory —

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A method of regular formulation of non relativistic quantum field theory is presented, using the well defined regular quantities and the rigorous analysis only.

§ 1 Introduction

The difficulties appearing in the ordinary quantum field theory seem to arise from the following two defects: the physical defect that we have no reasonable formulation in extended particle aspects and the mathematical defect that we have not any correct method to deal with ordinary singular functions at the singular points themselves. If one of these defects can be removed, the difficulties will be relieved to a great extent.

About the mathematical defect, the Dirac's delta symbol, for example, has an extraordinary nature as a function and its mathematically rigorous meaning is a measure rather than a function. The distribution analysis, introduced by Schwartz¹⁾, offers a rigorous method to treat such extraordinary quantities correctly. In order to formulate the field theory rigorously, it seems to be necessary to try its distribution analytical formulation. However ordinary products and definite integrals of distributions are not well defined in general. So a straightforward translation of ordinary field theory will be difficult, as pointed out in our previous report²⁾. Recently Güttinger³⁾, Stueckelberg and Petermann⁴⁾ and Jordan⁵⁾ developed interesting methods by using the distribution analysis. But there are some obscure points which seem to arise from the lack of rigorous multiplications. Indeed it is shown by mathematicians⁶⁾ that the products can be defined merely with respect to the extremely restricted class of distributions and that if we use projections of products on the distribution space as substitutes for the products they are neither transitive, commutative nor unambiguous.

About the physical defect, we must use the delta type quantization method in order to gain reasonable results, even if we cannot perform point type observations of any kind. In our opinion, all the theories based upon the point particle aspects seem to treat only the limiting cases. They are approximations to the phenomena played by extended objects, as a planet was approximated as a point in the classical dynamics. But such formulation of the quantized field theory is not easy, if we use the ordinary mathematics such as the function analysis.

In the present report we will give a method of such formulation by using the dis-

tribution analysis. The physical laws are described by distribution equations. Observable quantities are described by the values of distributions at a certain function called a testing function and this will give an extension to the point concept attributed to the position of a particle. About the multiplication two sorts of products are introduced. By using them properly, it is shown that the ordinary theory can be reformulated exactly and all the results have the definite meanings of finite values, though in this paper it is restricted in the non relativistic theory. In the limit of the delta type testing function the whole system reduces to the ordinary one and all the usual divergence difficulties occur.

This method will give a better approximation to the complicated structure of the nature at small distances, which may come from various origins such as non locality for instance, rather than the ordinary point particle aspect. Moreover this will mean a relief of the ordinary field theory in both physical and mathematical meanings, using well defined quantities and rigorous mathematics only without any manipulations such as cut-off, damping factor or renormalization by infinity.

§ 2. Distributional representation of physical quantities

In the ordinary quantum theory of wave fields, physical quantities are represented by functions—involving operators, which operate on functions, and matrices, whose elements are functions—of space time point x . In our distribution formulation all physical quantities should be represented by distributions. The distribution is defined as a linear functional of the functions involved in a well defined class in the following way: the values of distribution F attributed to f at the function φ is

$$F[\varphi] = \int f(\hat{z}) \varphi(\hat{z}) d\hat{z} \quad (\text{Lebesgue integral}), \quad (2.1)$$

where φ is an element of a certain well defined class, F is called as a distribution on that class and the distribution equation is a relation holding identically with respect to arbitrary φ in the class. Thus in the case of distributions their values are determined with respect to the argument-functions, on the contrary to the case of functions where the values are determined with respect to the values of the argument-variables. In our case we must establish a method to assign a function to the point which was usually described by a number x and to assign a distribution to a physical quantity.

We define a class of φ_x involving a parameter x in such a way that all the quantities can be well defined as distributions on the class. In order to represent the quantity, which was described with function $f(x)$ in the usual theory, let us assign to it the distribution F and interpret the value

$$F[\varphi_x] = \int f(\hat{z}) \varphi_x(\hat{z}) d\hat{z} \quad (2.2)$$

as the value of the quantity with respect to the testing function φ_x . The physical law described by a distribution equation means a sort of universal relation holding without regard to any kind of observation φ_x .

As pointed out by T. Ishihara and others⁷⁾, the so-called singular functions appearing in the ordinary field theory can be treated as temperal distributions, namely distributions defined on the class \mathfrak{H} introduced by Schwartz¹⁾. So we can give a method to introduce the class of φ_x . The so-called carrier space $\hat{\xi}$ is assumed to have the same structure as the space x . The class is defined as those composed of functions satisfying the conditions:

- (i) It is differentiable infinitely many times,
- (ii) The values of itself and all of its differential coefficients converge to zero rapidly enough in the limit of infinite $\hat{\xi}$, so that

$$\lim_{\xi \rightarrow \infty} \hat{\xi}^n \varphi(\hat{\xi}) = \lim_{\xi \rightarrow \infty} \hat{\xi}^n \varphi'(\hat{\xi}) = \lim_{\xi \rightarrow \infty} \hat{\xi}^n \varphi^{(j)}(\hat{\xi}) = \dots = 0 \quad (2.3)$$

The wave equations are given as distribusional equations on \mathfrak{H} , and the physical values are regarded as the values at the special testing functions $\varphi_x(\hat{\xi})$. The class of $\varphi_x(\hat{\xi})$ can be defined as those composed of such elements of \mathfrak{H} that satisfy the following conditions:

- (iii) It has an invariant meaning with respect to coordinate transformation of the space.
- (iv) It is normalized so that the value of the integration over the space becomes 1,

$$\int \varphi_x(\hat{\xi}) d\hat{\xi} = 1. \quad (2.4)$$

- (v) It has a parameter, say l , and approaches to the Dirac's delta measure in the limit of vanishing l

$$\lim_{l \rightarrow 0} \varphi_x^l(\hat{\xi}) = \delta(\hat{\xi} - x). \quad (2.5)$$

An example of $\varphi_x(\hat{\xi})$ for one dimensional x , $\hat{\xi}$ is

$$\begin{aligned} \varphi_x(\hat{\xi}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \{ -l^2 \alpha^2 / 4 + i(\hat{\xi} - x) \alpha \} d\alpha \\ &= \exp \{ -(\hat{\xi} - x)^2 / l^2 \} / \sqrt{\pi} l. \end{aligned} \quad (2.6)$$

An example for the three dimensional Euclidean space is

$$\begin{aligned} \varphi_r(\hat{\xi}) &= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \exp \{ -l^2 (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) / 4 + i(u_1 \alpha_1 + u_2 \alpha_2 + u_3 \alpha_3) \} d\alpha_1 d\alpha_2 d\alpha_3 \\ &= \exp \{ -(u_1^2 + u_2^2 + u_3^2) / l^2 \} / (\sqrt{\pi} l)^3 = \exp \{ -(\hat{\xi} - \mathbf{x})^2 / l^2 \} / (\sqrt{\pi} l)^3 \\ &\quad (u_\lambda = \hat{\xi}_\lambda - x_\lambda). \end{aligned} \quad (2.7)$$

A non relativistic example for the four dimensional space time is

$$\varphi_{(x,t)}(\hat{\xi}, \tau) = \varphi_x(\hat{\xi}) \varphi_t(\tau), \quad (2.8)$$

where $\varphi_x(\hat{\xi})$ is the same as that of (2.7) and $\varphi_t(\tau)$ is the same as that of (2.6) with t , τ , l/c in places of x , $\hat{\xi}$, l respectively. Concerning relativistic cases we will discuss them in the next paper. Anyhow the quantity $F[\varphi_x(\hat{\xi})]$ approaches in the limit of $\varphi_x(\hat{\xi}) \rightarrow \delta(\hat{\xi} - x)$ to the original function $f(x)$:

$$F[\varphi_x(\hat{\xi})] \rightarrow \int f(\hat{\xi}) \delta(\hat{\xi} - x) d\hat{\xi} = f(x). \quad (2.9)$$

We will proceed in this way so that the whole system reduces to the ordinary one in such a limit.

The products of two quantities which were described by functions f, g in the usual theory are represented in our case with products composed by some means from F, G corresponding to f, g . This may be given in the following two ways.

$$F \cdot G[\varphi] = \int f(\hat{\xi}) g(\hat{\xi}) \varphi(\hat{\xi}) d\hat{\xi}, \quad (2 \cdot 10)$$

$$F^{(1)} \times G^{(2)}[\varphi^{(1)} \varphi^{(2)}] = \int f(\hat{\xi}^{(1)}) g(\hat{\xi}^{(2)}) \varphi(\hat{\xi}^{(1)}) \varphi(\hat{\xi}^{(2)}) d\hat{\xi}^{(1)} d\hat{\xi}^{(2)}. \quad (2 \cdot 11)$$

These quantities approach both to the ordinary function products in the limit of $\varphi \rightarrow \delta$

$$F \cdot G[\varphi_x] \rightarrow \int f(\hat{\xi}) g(\hat{\xi}) \delta(\hat{\xi} - x) d\hat{\xi} = f(x) g(x), \quad (2 \cdot 12)$$

$$F \times G[\varphi_x \varphi_x] \rightarrow \int f(\hat{\xi}^{(1)}) g(\hat{\xi}^{(2)}) \delta(\hat{\xi}^{(1)} - x) \delta(\hat{\xi}^{(2)} - x) d\hat{\xi}^{(1)} d\hat{\xi}^{(2)} = f(x) g(x),$$

$$F \times G[\varphi_x \varphi_{x'}] \rightarrow \int f(\hat{\xi}^{(1)}) g(\hat{\xi}^{(2)}) \delta(\hat{\xi}^{(1)} - x) \delta(\hat{\xi}^{(2)} - x') d\hat{\xi}^{(1)} d\hat{\xi}^{(2)} = f(x) g(x').$$

About the properties of these products some consideration was given in our previous report²⁾. Of course the existence of $F \cdot G$ is limited, but $F \times G$ exists always and it is a special case of the direct product of F and G . For example in the case of Dirac's δ the product $\delta \cdot \delta$ has no meaning but the product $\delta \times \delta$ is meaningful:

$$\delta \times \delta[\varphi_1 \varphi_2] = \varphi_1(0) \varphi_2(0). \quad (2 \cdot 13)$$

The displacement character of δ as an integral kernel, such as $\int \delta(x - x') A(x') dx' = A(x)$ holds for $\delta \cdot A$ only, but not for $\delta \times A$. However from a distribution equation $F = 0$ we can deduce only

$$F \times G = 0 \quad (2 \cdot 14)$$

but not $F \cdot G = 0$ in general. These circumstances require us much carefulness in the course of distributional formulation of the theory.

The differentiation is defined as

$$D_\mu F[\varphi] = - \int f(\hat{\xi}) (\partial \varphi / \partial \hat{\xi}_\mu) d\hat{\xi}. \quad (2 \cdot 15)$$

For the products we have

$$D_\mu (F \cdot G)[\varphi] = - \int f(\hat{\xi}) g(\hat{\xi}) (\partial \varphi / \partial \hat{\xi}_\mu) d\hat{\xi}, \quad (2 \cdot 16)$$

$$\begin{aligned} D_\mu (F \times G)[\varphi^{(1)} \varphi^{(2)}] &= (D_\mu^{(1)} + D_\mu^{(2)}) (F^{(1)} \times G^{(2)}) [\varphi^{(1)} \varphi^{(2)}] \\ &= \{ (D_\mu^{(1)} F^{(1)}) \times G^{(2)} + F^{(1)} \times (D_\mu^{(2)} G^{(2)}) \} [\varphi^{(1)} \varphi^{(2)}] \\ &= - \int f(\hat{\xi}^{(1)}) g(\hat{\xi}^{(2)}) (\partial \varphi^{(1)} / \partial \hat{\xi}_\mu^{(1)} + \partial \varphi^{(2)} / \partial \hat{\xi}_\mu^{(2)}) d\hat{\xi}^{(1)} d\hat{\xi}^{(2)} \end{aligned} \quad (2 \cdot 17)$$

These differentiations approach to the ordinary differentiations of functions in the limit of $\varphi \rightarrow \delta$,

$$\begin{aligned} D_\mu F[\varphi_x] &= [-f(\hat{\xi})\varphi_x(\hat{\xi})]_{-\infty}^{+\infty} + \int \partial f(\hat{\xi}) / \partial \hat{\xi}_\mu \varphi_x(\hat{\xi}) d\hat{\xi} \\ &\rightarrow \int \partial f(\hat{\xi}) / \partial \hat{\xi}_\mu \delta(\hat{\xi} - x) d\hat{\xi} = \partial f(x) / \partial x_\mu, \\ D_\mu (F \cdot G)[\varphi_x] &\rightarrow \partial (f(x)g(x)) / \partial x_\mu, \\ D_\mu (F \times G)[\varphi_x \varphi_x] &\rightarrow \partial (f(x)g(x)) / \partial x_\mu. \end{aligned} \quad (2.18)$$

About the definite integral, we take, for it, the value of the distribution at the integrated testing function with respect to the parameter x . This coincides with the integration, with respect to x , of the value at a testing function involving x if exists. Of course such kind of integration exists only in the case where the integration of the testing function with respect to x has also the properties of convergence with respect to $\hat{\xi}$'s.

§ 3. Non relativistic quantum theory

In the case of non relativistic quantum theory for general field, our program to interpret all the field equations as distributional equations can be performed in the following way. In the usual theory the field equations can be derived in general from a Lagrangian function L , which is a function of field quantities $Q_\alpha(x)$ their space derivatives $Q_{\alpha;k} = \partial Q_\alpha(x) / \partial x_k$ and time derivatives $\dot{Q}_\alpha(x)$. In our case the quantities are distributions with respect to the three dimensional x space involving the time coordinate t as a parameter.

The distribution corresponding to the Lagrangian function is introduced as the corresponding distribution to L in which the product is taken with \times multiplication. For example the term corresponding to $(Q_\alpha)^2$ is $Q_\alpha^{(1)}(\hat{\xi}^{(1)})Q_\alpha^{(2)}(\hat{\xi}^{(2)})$.

Thus our variational principle is established as

$$\delta \int L(Q_\alpha^{(3)}(\hat{\xi}^{(3)}), Q_{\alpha;(k)}^{(3)}(\hat{\xi}^{(3)}), \dot{Q}_\alpha^{(3)}(\hat{\xi}^{(3)})) [\varphi_x(\hat{\xi})] d^4x = 0, \quad (3.1)$$

where β indicates the various $\hat{\xi}^{(\beta)}$ arising from the \times products and the notations (k) mean the distributional differentiations

$$Q_{\alpha;(k)}^{(\beta)}[\varphi] = D_k^{(\beta)} Q_\alpha^{(\beta)}[\varphi] \quad (3.2)$$

and lastly $\varphi_x(\hat{\xi})$ is defined as

$$\varphi_x(\hat{\xi}) = \prod_{\beta} \varphi_x^{(\beta)}(\hat{\xi}^{(\beta)}) \phi(\hat{\xi})$$

with arbitrary $\phi(\hat{\xi}) = \phi(\hat{\xi}^{(1)}, \dots, \hat{\xi}^{(\beta)}, \dots)$ in the class whose elements allow the convergence of $L[\varphi]$. The derivative $Q_{\alpha;(k)}^{(\beta)}$ has the same meaning as $Q_{\alpha;k}^{(\beta)}$ for the case of a continuous $Q_\alpha^{(\beta)}$.

The Euler equation of the principle (3.1) is

$$\{\partial L / \partial Q_\alpha^{(\beta)} - \sum_k D_k^{(\beta)} (\partial L / \partial Q_{\alpha;(k)}^{(\beta)}) - D_0^{(\beta)} (\partial L / \partial \dot{Q}_\alpha^{(\beta)})\} [\varphi_x] = 0. \quad (3.3)$$

This equation has a distributional meaning, and so we have obtained a distributional field equation.

By introducing the conjugate momentum $P_\alpha^{(3)}$ of $Q_\alpha^{(3)}$ we can develop the Hamiltonian formalism in the usual way, and we have the canonical equation

$$cD_0 Q_\alpha^{(3)} = \partial \bar{H} / \partial P_\alpha^{(3)}, \quad cD_0 P_\alpha^{(3)} = -\partial \bar{H} / \partial Q_\alpha^{(3)}, \quad (3.4)$$

where

$$\bar{H} = \int_{\mathbb{R}} H(\xi^{(3)}) (H\varphi_x^{(3)}) d^3x \quad (3.5)$$

$$\partial \bar{H} / \partial Q_\alpha^{(3)} = \partial H / \partial Q_\alpha^{(3)} - \sum_k D_k (\partial H / \partial Q_{\alpha; (k)}^{(3)}). \quad (3.6)$$

The quantization of the field can be performed by assuming the commutation relations between kernels $Q(\xi, t)$ and $P(\xi', t)$ at the same value of the parameter t

$$[Q_\alpha^{(3)}(\xi, t), P_{\alpha'}^{(3')}(\xi', t)] = i\hbar \partial_{\alpha\alpha'} \partial_{\xi\xi'} \partial(\xi - \xi'), \quad (3.7a)$$

$$[Q_\alpha^{(3)}(\xi, t), Q_{\alpha'}^{(3')}(\xi', t)] = 0, \quad (3.7b)$$

$$[P_\alpha^{(3)}(\xi, t), P_{\alpha'}^{(3')}(\xi', t)] = 0. \quad (3.7c)$$

These equations have rigorous meanings as relations between kernels, for ∂ -symbol has a rigorous meaning as an element of kernels. The distributional meaning of (3.7a) is

$$[Q_\alpha^{(3)}(t), \times P_{\alpha'}^{(3')}(t)] = i\hbar \partial_{\alpha\alpha'} \partial_{\xi\xi'} \partial^{(3)-(3')} \times 1^{(3)+(3')}, \quad (3.8)$$

where $\partial^{(3)-(3')}$ and $1^{(3)+(3')}$ are the delta symbol in the $(\xi^{(3)} - \xi^{(3')})/2$ space and the unity in the $(\xi^{(3)} + \xi^{(3')})/\sqrt{2}$ space respectively and the bracket means

$$[A, \times B] = (A \times B) - (B \times A). \quad (3.9)$$

However the expression such as (3.7a) is more familiar to us, so we will use it always hereafter.

In the physically meaningful cases H is a polynomial of Q and P , so we can derive the quantum mechanical equation of motion in the usual way by using the commutation relations and we have

$$i\hbar cD_0 Q_\alpha^{(3)} = [Q_\alpha^{(3)} \circ H] \quad (3.10a)$$

$$i\hbar cD_0 P_\alpha^{(3)} = [P_\alpha^{(3)} \circ \bar{H}] \quad (3.10b)$$

where the bracket means

$$[A^{(3)} \circ B^{(3')}] = (A^{(3)} \circ B^{(3')} - B^{(3')} \circ A^{(3)}) \quad (3.11)$$

$$A^{(3)} \circ B^{(3')} = 1/\sqrt{n} \cdot \sum_{(\gamma)} \partial^{(3)-(3')} A^{(3)} \cdot B^{(3')} \quad (3.12)$$

The summation \sum' being taken with respect to all r 's permitting the meaningful multiplication $\partial^{(3)-(3')} A^{(3)} B^{(3')}$ and n being the number of such γ 's. The relations as (3.10) do not hold with respect to the other sorts of bracket $[\times]$. The fact that we have no means to obtain the equation of motion by using the \times product and \bar{H} is

consistent with the Lehmann's investigation⁸⁾ which shows that the propagator does not become less singular if such type of equation of motion exists. Indeed, Heisenberg⁹⁾ gave up such equation of motion of the so-called Heisenberg's type in his trial of the non-linear field theory in the Hilbert space II.

The Schrödinger equation for the state vector Ψ of the quantized field is

$$i\hbar \partial \Psi / \partial t = \bar{H} \Psi. \quad (3 \cdot 13)$$

The meaning of this equation is the usual differential equation with respect to t . In our treatment t is a parameter involved in distributions and so the differentiation with respect to t is the usual one in the ordinary function analysis. Since the products in \bar{H} are taken always by means of the \times multiplication, the effect of the quantization occurs as the \times products of $\hat{\phi}$ symbols. Such products have rigorous meanings always, so we have no origin which might cause divergences in the Hamiltonian \bar{H} . When the limit $l \rightarrow 0$ is taken, the whole formulation reduces to the ordinary one or every equation becomes usual one and so all the ordinary divergence difficulties occur, the \times and \cdot products losing their peculiar meanings.

For example all the terms of finite order perturbations converge in our case. Since our equations have the ordinary meanings with respect to t , we can develop the perturbation theory exactly in the same way as the usual theory. In the matrix elements of the perturbing Hamiltonian H'_{ab} , the factors such as $\varphi(\hat{z}^{(1)})\varphi(\hat{z}^{(2)})\dots$ are involved and such factors become $\phi(k^{(1)})\phi(k^{(2)})\dots$ when the calculation of the matrix elements is performed in the k space. Since $\phi(k)$ has also the properties (i), (ii) in § 2 when $\varphi(\hat{z})$ has the same property, this factor gives a convergent result to the integration in the k -space. However in the limit of vanishing l , φ approaches to $\hat{\phi}$ and so its Fourier transform ϕ approaches to 1, thus in this limit the integral turns to the ordinary one and the divergence occurs.

§ 4. Non relativistic electrodynamics

In the ordinary theory, due to Fermi or Heisenberg-Pauli, of the quantized electromagnetic field interacting with electrons as point charges

$$\rho(\mathbf{r}) = \sum_s e_s \delta(\mathbf{r} - \mathbf{r}_s), \quad \mathbf{j}(\mathbf{r}) = \sum_s \mathbf{j}_s \delta(\mathbf{r} - \mathbf{r}_s) \quad (4 \cdot 1)$$

the well-known divergencies occur in the photon zero point energy $\sum_{\lambda} \hbar \omega_{\lambda} / 2$ and the Coulomb- and the transverse-electron self-energies $\sum_s e_s^2 / 2r_{ss}$, $1/137\pi\mu \cdot \int k dk$ (second order). In this section we will show how these cases can be formulated in our treatment and how these quantities gain definite finite values.

The fundamental field equations in this case are the Maxwell equations

$$\square A_0 = -4\pi\rho, \quad (4 \cdot 2)$$

$$\square \mathbf{A} = -4\pi\mathbf{j}/c. \quad (4 \cdot 3)$$

and the Lorentz condition

$$D_{\mu} A_{\mu} = 0 \quad (\mu = 0, 1, 2, 3). \quad (4 \cdot 4)$$

However it must be noticed that these equations in our case are distribution equations. The quantities such as $A_\mu = (A_0, \mathbf{A})$, ρ and \mathbf{j} are distributions and the differential operators D_μ and $[\] = D^\mu D_\mu$ are the distributional ones. The analytical meanings of these distributional equations may be obvious from the definition of § 2 and they can be described as relations between integrals involving φ . For example the singular particle density $\rho_s = e_s \delta(\mathbf{r} - \mathbf{r}_s)$ in the ordinary point particle aspect becomes a continuously extended density

$$\begin{aligned} \rho_s[\varphi_r] &= \int e_s \delta(\mathbf{r}' - \mathbf{r}_s) \varphi_r(\mathbf{r}') d\mathbf{r}' \\ &= e_s \varphi_r(\mathbf{r}_s) \end{aligned} \quad (4.5)$$

in our treatment in the extended particle aspect.

The solutions of our equations are also distributions as a matter of course. For example, as a spherically symmetrical solutions of the electrostatic field due to a single charge at the origin we have, as shown by L. Schwartz, the distribution¹⁾

$$A_0[\varphi] = e/r[\varphi] \quad (4.6)$$

corresponding to the Coulomb potential and its value at φ_r is

$$A_0[\varphi_r] = \int e/r' \varphi_r(\mathbf{r}') d\mathbf{r}' \quad (4.7)$$

which has finite values everywhere on account of the radial part $(r')^{-2} dr'$ of $d\mathbf{r}'$; the value at the origin $\mathbf{r} = 0$ being

$$A_0[\varphi_0] = 2e/\sqrt{\pi}l \quad (4.8)$$

for the choice of φ_r given in (2.7). This solution may correspond to the so-called particle like solution discussed by Rosenstock¹⁰⁾ and Finkelstein¹¹⁾, and it is remarkable that such a solution can be obtained from a linear equation in the case of the distribution equation on the contrary to the case of the function equation.

As a second example, in the case of chargeless field we have the solution

$$A_{(k)}[\varphi_x] = \int e^{ik_\mu x_\mu} \varphi_x(\hat{\xi}) d\hat{\xi} \quad (k_\mu k^\mu = 0) \quad (4.9)$$

corresponding to the plane wave. This solution satisfies the following orthogonality relation

$$\int A_{(k)}^{(1)} A_{(-k')}^{(2)} [\varphi_x^{(1)} \varphi_x^{(2)}] dx = \delta(k - k') \phi(k') \phi(-k') \quad (4.10)$$

for $\varphi_x(\hat{\xi}) = \varphi(\hat{\xi} - x)$, where $\phi(k)$ is the Fourier transform of $\varphi(\hat{\xi})$

$$\phi(k) = \int e^{-ik_\mu x_\mu} \varphi(\hat{\xi}) d\hat{\xi} \quad (4.11)$$

and has the properties (i), (ii) in § 2 as a function of k whenever $\varphi(\hat{\xi})$ has the same properties as a function of $\hat{\xi}$. Since the quantity $A_{(k)}(\hat{\xi}) = e^{ik_\mu x_\mu}$ is the so-called plane wave we have as the orthogonal relation of kernels

$$\int A_{(k)}(\xi) A_{(-k)}(\xi') dk = \delta(\xi - \xi'). \quad (4.12)$$

Now let us consider the fundamental field equations. The field strength can be defined in the usual way as

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu. \quad (4.13)$$

The components of $F_{\mu\nu}$ are arranged as \mathbf{E} and \mathbf{H} as usual. The quantization of the field can be performed by assuming the commutation relations between the kernels $A(\xi, t)$, $E(\xi', t)$

$$[A_i(\xi, t), E_j(\xi', t)] = 4\pi i \hbar c \delta_{ij} \delta(\xi - \xi') \quad (4.14)$$

the other commutators all vanishing.

The Schrödinger equation for the state vector of the quantized field is

$$i\hbar \partial \Psi / \partial t = \bar{H} \Psi, \quad (4.15)$$

with the subsidiary condition

$$(D_\mu A_\mu) \Psi = 0 \quad (4.16a)$$

$$(\text{Div } \mathbf{E} - 4\pi \rho) \Psi = 0 \quad (4.16b)$$

which means the state vector is restricted so that the Lorentz condition holds always automatically. The compatibility of these equations can be shown as usual by

$$[D_\mu A_\mu, \bar{H}] = [\text{Div } \mathbf{E} - 4\pi \rho, \bar{H}] = [D_\mu A_\mu, \text{Div } \mathbf{E} - 4\pi \rho] = 0. \quad (4.17)$$

By virtue of the subsidiary conditions the term involving A_0 vanishes from H in the well-known way and thus \bar{H} becomes

$$\bar{H} = H_0 + H, \quad (4.18)$$

$$H_0 = 1/8\pi \int \{(\mathbf{E})^2 + (\mathbf{H})^2\} [\varphi_r] d\mathbf{r} + \sum_s c(\alpha_s \mathbf{p} + \beta_s mc) \quad (4.19a)$$

$$H = - \sum_s e_s \{ \alpha_s A[\varphi_{r_s}] \}. \quad (4.19b)$$

In order to separate the transverse and the longitudinal parts of the field, we set as usual

$$\mathbf{A} = \mathbf{A}^{(t)} + \mathbf{A}^{(l)}, \quad (4.20)$$

$$\text{Div } \mathbf{A}^{(t)} = 0, \quad \text{Rot } \mathbf{A}^{(l)} = 0. \quad (4.21)$$

Among them we can choose the oscillating parts with $k\mathbf{A}_k^{(t)}, \mathbf{A}_k^{(l)}$ satisfying

$$\Delta \mathbf{A}_k + k^2 \mathbf{A}_k = 0 \quad (4.22)$$

and the corresponding potential χ_k satisfying

$$\mathbf{A}_k^{(l)} = 1/k \cdot \text{Grad } \chi_k \quad (4.23)$$

with the normalization conditions

$$1/(2\pi\hbar c)^3 \int A[\varphi_r] A_{-r'}[\varphi_{r'}] d\mathbf{r} = 4\pi c^2 \delta(k-k') \phi(k') \phi(-k'), \quad (4.24)$$

$$1/(2\pi\hbar c)^3 \int \chi_k(\hat{\xi}) \chi_k(\hat{\xi}') d\mathbf{k} = 4\pi c^2 \delta(\hat{\xi}-\hat{\xi}'). \quad (4.25)$$

The notations Δ , Grad , Div and Rot are, of course, the distributional operators corresponding to the ordinary Laplacian, grad, div and rot, namely the ones arising after the substitution with D_i for ∂_i . Using these quantities we assume as

$$\mathbf{A} = 1/(2\pi\hbar c)^3 \left\{ \int q_k \mathbf{A}_k^{(0)} d\mathbf{k} + \int Q_k \mathbf{A}_k^{(0)} d\mathbf{k} \right\}, \quad (4.26a)$$

$$\mathbf{E} = -1/c \cdot (2\pi\hbar c)^3 \left\{ \int p_k \mathbf{A}_k^{(0)} d\mathbf{k} + \int P_k \mathbf{A}_k^{(0)} d\mathbf{k} \right\}. \quad (4.26b)$$

The commutation relations can be transposed with

$$[q_\lambda, p_{\lambda'}] = i\hbar \delta(\lambda - \lambda') \quad (4.27a)$$

$$[Q_\lambda, P_{\lambda'}] = i\hbar \delta(\lambda - \lambda') \quad (4.27b)$$

as relations between kernels, all other commutators vanishing.

Thus we can proceed exactly in the same way as Fermi's treatment and we can arrive at the representation where the longitudinal waves are eliminated. Then the Hamiltonian becomes

$$\begin{aligned} \bar{H} = & 1/(2\pi\hbar c)^3 \int \hbar c k \phi(k) \phi(-k) dk \\ & + \sum_s e_s e_s / 2 |\mathbf{r}'_s - \mathbf{r}_s| \cdot [\varphi_{p_s}(\mathbf{r}'_s) \varphi_{p_s}(\mathbf{r}_s)] \\ & + \sum_s c (\alpha_s \mathbf{p}_s + \beta_s mc) - \sum_s e_s / (2\pi\hbar c)^3 \cdot \int d\mathbf{k} (q_k \alpha_s \mathbf{A}_k [\varphi_{p_s}]). \end{aligned} \quad (4.28)$$

The four terms in this expression mean the zero point energy of the radiation field, the Coulomb energy of electrons, kinetic energy of electrons and the interaction energy respectively. These are all convergent quantities. For example for the choice of φ in (2.7) the zero point energy becomes

$$2/(\pi\hbar c)^2 l^4, \quad (4.29)$$

the Coulomb self energy of the electron becomes

$$2\sqrt{2} e^2 / \sqrt{\pi} l. \quad (4.30)$$

From the interaction term the so-called transverse self energy of the electron arises and it becomes in the second order approximation of perturbation

$$1/(137\pi m \cdot 4l^2). \quad (4.31)$$

When we proceed in this manner, we see that all the results for any process in an approximation of a finite order perturbation procedure become finite unambiguously. Indeed we can obtain unambiguous results in the domains where divergences arose in the usual

theory and special manipulations were taken in order to obtain finite results. Thus some positive results are obtained in applications of the present theory to the non relativistic treatments of the phenomena where ordinary divergent processes are looked upon to play important roles, for example to the problems of electrons interacting with the lattice¹²⁾. However these detailed results for special problems will be reported separately and in the next report a relativistic theory in our method will be given where the S matrix theory can be developed in parallel to the ordinary theory²⁾.

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Note added in proof. On an application to the roton self energy problem; We will notice an example that the convergent rigorous solution of our formalism can give reasonable results for the case where ordinary cut off method is doubtful. A result obtained by cut off is not a rigorous solution unless the following two conditions hold; (i) the restriction of momenta must be consistent with the formalism, (ii) the restriction of the cut off momentum must have a physical reason. Debye's original cut off satisfies these conditions since each oscillator has an independent physical meaning. But they do not hold sometimes for other applications such as the roton self energy problem in the quantum hydrodynamics, where Dirac's δ (involving all the frequencies) occurs, oscillators are not independent solutions (of the non linear equation) and the separation into phonons and rotons is not invariant (this makes discussions using the number of freedom obscure). The self energy in the one-roton state was calculated by Ziman (*P.R.S.* **219** (1953), 543) and Allcock-Kuper (*P.R.S.* **231** (1955), 228), but the cut off used by them involves the above stated unrigor. We can easily reformulate them consistently in our method, by interpreting the hydrodynamical field as a collective description of the system and the non locality l as a measure of the collective corpuscularity (such as interatomic distances), and the convergent rigorous solution is obtained: $6\hbar^2(\sqrt{\pi})^3/\rho_0 l^5$ and $80\hbar^2(\sqrt{\pi})^5/3sp_0^2 l^9$ for the one roton self energy corresponding to them respectively. To get agreement with the experimental value 9°K for He II, l must be $\sim 4.3\text{\AA}$ and $\leq 5.2\text{\AA}$. These values for l are reasonable, because the experimental interatomic distance is $\sim 4\text{\AA}$. This shows that our treatment is a rigorous and practical method to introduce a kind of non locality.

Letters to the Editor

On Solutions of He^3 and He^4

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December 21, 1955

In order to account for their vapour pressure measurements on dilute solutions of He^3 in He^4 , Taconis, Beenakker, Nier and Aldrich¹⁾ have suggested that He^3 dissolves *only* in the normal part of He^4 . On the basis of this assumption, the statistical mechanics of these solutions has been developed by de Boer and Gorter²⁾. This assumption by Taconis et al., however, must be regarded with a due amount of caution, especially because the predictions of de Boer and Gorter's theory are not in good agreement with the subsequent experimental results³⁾⁴⁾⁵⁾. Moreover, their assumption appears too artificial because, taken literally, this would imply that the normal part of He^4 is separated in space from the superfluid part. This interpretation of the two fluid theory leads us into difficulties with the explanation of the basic properties of He II such as second sound, where the superposition of the two fluids is always taken granted for.

We have, therefore, tried to develop a more consistent theory of He^3 and He^4 solutions on the basis of the following assumptions:

(i) He^3 mixes with the whole of He^4 (normal as well as superfluid).

(ii) The contribution to the entropy of mixing is due only to the normal part of He^4 .

(iii) The solution of He^3 in He^4 is not an "Ideal Solution" even above the λ -temperature. It obeys laws of a "Strictly Regular Solution".

On the basis of these assumptions, the Gibbs' function ($G=U-TS+PV$) for the solution can be written in the form:

$$G = (1-X)G_1 + XG_3 + RT[X \ln X + x(1-X) \ln(1-X)] + xX(1-X)W, \quad (1)$$

where X is the concentration of He^3 in liquid phase. G_1 and G_3 are the Gibbs' functions for pure He^4 and He^3 respectively. x is the fraction of the total number of atoms of He^4 which constitutes the normal fluid, while $w=W/N$ is the interchange energy⁶⁾, and N is Avogadro's number. When $x=1$, eq. (1) becomes identical with the Gibbs' function of a "Strictly Regular Solution".

Following de Boer and Gorter, we apply the condition of equilibrium

$$(\partial G / \partial x)_{T, X} = 0. \quad (2)$$

Further, we assume the following quadratic expression for G_4 :

$$G_4 = -4.5/13.0 \cdot S_\lambda T_\lambda (1 - x^{6.5/5.5}) - 1/2 \cdot x^{4.5/5.5} \cdot T^2 (S_\lambda / T_\lambda), \quad (3)$$

where

$$S_\lambda = 1.622 \text{ (cal/deg. mole)}$$

$$\text{and } T_\lambda = 2.182^\circ \text{K.}$$

Substitution of eq. (1) in eq. (2) leads

to the following implicit equation for $x = x(T, X)$:

$$(\partial G_4 / \partial x)_{T, X} + RT \ln(1 - X) + XW = 0. \quad (4)$$

Using eq. (3) the λ -temperature for the solution, ($T = T_{\lambda X}$, when $x = 1$) is given by

$$(T_{\lambda X} / T_\lambda)^2 - 6.896 (T_{\lambda X} / T_\lambda) \log_{10}(1 - X) - 1 - 0.6905 XW = 0. \quad (5)$$

Using the experimentally observed value by Dash and Taylor⁵⁾ of $(\partial T_{\lambda X} / \partial X)_{X \rightarrow 0} = -1.455$, W is found out to be 2.406 cal/mole. λ -temperature of the solution, as

calculated from eq. (5), is compared in Fig. 1 with the empirical relation

$$T_{\lambda X} / T_\lambda = (1 - X)^{2/3}, \quad (6)$$

proposed by Dash and Taylor for X up to 10 per cent. It will be noticed that for X up to 20 per cent the two formulae are in excellent agreement.*

*) We have been informed that the unpublished results of the Los Alamos Group give support to the empirical relation (6) even for higher values of X . We are thankful to the Los Alamos Low Temperature Group for forwarding to us their results before publication.

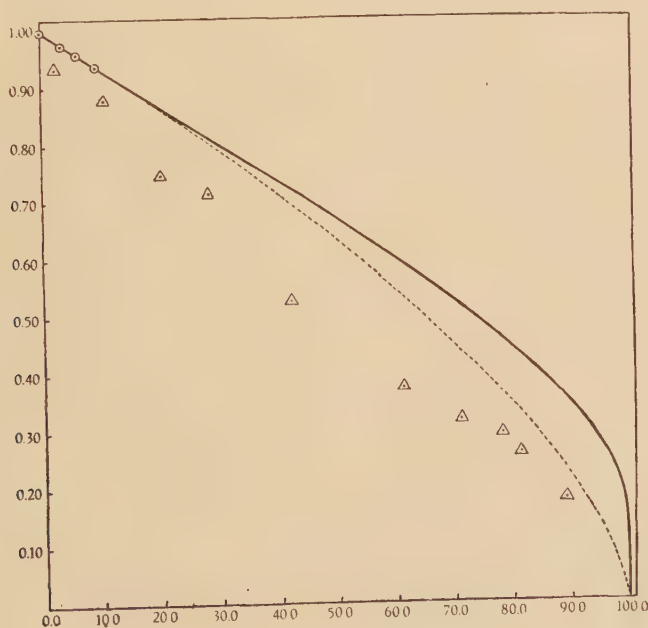


Fig. 1.

Fig. 1. The ratio of the λ -temperature of $\text{He}^3 - \text{He}^4$ solution, $T_{\lambda X}$, to that of pure He^4 , T_λ , as a function of the He^3 concentration X .

—: Present Theory, eq. (5).

.....: Empirical relation $T_{\lambda X} / T_\lambda = (1 - X)^{2/3}$, proposed by Dash and Taylor⁵⁾ for X upto 10 per cent.

○: Experimental results of Dash and Taylor.⁵⁾

●: Leiden specific heat result (Dokoupil et al.).⁴⁾

△: Earlier Experimental results of Abraham et al.⁹⁾ and of Daunt and Heer¹⁰⁾

We have also calculated molar specific heat of mixing for 2.50 per cent solution of He^3 in He^4 . It is found to be in good agreement with the experimental data given by Dokoupil et al.⁴⁾ The calculated vapour pressures of the solution and vapour-liquid concentration ratio for various He^3 concentrations ranging from 0.58 to 13.00 per cent are found to be in good agreement with the experimental results of Sommers³⁾. The second sound velocity for 0.8 per cent solution in the temperature range above 1.0°K , is also found to be in good agreement with the experimental results of Lynton and Fairbank⁸⁾ under the assumptions: (i) He^3 atoms partake in the motion of the normal fluid and (ii) the effective mass of He^3 atoms in solution is 2.17 times the actual mass.

Details of the above work will be published elsewhere. Our grateful thanks are due to Prof. D. S. Kothari for his interest in this investigation.

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Note on the Decay Interactions of Hyperons and Heavy Mesons

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December 22, 1955

The nature of hyperons and heavy mesons is qualitatively well understood on the charge independence hypothesis¹⁾. In such a theory all elementary interactions are classified according to their strengths into three categories, i.e., a) charge independent interactions, b) electromagnetic interactions, and c) weak decay interactions. For the former two, a) and b), the so-called η -charge is conserved, i.e., $\Delta\eta=0$, but for the last one, c), the η -charge obeys the selection rule $\Delta\eta=0, \pm 1$, which is also valid to decays involving leptons in the decay products provided that one assigns $\eta=0$ to all leptons.

In this note we propose a further specified classification of decay interactions into two classes.

c') Charge independent decay interactions

When all particles participating in a decay process have definite isotopic spins, we are naturally concerned with the transformation property of the decay interaction in isotopic space. In such a case the selection rule $\Delta\eta=0, \pm 1$ can be rewritten as $\Delta I_\eta=0, \pm 1/2$, and we may assume in most cases that the decay interactions are *spinors* in isotopic space. This assignment is con-

sistent with the transformation properties of stronger interactions in isotopic space, i.e., a) (*scalar*), and b) (*scalar*) + (*vector*)₃₃ in the sense that weaker interactions cannot be composed of stronger interactions because of the difference of the transformation properties in isotopic space. If we take this hypothesis for granted, the selection rule governing such decays are given by $\Delta I_3 = \pm 1/2$, which has previously been proposed by Gell-Mann and Pais.²⁾ We further assume here that such interactions lead to lifetimes of the order of 10^{-10} sec. for two-body decays. For three body decays the life time are considerably longer.

c'') Weaker decay interactions

It is clear, however, that not all decays are governed by the above selection rule, e.g., decays involving photons or leptons cannot be covered. Hence, there must be weaker decay interactions which may lead to lifetimes of the order of 10^{-8} sec. for decay processes.

In what follows, we shall exhibit the results derived on the hypothesis of "charge independent decay interactions."

(1) Hyperons

Process	I_i	I_f	Remarks
$\Lambda^0 \rightarrow p + \pi^-$ $\Lambda^0 \rightarrow n + \pi^0$	0	1/2	$\frac{(\Lambda^0 \rightarrow p + \pi^-)}{(\Lambda^0 \rightarrow n + \pi^0)} = 2$
$\Sigma^+ \rightarrow p + \pi^0$ $\Sigma^+ \rightarrow n + \pi^+$	1	1/2, 3/2	$3\tau(\Sigma^-) \geq \tau(\Sigma^+)$
$\Sigma^- \rightarrow n + \pi^-$	1	3/2	

I_i : isotopic spin in the initial state.
 I_f : Isotopic spin in the final state.

All these hyperons must decay with lifetimes of the order of 10^{-10} sec. in conformity with experimental results.

(2) θ -Mesons

The discussions differ drastically according to whether the θ -meson has even spin and even parity or odd spin and odd parity.

(i) Case of even spin and even parity

Process	I_i	I_f	Remarks
$\theta^0 \rightarrow \pi^+ + \pi^-$ $\theta^0 \rightarrow \pi^0 + \pi^0$	1/2	0	$\frac{(\theta^0 \rightarrow \pi^+ + \pi^-)}{(\theta^0 \rightarrow \pi^0 + \pi^0)} = 2$
$\theta^+ \rightarrow \pi^+ + \pi^0$	1/2	forbidden	

As seen in the above table, the decay of θ^+ into two pions through c') is forbidden so that the decay of a θ^+ must take place through the weaker interaction $c'')$.* Hence the lifetime of θ^+ must be of the order of 10^{-8} sec. in accordance with the experimental data, whereas θ^0 can decay as fast as in 10^{-10} sec. which is also consistent with experimental information.

(ii) Case of odd spin and odd parity

Process	I_i	I_f	Remarks
$\theta^0 \rightarrow \pi^+ + \pi^-$ $\theta^0 \rightarrow \pi^0 + \pi^0$	1/2	1	forbidden
$\theta^+ \rightarrow \pi^+ + \pi^0$	1/2	1	$\tau(\theta^0) = \tau(\theta^+)**$

As is well known, the lifetime of θ^0 is much shorter than that of θ^+ in contradiction to the above result $\tau(\theta^0) = \tau(\theta^+)$.

Thus the hypothesis of "charge independent decay interactions" is reconciled with experiments only if the θ meson has even spin and even parity.

* The decay interaction obeys the selection rule $\Delta I = \pm 3/2$, $\Delta I_3 = \pm 1/2$ in this case.
 ** According to Gell-Mann and Pais (Phys. Rev. 97 (1955), 1387) we have two kinds of lifetimes of θ^0 , the shorter one of which is $T(\theta^0)$ here.

(3) τ -Mesons

Process	I_i	I_f	Remarks
$\tau^+ \begin{cases} \nearrow 2\pi^+ + \pi^- \\ \searrow 2\pi^0 + \pi^+ \end{cases}$	1/2	1	$I \geq \frac{(\tau^+ \rightarrow 2\pi^0 + \pi^+)}{(\tau^+ \rightarrow 2\pi^+ + \pi^-)} \geq 1/4$

The branching ratio has ever been calculated by Dalitz for the case $I_i = I_f = 1$ and our result agrees with his one in spite of the different choice of I_i .

One of the authors (M.K.) is much obliged to the Yukawa Fellowship of Osaka University for the financial aid.

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Note on the Decays of Σ Particles

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In a paper by one of the present authors (M. K.) and Minami,¹⁾ it has been shown that the unitarity condition of the S-matrix is useful to relate the phases of the matrix elements of the photo-pion production to the phase shifts for pion-nucleon scattering. In the present note we shall discuss the

application of this idea to the decays of Σ particles, especially in connection with the determination of the type of Σ particles.

Let us consider the decay processes

$$\sum^+ \nearrow p + \pi^0 \quad (1a)$$

$$\sum^+ \searrow n + \pi^+ \quad (1b)$$

and denote their transition matrices by R_a and R_b , respectively. It is sometimes more convenient to reexpress R_a and R_b in terms of the matrix elements for decays of \sum^+ into eigenstates of the isotopic spin, i.e., R_1 for $\sum^+ \rightarrow N + \pi$ ($I = 1/2$), and R_3 for $\sum^+ \rightarrow N + \pi$ ($I = 3/2$). They are given by

$$R_a = \sqrt{2/3} R_3 - \sqrt{1/3} R_1,$$

$$R_b = \sqrt{1/3} R_3 + \sqrt{2/3} R_1. \quad (2)$$

From the unitarity condition of the S-matrix, one readily arrives at*

$$e^{i\delta_1} R_1^* = -R_1, \quad e^{i\delta_3} R_3^* = -R_3, \quad (3)$$

where δ_1 and δ_3 are the phase shifts for the pion-nucleon scattering in $I = 1/2$ and $I = 3/2$ states, respectively. The parity and orbital angular momentum of the states are specified subject to the spin and parity of the Σ particle.

Combining (2) and (3), one obtains

$$\frac{|R_a|^2}{|R_b|^2} = \frac{2 - 2\sqrt{2}\rho x + x^2}{1 + 2\sqrt{2}\rho x + 2x^2}, \quad (4)$$

where $x = \pm |R_1|/|R_3|$ and $\rho = \cos(\delta_3 - \delta_1)$. The branching ratio of (1a) to (1b) is determined by this equation.

The \sum^+ particle may decay at a small rate as

$$\sum^+ \rightarrow p + \gamma, \quad (5)$$

the transition matrix of which will be denoted

* For details see Ref. 1).

by R_T . The unitarity condition provides us with the following relation

$$M_a R_a^* + M_b R_b^* = -2 \text{Re } R_T, \quad (6)$$

where

$$M_a = \langle \pi^0 p | R | p \gamma \rangle \text{ and } M_b = \langle \pi^+ n | R | p \gamma \rangle.$$

In a similar way to eq. (2), M_a and M_b can be expressed in terms of M_1 and M_3 , which satisfy similar equations to (3). Hence it is not hard to prove

$$\frac{|\text{Re } R_T|^2}{|R_b|^2} = \frac{3}{4} \cdot \frac{|M_1|^2 x^2 + 2|M_1| \cdot |M_3| x + |M_3|^2}{1 + 2\sqrt{2} \sqrt{x + 2x^2}}. \quad (7)$$

Fixing the type of Σ^+ , it is possible to draw a curve $|\text{Re } R_T|^2/|R_b|^2$ versus $|R_a|^2/|R_b|^2$ by choosing x as a parameter. Since $|R_T|^2 > |\text{Re } R_T|^2$ the experimental point $((1a)/(1b), (5)/(1b))$ must fall above this curve. This criterion might serve to exclude certain choices of the type of Σ^+ . The numerical values of δ_1, δ_3, M_1 and M_3 are given in Table I, and the curves are given in Fig. 1.

If we take the hypothesis of charge independent decay interactions²⁾ for granted, we can propose another more favourable method to determine the type of Σ rather than the above.

According to the hypothesis, the transition matrix for

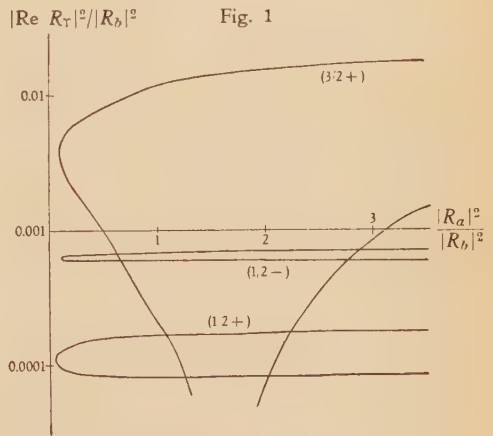
$$\Sigma^- \rightarrow n + \pi^- \quad (8)$$

should be equal to $\sqrt{3} R_3$ and we have

$$\frac{\tau(\Sigma^-)}{\tau(\Sigma^+)} = \frac{|R_3|^2 + |R_1|^2}{3|R_3|^2} = \frac{1+x^2}{3}. \quad (9)$$

Fixing the type of Σ , we can plot $\tau(\Sigma^-)/\tau(\Sigma^+)$ versus $|R_a|^2/|R_b|^2$, and we have many curves corresponding to the possible choices of the type of Σ . This time the experimental point should fall on one of such curves and one might determine the type of the hyperon Σ .

One of the authors (M.K.) is much obliged to the Yukawa Fellowship of Osaka University for the financial aid.



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Table I**

Type of Σ^+	State of emitted pion-nucleon system	δ_3	δ_1	Emitted radiation	M_3	M_1
1/2-	$s_{1/2}$	-12.5°	11.0°	E1	-0.03	-0.04
1/2+	$p_{1/2}$	5.4°	-4.5°	M1	-0.01	-0.02
3/2+	$p_{3/2}$	38.1°	2.3°	M1, E2	0.14	0.05
3/2-	$d_{3/2}$			E1, M2		
⋮	⋮					

** It must be noted that the calculation of M 's, which are equal to a and b of Reference 1 including the sign from experimental data, is susceptible of some ambiguities. We neglect the contribution from E2.

On the Energy Dissipation of Conduction Electrons undergoing Elastic Scattering by Impurities

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The origin of the residual resistance has been ascribed to the presence of impurities by which the conduction electrons undergo elastic scattering. It goes without saying that the electrons cannot lose their energy through elastic scattering. On the other hand, when a steady current is flowing in a metal, the conduction electrons are continuously accelerated by applied electric field. In order that a current flows steadily, therefore, some mechanism must be acting to throw out rapidly the additional energy supplied by applied field.¹⁾ We shall propose a mechanism responsible for the energy dissipation in question.

The motion of electrons is considered to consist of the following three processes: First, the equilibrium Fermi sphere is displaced by the field, a *p*-type distortion being produced on the electron distribution in momentum space. Secondly, due to the elastic scattering by impurities this distortion is brought into an *s*-type, to the effect that the net momentum of the electron gas vanishes (momentum dissipation). Finally, the *s*-type distortion relaxes through the interaction with the phonon-field and the

original Fermi distribution recovers (energy dissipation).

If the relaxation time τ_s of the third process is sufficiently short, the electron gas can quickly give the part of its energy supplied by the field to the phonon field. In this case, we can expect that a steady current will flow. Therefore we must find the value of τ_s .

In order to know the order of magnitude of τ_s , we calculate the relaxation time of a Fermi distribution which corresponds to a temperature, a little different from that of the phonon field. In this approximation, τ_s is given by

$$\frac{1}{4\pi^3} \int \frac{\hbar^2 k^2}{2m} \left\{ \frac{\partial}{\partial t} f(k, T + \Delta T) \right\}_{\text{phonon}} dk = -\frac{C_e}{\tau_s} \Delta T, \quad (1)$$

where f , C_e and ΔT stand for the Fermi distribution function, the electronic specific heat and the temperature difference between the electron gas and the phonon field, respectively. Here we have assumed electrons are nearly free and the phonon field has an infinite heat capacity.

Making use of the formula (34-40) in Sommerfeld-Bethe's text²⁾, we can easily carry out the integration of the left-hand side of (1). The result is, when $\theta \gg T$,

$$1/\tau_s = (160C^2/3\pi u_0 M n \gamma) (km/\hbar^2)^2 (T/\theta)^3, \quad (2)$$

where the notations are the same as in reference 2 (γ is the Sommerfeld constant). As a numerical example, we cite the data of metallic sodium: $M = 3.84 \times 10^{-23}$ gr, $\theta = 202^\circ \text{K}$, $n = 2.37 \times 10^{22}$ c.c., $C = 5.25 \times 10^{-12}$ erg, $u_0 = 2.3 \times 10^5$ cm/sec. For this metal we have

$$\tau_s = 2.5 \times 10^{-9} \times (1/T^3) \text{ sec.} \quad (3)$$

We can now ask whether τ_s found here is so short that the electron gas can quickly deliver the work done by the electric field to the lattice. For this purpose, suppose an electric field F be applied to the specimen under discussion. Then we should generally expect that a temperature difference ΔT is set up between the electron gas and the lattice. Denoting the residual resistance by R , this difference will obey the following equation of motion:

$$\frac{d\Delta T}{dt} = \frac{F^2}{R} \cdot \frac{1}{n\gamma T} - \frac{\Delta T}{\tau_s}. \quad (4)$$

After a steady state is established, the temperature difference becomes

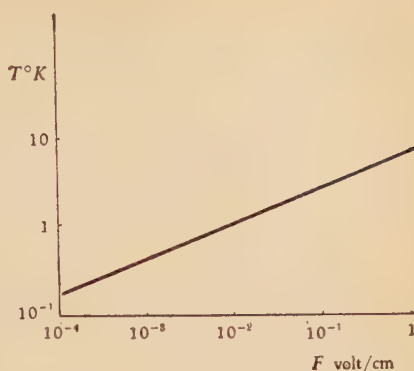
$$\Delta T = F^2/R^2 \cdot \tau/n\gamma T. \quad (5)$$

In order to take a general view of this situation, it is worth-while finding the temperature at which $\Delta T \approx T$ for a given F . This is given by

$$T = F^2/R \cdot \tau_s/n\gamma T. \quad (6)$$

Roughly speaking, Eq. (6) means that under the influence of the electric field F , the electron gas can no longer be regarded to remain in a steady state with the same temperature as that of the phonon field, unless the temperature is higher than that given by (6). This critical temperature versus field curve is drawn in Fig. 1 for Na. It is seen that for not very large field the critical temperature is of the same order as the liquid helium temperatures. Therefore it may be possible to detect the temperature difference experimentally.

The authors would like to express their thanks to Mr. H. Matsuda with whom they have had many fruitful discussions.



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Interaction of Antinucleons in Matter, I

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The development of high energy accelerator physics is making it possible to get informations about the properties of antinucleons. Since the production processes of antinucleons have been attacked from various sides¹⁾, we shall summarize here our knowledge about the properties of antinucleons other than the production processes.

i) *Selection rules on annihilation* The conservation laws of angular momentum and parity forbid the 2π -annihilation of nucleon (N) and antinucleon (\bar{N}) in singlet states. The invariance under charge conjugation and charge symmetry transformation gives

further restriction. The results are shown in Table 1²⁾. Probable number of produced pions in the annihilation is estimated by Fermi's statistical theory³⁾ with conservation of isospin. It is found to be ~ 3.6 in both ($p-\bar{p}$) and ($n-\bar{p}$) cases. Annihilation to two pions seems not to be the most probable one statistically, therefore selection rules will operate mainly on the number of pions (even or odd) produced.

ii) *Nuclear forces* The nuclear forces between $N-\bar{N}$ pair will have different character from that for $N-N$ pair at short distances of the order of \hbar/Mc , since many processes through annihilation of initial pair will contribute to the former case. While, the outer potential for $N-\bar{N}$ pair will be obtained on the same basis as that for $N-N$ pair.

A potential due to pion exchange is obtained from the following interaction Hamiltonian :

$$H' = 1/2 \{ iG \{ \bar{\psi}(x) \gamma_5 \tau_a \psi(x) \phi_a(x) dx - iG \{ \bar{\psi}'(x) \gamma_5 \tau_a \psi'(x) \phi_a(x) dx \} \cdots P_S(p_S) \quad (1)$$

or

$$H' = 1/2 \{ i \frac{g}{\mu} \int \bar{\psi}(x) \gamma_5 \gamma_\mu \tau_a \psi(x) \frac{\partial}{\partial x_\mu} \phi_a(x) dx - i \frac{g}{\mu} \int \bar{\psi}'(x) \gamma_5 \gamma_\mu \tau_a \psi'(x) \frac{\partial}{\partial x_\mu} \phi_a(x) dx \} + (\text{normal dependent term}) \cdots P_S(pv)$$

where $\bar{\psi}'(x)$ and $\psi'(x)$ are obtained from

$\bar{\psi}(x)$ and $\psi(x)$ by successive operations of charge conjugation and charge symmetry transformations, i.e.,

$$\psi = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \bar{\psi} = \begin{pmatrix} \bar{\psi}_p \\ \bar{\psi}_n \end{pmatrix},$$

$$\psi' = \begin{pmatrix} \psi_{nc} \\ -\psi_{pc} \end{pmatrix}, \bar{\psi}' = \begin{pmatrix} \bar{\psi}_{nc} \\ -\bar{\psi}_{pc} \end{pmatrix}. \quad (2)$$

One finds that the n -pion exchange potential for $\bar{N}-N$ pair is the same as that for $N-N$ pair except the factor $(-1)^n$.

The signs of "no pair" potentials are listed in Table 1 for two cases of Taketani-Machida-Onuma (T.M.O.) and Brueckner-Watson (B. W.) approximations⁴⁾. There are eight states in $\bar{N}-N$ case compared with only four states in $N-N$ case. The possibilities of nuclear (not electrostatic) bound states are conjectured assuming that inner potentials ($r < 0.7/\mu$) are not so strongly attractive. Scattering experiments of $p-\bar{p}$ and $\bar{p}-d$ are desirable to check the outer potentials predicted above.

iii) *Behavior of slow antiprotons* We considered competition among various processes supposed to be possible in matter (say liquid hydrogen). It takes about 10^{-10} second for a \bar{p} of several Mev to be slowed down and trapped into the K -orbit of Coulomb potential⁵⁾. On the other hand, the cross section of 2π -annihilation of slow $p-\bar{p}$ pair is estimated in the lowest order, as⁶⁾

Table 1.

T	S	orbital angular momentum	2π annihilation	number of pions produced	Potential for NN of T.M.O. (B.W.)						Possibility of nuclear bound state	
					central			tensor			NN	NN
					G ²	G ⁴	G ⁶	G ²	G ⁴	G ⁶		
0	0	even	No	even	-9	+(+)	+(+)				?	/
		odd	No	odd							No	No.
1	0	even	No	odd	+3	-(-)	+(+)				No	No.
		odd	No	even							No	/
0	1	even	No	odd	+3	+(-)	-(+)	+3	-(+)	-(+)	No	Yes
		odd	$J=L\pm 1$	even							No	/
1	1	even	$J=L\pm 1$	even	-1	-(-)	+(+)	-1	-(-)	-(-)	?	/
		odd	No	odd							No	No.

$$\sigma(p + \bar{p} \rightarrow \pi^+ + \pi^-) = \pi \left(\frac{G^2}{4\pi} \right)^2 \frac{1}{M^2 v},$$

$$\frac{G^2}{4\pi} \frac{\mu}{2M} = 0.08 \quad (3)$$

which suggests the time required for a \bar{p} to be annihilated in plane wave in matter to be of the order of 10^{-9} sec. It is interesting to know whether a \bar{p} goes into a nuclear bound state or is annihilated directly from these Coulomb orbits. Tentatively accepting (3) as giving only the order of magnitude, an estimation gives a lifetime of $\sim 10^{-19}$ sec for a p to be annihilated with \bar{p} in an orbit with radius $\sim 10^{-11}$ cm (say, K -orbit radius). While, the transition from Coulomb to nuclear orbits will take $\sim 10^{-14}$ sec, provided that this goes through an emission of magnetic dipole radiation of a few Mev. Thus, the formation of nuclear bound states seems to be rare, unless the nuclear forces are strongly repulsive at short distances. The presence of these states might be detected by observing the possible monochromatic γ -rays.

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- Their coupling constant g is in the following relation with our G , $g = \sqrt{2/4\pi} G$.

ERRATA

On the Bose-Einstein Liquid Model for Liquid Helium, IV

—A Revised Model—

ZIRO MIKURA

Prog. Theor. Phys. **14** (1955), 337.

In Fig. 2, Curves C (de Boer-Gorter theory) and D (Heer-Daunt theory) have been found to be in error on account of the wrong value put for the specific heat of He^3 in a mixture. The values for Curve C should be increased by about 10% at all temperatures, and those for D by about 60~80% for $2.1^\circ \sim 1.2^\circ \text{K}$. Curve A (modified B. E. theory) need not be altered, as calculations are found to be correct in this case. These corrections, however, do not affect the earlier conclusion that the modified B. E. theory provides the best result for the specific heat of a He^3 - He^4 mixture.

The author is deeply indebted to Dr. B. K. Agarwal of the Allahabad University, India, for pointing out the above-mentioned error.

Nuclear Reactions at Moderate Energies and Fermi Gas Model

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Prog. Theor. Phys. 13 (1955), 415

	originally	should be read
p. 416 l.6	complicated	complicated
p. 416 l.30, 34	Perganini	Patergnani
p. 419 l.4	$\mathcal{Q}' \leq 0$	$\mathcal{Q}' \geq 0$
p. 419 eq. (2.7)	$\int_{-1}^1 \frac{dx}{\sqrt{P_1^2 + P_2^2 - 2P_1 P_2 x}}$	$\int_{-1}^1 \frac{dx}{\sqrt{P_1^2 + P_2^2 + 2P_1 P_2 x}}$
p. 419 eq. (2.10)	σ	σ
p. 422 eq. (3.2)	$\delta(P^2 - P^2)$	$\delta(P'^2 - P^2)$
p. 422 eq. (3.4)	$\delta(z - z_0)$	$\delta(z - z_0)$
p. 422 eq. (3.6)	$1/2q$	$1/q$
p. 423 l.5	σ_L	$2\sigma_L/P_1^2$
p. 423 eq. (3.8)	$\sin \zeta \cos \varphi$	$(\rho/P_2) \sin \zeta \cos \varphi + (z_0/P_2) \cos \zeta$
p. 423 eq. (3.9)	$2\sigma_L$	$4\sigma_L$
p. 423 eq. (3.10)	$b = P_1^2 + z_0^2 - 2P_1^2 \sin^2 \zeta$ $c = P_1^2 + z_0^2 - 4P_1^2 z_0^2 \sin^2 \zeta$	$b = P_1^2 + z_0^2 + 2P_1 z_0 \cos \zeta - 2P_1^2 \sin^2 \zeta$ $c = (P_1^2 + z_0^2 + 2P_1 z_0 \cos \zeta)^2$
p. 423 eq. (3.11)	$P_2^2 \geq p_F^2$	$P_2^2 \geq P_F^2$
p. 423 eq. (3.14)	$\sigma_t(E)$	$\sigma_t(E_1)$
	$d\mathbf{P}_f$ should be multiplied by the expression in the right hand side.	
p. 423 foot note	P_0^0	P_0^2
p. 425 eq. (3.18)	$\int \cdots d\mathbf{P}_f$	$\int \cdots d\mathbf{P}_2$
p. 426 eq. (3.22)	\tanh	\tanh^{-1}
p. 427 eq. (4.4)	$\int d\omega \int d\alpha_i \cdots \frac{L_i}{\lambda_i}$	$\int d\omega \int d\alpha_i \int \frac{dL_i}{\lambda_i} \cdots$
p. 431 eq. (5.5)	$\sum_k F_k(E_{ex} - S_k)$	$\sum_k F_k(E_{ex} - S_k)$
p. 431	The equation between (5.5) and (5.7) should be numbered as (5.6).	
p. 431 eq. (5.6)	$\sigma_i^{(1)}(E_i)$	$\sigma_c^{(1)}(E_i)$
p. 437 eq. (6.1)	L_i/λ_i	dL_i/λ_i
p. 437 eq. (6.2)	rL_i/λ_i	rdL_i/λ_i
p. 441 reference 12)	Perganini	Patergnani

On the One Pion Exchange Potential

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It is shown by a general argument that the one pion exchange potential is not changed but the unrenormalized coupling constant must be replaced by the renormalized one, in the region far from the origin, where the static approximation is allowed, by the inclusion of the radiative effects.

It is found in the analysis of two nucleon problems that the one pion exchange potential given in the meson theory of nuclear forces plays the main role in the low energy region and gives excellent agreements with experiments¹⁾.

The importance of the one pion exchange potential in the low energy phenomena may be attributed to its long force range. The range is longer than the ones of the potentials obtained by two or more pion exchange processes. Therefore the one pion exchange potential has the most decisive effects on the behaviours of the loosely bound state such as the deuteron as well as on the phase shift of higher partial wave scattering where the centrifugal force pushes the wave function outside.

Accordingly it may be desirable to examine if higher order radiative corrections to the one pion exchange potential are appreciable in the distances far from the origin. In the present paper, using a general argument, we would like to show that the one pion exchange potential is not changed but the unrenormalized coupling constant must be replaced by the renormalized one, in the region far from the origin, where the static approximation may be plausible, by the inclusion of the radiative effects.

Let us consider the process which is illustrated in Fig. 1, where A and B are the self-energy part of nucleon and pion line respectively and C is the involved vertex part. We consider the problem in center of mass system and assume both incoming and outgoing nucleons free, whose momenta are represented as (\mathbf{p}, E_p) , $(-\mathbf{p}, E_p)$,

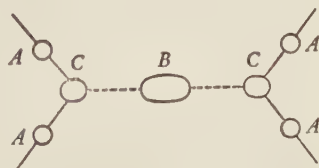


Fig. 1 Diagram of the one pion exchange process

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and (\mathbf{p}', E_p) and $(-\mathbf{p}', E_p)$, where E_p is $(p^2 + M)^{1/2}$ and M is the mass of a nucleon. The corrections due to the deviation of nucleon states from their free states may be taken into account in the course of evaluating the two or more pion exchange potentials,²⁾ so that they result in the modification of potential with range shorter than $(2\mu)^{-1}$, where μ is the mass of a pion.

First we consider the effect of the insertion of self energy parts in nucleon lines. Each insertion simply supplies a factor $Z_2^{1/2}$, which will be incorporated into coupling constant renormalization.

Secondly, concerning the vertex correction the general form of the renormalized vertex operator is given in the symmetrical pseudoscalar meson theory with pseudoscalar coupling as follows³⁾:

$$\begin{aligned} \Gamma_{5\alpha}(p_1, p_2) = & \gamma_5 \tau_\alpha \left[1 + \frac{(p_1 + p_2)^2}{M^2} f_1 + \frac{(p_1^2 + M^2)}{M^2} f_2 + \frac{(p_2^2 + M^2)}{M^2} f_3 \right] \\ & + \frac{(i\gamma p_1 + M) \gamma_5 \tau_\alpha (i\gamma p_2 + M)}{M^2} f_4 \\ & + \frac{(i\gamma p_1 + M)}{2M} \gamma_5 \tau_\alpha f_5 + \gamma_5 \tau_\alpha \frac{(i\gamma p_2 + M)}{2M} f_6, \end{aligned} \quad (1)$$

where the functions f_1, \dots, f_6 are scalar functions of $p_1^2 + M^2$, $p_2^2 + M^2$ and $(p_1 - p_2)^2$, and are finite in the limit $p_1 \rightarrow p_2$ and $i\gamma p_1 + M = 0$. In the present case the nonvanishing contribution is

$$\Gamma_{5\alpha}(\pm \mathbf{p}, E_p; \pm \mathbf{p}', E_p) \rightarrow \gamma_5 \tau_\alpha \left[1 + \frac{(\mathbf{p}' - \mathbf{p})^2}{M^2} f_1 \right], \quad (2)$$

where we have used the fact that this operator operates on the free states of nucleon.

Finally we shall consider the self energy part of pion line. By the insertion of the self energy part the propagator of pion J_F becomes to the modified propagator $J_{F'}$. The form of the renormalized $J_{F'}$ can be expressed generally as follows⁴⁾:

$$J_{F'}(k) = -2i \int \frac{\rho(\kappa^2)}{k^2 + \kappa^2 - i\epsilon} d(\kappa^2), \quad (3)$$

where

$$\rho(\kappa^2) = \delta(\kappa^2 - \mu^2) + \sigma(\kappa^2). \quad (4)$$

On account of the pseudoscalar property of pion $\sigma(\kappa^2)$ satisfies the following conditions:

$$\sigma(\kappa^2) = 0 \quad \text{for} \quad (3\mu)^2 > \kappa^2 \geq 0, \quad (5)$$

$$\sigma(\kappa^2) \geq 0 \quad \text{for} \quad \kappa^2 \geq (3\mu)^2. \quad (6)$$

In fact, if we calculate the lowest order correction in the symmetrical $P_\pi(p_5)$ theory, the following expression for $\sigma(\kappa^2)$ is obtained and it satisfies the above conditions:

$$\sigma_2(\kappa^2) = 2 \frac{G_\pi^2}{4\pi} \theta(\kappa^2 - 4M^2) \frac{\kappa(\kappa^2 - 4M^2)^{1/2}}{\kappa^2 - M^2},$$

where

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Consequently the R matrix is given as

$$\begin{aligned} (\mathbf{p}', -\mathbf{p}' | R | \mathbf{p}, -\mathbf{p}) = & -\frac{i}{2} \frac{G^2}{(2\pi)^4} (\bar{u}(\mathbf{p}') | \gamma_5 \tau_a | u(\mathbf{p}))^{(1)} \cdot (\bar{u}(-\mathbf{p}') | \gamma_5 \tau_a | u(-\mathbf{p}))^{(2)} \\ & \cdot \left(1 + f_1 \frac{(\mathbf{p}' - \mathbf{p})^2}{M^2} \right)^2 \cdot \int \frac{\rho(\kappa^2)}{(\mathbf{p}' - \mathbf{p})^2 + \kappa^2 - i\epsilon} \alpha(\kappa^2), \end{aligned} \quad (7)$$

where G is the renormalized coupling constant, $G = Z_1^{-1} Z_2^{1/2} G_0$, G_0 being the unrenormalized coupling constant, and $u(\mathbf{p})$ is the spinor wave function of the free nucleon with momentum \mathbf{p} . We can construct the one pion exchange potential as

$$V(\mathbf{r}) = 4\pi/i \cdot \int (\mathbf{p}', -\mathbf{p}' | R | \mathbf{p}, -\mathbf{p}) \exp \{i(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{r}\} d(\mathbf{p}' - \mathbf{p}). \quad (8)$$

For large values of r , the appreciable contribution of the integrand of (8) comes only from the small values of $|\mathbf{p}' - \mathbf{p}|$ due to the rapid oscillation of the exponential factor. Hence we can neglect $(\mathbf{p}' - \mathbf{p})^2 f_1/M^2$ in (7) for the large separation r . Then we obtain the potential

$$V(r) = \frac{G^2}{(2\pi)^3} \left(\frac{1}{2M} \right)^2 (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \int d\mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} \int d(\kappa^2) \frac{(i\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k})(i\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k})}{k^2 + \kappa^2 - i\epsilon} \rho(\kappa^2) \quad (9)$$

$$\begin{aligned} = & \frac{G^2}{4\pi} \left(\frac{\mu}{2M} \right)^2 \frac{(\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)})}{3} \left[\left\{ (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + S_{12} \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) \right\} \frac{e^{-\mu r}}{r} \right. \\ & \left. + \int_{3\mu}^{\infty} \left(\frac{\kappa}{\mu} \right)^2 \sigma(\kappa^2) \left\{ (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + S_{12} \left(1 + \frac{3}{\kappa r} + \frac{3}{(\kappa r)^2} \right) \right\} \frac{e^{-\kappa r}}{r} \cdot d(\kappa^2) \right], \end{aligned} \quad (10)$$

where we have invoked the interchange of the order of κ^2 - and \mathbf{k} -integration. The second term of (10), which is the contribution from B part, has the range shorter than $(3\mu)^{-1}$ so that its effects are negligible when $r \gg (3\mu)^{-1}$.

We have shown above, by a general argument, that the one pion exchange potential is not changed but G_0^2 must be replaced by G^2 in the region far from the origin, where the static approximation may be plausible, by the inclusion of the radiative effects. This conclusion is unaffected by the coupling of the nucleons to heavier bosons or fermions.

The above argument holds also in the case of the pseudovector coupling, although the cut-off prescription is necessary to avoid the divergences.

Adding to our conclusion we would like to make a remark on Henley and Ruderman's result.⁽⁵⁾ They have calculated the nuclear forces taking into account the mass and coupling constant renormalization which comes from the self energy part of nucleon only. According to them, the one pion exchange potential has the following factor besides the renormalized coupling constant, g_p^2

$$\left[1 + \frac{1}{3} (g_v^2/\mu^2) (1/2\pi)^3 \int \frac{k^2 d\mathbf{k}}{2\omega^3} \right]^2 = (1 + 0.07)^2.$$

However, this extra factor corresponds to Z_1^{-2} so that it can be absorbed into the correctly renormalized coupling constant as we have shown above. This fact can also be shown by Chew's nonrelativistic renormalization theory.⁶⁾

The authors are grateful to the members of nuclear force group in Japan for their stimulating discussions.

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Electromagnetic Radiation from Electron Plasma

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Theory of plasma oscillation developed by Bohm and Pines is extended to involve the coupling between longitudinal and transverse oscillations. The coupling arises from the quantum fluctuation of electrons interacting with electromagnetic field. The coupling energy is found to be $\hbar kv$ approximately, where $\hbar k$ represents the recoil momentum of an electron due to the absorption (emission) of a longitudinal quantum with wave number k and v the velocity of the electron. This coupling may account for the solar outburst of radio waves, if the plasma oscillation is excited in a stream of electrons injected into corona with high velocity.

§ 1. Introduction

It is well known that an electron gas of high density undergoes an organized oscillation. This oscillation, the so-called plasma oscillation, results from the effects of the long-range correlation of electron positions brought about by Coulomb interactions. In a metal, for instance, the energy of a quantum of the plasma oscillation is so high that this will not be excited at ordinary temperatures. However, it may be highly excited in the solar atmosphere due to a strong disturbance. It has been suggested that the plasma oscillation excited in the outer layers of solar atmosphere is responsible for the observed outburst of radio waves received from the sun. Several authors¹⁾ have been concerned with this problem and explained the growth of the plasma oscillation from various points of view. However, no satisfactory explanation has been given of the radiation of electromagnetic waves. A difficulty lies in the problem of how the energy of plasma oscillations can be converted into the radiation, since such oscillations are irrotational and therefore do not radiate electromagnetic waves directly. In the present paper we would like to point out that the plasma oscillation, the collective oscillation of an electron gas as a whole, couples with the electromagnetic field through the agency of individual electrons.

The collective or medium-like behavior of the plasma has extensively been discussed by Bohm and his collaborators²⁾. They were able to show the origin of organized motion of the electron gas, expressed by longitudinal waves like sound waves. They also considered the transverse oscillation that is nothing but an electromagnetic wave modified by the field arising from collective response of a large number of electrons. However, the coupling between these two modes, the longitudinal and transverse oscillations, has thus far been overlooked. It is the purpose of the present paper to derive this coupling and to see if this can play a role in the outburst of solar radiations.

In the way of derivation we use the technique of quantum electrodynamics. The plasma oscillations can effectively be redescribed in terms of longitudinal photon variables, since there is a close parallelism between the behavior of the density fluctuation of the electron gas and the longitudinal field variables. The field variable oscillates with a frequency equal to that of the plasma oscillation. The electromagnetic waves in the electron gas are described in terms of transverse oscillator and the plasma waves of longitudinal oscillators, as if they would consist of longitudinal photons. With these collective oscillations are associated quanta which we call collectons, for short. These collectons interact with one another through the intermediary of electrons, since successive collecton processes (emission and absorption of collectons are mutually correlated owing to the recoil of electrons. In other words, for example, the absorption of a longitudinal collecton by an electron gives rise to the recoil of the electron. The uncertainty (fluctuation) in the motion of the electron due to the virtual absorption (emission) of longitudinal collectons already present produces, in turn, an uncertainty in the transverse field which the electron radiates, that is, an extra radiation.

The coupling energy for this process can be estimated as follows. Since the coupling of longitudinal waves with an electron is proportional to the momentum of the latter, the coupling with recoil is proportional to $\hbar k$, where k is the wave number of the longitudinal collecton absorbed. The coupling of the electron with a transverse collecton is proportional to the velocity of the former, v . Consequently, the coupling energy becomes $\hbar k v$ with a dimensionless factor which will be found to be of the order of unity. The fact that \hbar appears indicates the importance of quantum fluctuation of electrons. If electrons were treated classically, the amplitude, for instance, for absorption of a longitudinal collecton and subsequent emission of transverse one would be cancelled by the amplitude for emission of the transverse collecton and the subsequent absorption of longitudinal one, because energy denominators for two processes appear with opposite signs.

Our way of derivation is close to that in the third paper of Bohm and Pines²⁾. Their method is recapitulated in Appendix, except that longitudinal and transverse components are treated on the same footing. Then the coupling between longitudinal and transverse collectons is derived by means of the canonical transformation in the same way as Bohm and Pines. The result is summarized in § 2 in connection with the radio emission from plasma oscillations. Application of our result to the solar outburst is discussed in § 3.

§ 2 Outline of the radiation mechanism

The detailed theory of deriving the interaction between longitudinal and transverse waves is worked out in Appendix. The interaction Hamiltonian necessary for us is given in (A·25a). For our practical purpose, the angular frequencies of longitudinal and transverse waves, $\omega_l(k)$ and $\omega_t(k)$, are only slightly larger than the plasma frequency ω_p , which is of the order of 10^8 sec^{-1} . Hence the wave number of these waves, k , is of the order of ω_p/c , so that the recoil momentum of an electron, $\hbar k$, is negligible compared with the momentum of an electron, P_i . Therefore, we can safely neglect the recoil terms in the denominators of (A·25a). Terms like $\mathbf{k} \cdot \mathbf{v}_i$, where $\mathbf{v}_i = \mathbf{P}_i/m$ is the velocity of the i -th

electron, are also negligible compared with ω_l and ω_t , as we are interested in the velocity of one hundredth of the light velocity at most. Hence we can also neglect such terms that express the doppler effect. Thus we obtain the interaction Hamiltonian as

$$H_{\text{int}} = \sum_{k\mu} (\hbar/4(\omega_l\omega_t)^{1/2}) (4\pi e^2/m) \{k/\omega_l + k/\omega_t\} \sum_i (\mathbf{E}_{k\mu} \cdot \mathbf{v}_i) (a_{k\mu} b_{k\mu}^* + b_{k\mu}^* a_{k\mu}) \quad (1)$$

where $\mathbf{E}_{k\mu}$ ($\mu=1, 2$) are the polarization vectors of transverse waves.

The summation over the velocities of electrons in (1) vanishes, as far as the motion of electrons is random. If a group of electrons with density ρ have essentially the same velocity \mathbf{v}_s , however, there arises a considerable contribution from the summation. This happens when a stream of ionized gas is injected in another ionized gas, say corona. The physical reason for this may be understood as follows. In the coordinate system in which the stream is at rest, the electrons in the corona have a common velocity component \mathbf{v}_s that is equal, but of opposite direction to the velocity of the stream. These coronal electrons are scattered by the crest or trough of the electric potential caused by the plasma oscillation excited in the wave front of the stream. Thereby the electrons pick up a momentum $\hbar k$ with slight deflection and make their way by a distance, l , from the front with velocity \mathbf{v}_s , until they suffer collisions with individual electrons in the stream. The layer of thickness l forms a shock front, in which the electrons can radiate electromagnetic waves due to the interaction with plasma waves.

In the new coordinate system we observe that the group velocity, $\partial\omega_l(k)/\partial k$, of plasma waves in the stream becomes small. This means that the electrons in the stream carry disturbances barely from one region of the stream to another. Hence excitations remain nearly localized near the front. In the original coordinate system this will appear as a shock front of thickness l traveling upward through the corona.

The magnitude of l may be estimated as substantially equal to the mean free path for electron-electron collisions, $l = (\pi d^2 N_s)^{-1}$, where N_s is the density of electrons in the stream and d is the Rutherford radius given by $d = e^2/2mv^2$. If the stream has a cross section in which its physical conditions are isolated from the surroundings, $l \times$ (cross section) is the effective volume in which the conversion process takes place.

With the above considerations we can put

$$\sum_i (\mathbf{E}_{k\mu} \cdot \mathbf{v}_i) = N_c (\mathbf{E}_{k\mu} \cdot \mathbf{v}_s), \quad (2)$$

where N_c is the density of electrons in the corona. On account of that the plasma frequency in the colliding medium is given by

$$\omega_p = (4\pi e^2 N/m)^{1/2}, \quad (3)$$

where N is the density of electrons in the medium consisting of the stream and the corona, the substitution of (2) and (3) into (1) gives us

$$H_{\text{int}} = \hbar \omega_p^2 \frac{N_c}{N} \sum_{k\mu} \frac{\omega_l + \omega_t}{4(\omega_l\omega_t)^{3/2}} k (\mathbf{E}_{k\mu} \cdot \mathbf{v}_s) (a_{k\mu}^* b_{k\mu} + a_{k\mu} b_{k\mu}^*). \quad (4)$$

Now one can see that the electromagnetic waves radiated are linearly polarized. However, the polarization could not be observed because the Faraday effect due to magnetic field would destroy the polarization.

As we are concerned with the order of magnitude of the conversion probability, both ω_i and ω_t are approximated by ω_p . Then (4) is reduced to

$$H_{it} = (N_e/2N) \sum_{k\mu} \hbar k (\mathbf{E}_{k\mu} \cdot \mathbf{v}_s) (a_k^* b_{k\mu} + a_k b_{k\mu}^*). \quad (5)$$

This is what we have expected in § 1 on the basis of a qualitative consideration. The probability per unit time and volume of converting longitudinal collectons into transverse ones is obtained from (5) as

$$w \simeq (N_e^2/3\pi N^2) (v_s/c)^2 \omega_p k^3 n_k^l \quad (6)$$

where n_k^l means the number of longitudinal quanta excited per oscillator with wave number k .

§ 3. Application to the solar outburst

We shall now apply the above result to the interpretation of the solar outburst of radio waves. The transverse waves generated as described in § 2 will escape into free space as radio waves, since the plasma density decreases as one leaves the atmosphere of the sun. We have, therefore, only to ask how much energy is converted into transverse waves by the mechanism described in § 2.

The energy of transverse waves radiated per unit volume is given by $\hbar\omega w$, where w is given by (6). The effective volume in which the radiation takes place can be obtained, if the thickness of the shock front, l , and the cross section of streams are known. The latter may be assumed as equal to the area of active sun spots that is the fraction, f , of the surface area of the sun, $f \cdot 4\pi R_s^2$, where R_s is the radius of the sun. The intensity received on the ground surface of the earth is obtained by dividing the radiated intensity at the sun by $4\pi R_{s-E}^2$, where R_{s-E} is the distance between the sun and the earth. Thus the power received per frequency band is obtained as

$$dP/d\omega_t = d \{ \hbar\omega w f l (R_s/R_{s-E})^2 \} / d\omega_t = 2 (N_e^2/3\pi N^2) (v/c)^2 \hbar\omega k^3 f l (R_s/R_{s-E})^2 n_k^l. \quad (7)$$

For the density of electrons, $N_s (\sim N_e) \simeq 10^8 \text{ cm}^{-3}$, and the velocity of electrons, $v_s \simeq 10^8 \text{ cm sec}^{-1}$,

$$l = (\pi d^2 N_s)^{-1} = (2mv_s^2/e^2) / \pi N_s \simeq 10^7 \text{ cm}.$$

The fractional area of active sun spots may be of the order of 10^{-3} . The density of whole electrons, N , seems to be about twice of N_s , so that $\omega_p \simeq 6 \times 10^8 \text{ sec}^{-1}$. Since ω_i is nearly equal to ω_p , $k \simeq 2 \times 10^{-2} \text{ cm}^{-1}$.

Substituting these figures into (7), we have

$$dP/d\omega_t \sim 10^{-33} n_k^l \text{ watt m}^{-2} \text{ cps}^{-1}. \quad (8)$$

This is compared with the observed power, $\sim 10^{-10} \text{ watt m}^{-2} \text{ cps}^{-1}$. Thus we have $n_k^l \sim 10^{11}$ or the equivalent temperature

$$T^* \sim n_e^* \hbar \omega_p / \kappa \sim 10^{12} \text{ deg}, \quad (9)$$

where κ is the Boltzmann constant.

That this is not unreasonably high may be seen in the following manner. In the coordinate system in which the stream is at rest, although the major part of the coronal electrons move past at the velocity of the stream, some ones will, for instance, stay at rest, and others will travel at lower velocities. Hence some electrons will be trapped by one of the troughs of the electric potential resulting from plasma waves already excited in the stream, while others go over the crest and move across the potential wave. A considerable amount of the kinetic energy of the trapped particles will be exhausted to further excite the plasma oscillation, oscillating back and forth many times inside the trough, until individual collisions throw them out of the trapped region. As the plasma waves are excited higher in the stream and larger electric potential is built up near the front, the electrons of higher velocities (in the moving system) will come to be trapped. The energy of plasma oscillations thus increases until equipartition of energy is accomplished between excitations and electrons. The maximum amount of available energy that can be derived from the kinetic energy of the coronal electrons which enter the space occupied by the stream is given by

$$(4\pi/3) \hbar \omega_p n_{\text{max}}^l (k_D/2\pi)^3 = (1/2) N_e m v_s^2,$$

where $1/k_D$ is the Debye length. Since $k_D \sim 1$ in our case, we have $n_{\text{max}}^l \sim 10^{16}$, larger than required in (8).

The observed higher harmonics of the outburst is to be implied in the spectrum of n_k^l . Our theory, however, cannot answer the spectrum of radio waves, but merely tells one the conversion of the plasma energy into the electromagnetic energy.

Our theory may be applied to other phenomena, such as the scattering of two beams of radio waves. Such problems will be discussed in later occasions.

We express our thanks to Messers Kawabata and Takakura for their helpful comments on our work.

Appendix. Derivation of interactions between electron plasma and electromagnetic waves

The Hamiltonian for a set of N non-relativistic electrons interacting with electromagnetic field is written as

$$H = (1/2m) \sum_{i=1}^N \left(\mathbf{P}_i + \frac{e}{c} \mathbf{A}(\mathbf{x}_i) \right)^2 + H_f, \quad (\text{A} \cdot 1)$$

where \mathbf{x}_i and \mathbf{P}_i are the position and the momentum of the i -th electron, respectively, and \mathbf{A} the electromagnetic potential. The Hamiltonian for electromagnetic field, H_f , consists of the transverse (\perp) and the longitudinal (\parallel) parts as

$$H_f = \frac{1}{8\pi} \int (E_{\perp}^2 + H^2) d\mathbf{x} + \frac{1}{8\pi} \int E_{\parallel}^2 d\mathbf{x}, \quad (\text{A} \cdot 2)$$

where \mathbf{E} and \mathbf{H} are the electric and magnetic field strengths. As we adopt such a gauge that electrostatic potential is set equal to zero, the subsidiary condition acting on the state vector for the whole system ψ , is given by

$$\Omega\psi=0,$$

$$\text{with } \Omega = \text{div } \mathbf{E}(\mathbf{x}) - 4\pi e \sum_{i=1}^N \delta(\mathbf{x} - \mathbf{x}_i). \quad (\text{A} \cdot 3)$$

Now we split H into three parts :

$$H = H_p + H_i + \mathcal{A}, \quad H_0 = H_p + H'_{osc} \quad (\text{A} \cdot 4)$$

where

$$H_p = (1/2m) \cdot \sum_i \mathbf{p}_i^2$$

$$H'_{osc} = \frac{1}{8\pi} \int (\mathbf{E}_\perp^2 + \mathbf{H}^2 + (\omega_p/c)^2 \mathbf{A}_\perp^2) d\mathbf{x} + \frac{1}{8\pi} \int (\mathbf{E}_\parallel^2 + (\omega_p/c)^2 \mathbf{A}_\parallel^2) d\mathbf{x}, \quad (\text{A} \cdot 4b)$$

$$H_i = (e/mc) \sum_i (\mathbf{p}_i \mathbf{A}(\mathbf{x}_i) + \mathbf{A}(\mathbf{x}_i) \mathbf{p}_i) \quad (\text{A} \cdot 4c)$$

and

$$\mathcal{A} = \frac{e^2}{2mc^2} \left(\sum_i \mathbf{A}^2(\mathbf{x}_i) - \frac{N}{L^3} \int \mathbf{A}^2(\mathbf{x}) d\mathbf{x} \right). \quad (\text{A} \cdot 4d)$$

Here ω_p is the so-called plasma frequency defined by

$$\omega_p = (4\pi e^2 N/m)^{1/2}, \quad (\text{A} \cdot 5)$$

where N is the density of electrons and the volume of a cube in which our system is enclosed, L^3 , is taken as unity. Three terms in H_i represent a set of free electrons, the electromagnetic wave in a medium and the longitudinal plasma wave respectively.

In extracting the medium-like behavior from this system, only the Fourier components of long wave lengths are taken into consideration. As we are mainly interested in this part, the field quantities, \mathbf{A} , \mathbf{E} and \mathbf{H} are understood by such that contain longitudinal waves of wave lengths longer than the Debye length, λ_D , which is of the order of distance traveled during the period of an plasma oscillation by an electron moving with the mean thermal speed, $\langle P/m \rangle_{Av}/\omega_p$ and that contain transverse waves of wave lengths longer than $\lambda_c = c/\omega_p$. Namely,

$$\mathbf{A}(\mathbf{x}) = (4\pi e^2)^{1/2} \left[\sum_{k < 2\pi/\lambda_D} \boldsymbol{\epsilon}_k q_k e^{i\mathbf{k}\cdot\mathbf{x}} + \sum_{k < 2\pi/\lambda_c} \sum_{\mu=1,2} \boldsymbol{\epsilon}_{k\mu} q_{k\mu} e^{i\mathbf{k}\cdot\mathbf{x}} \right], \quad (\text{A} \cdot 6)$$

where $\boldsymbol{\epsilon}_k$ and $\boldsymbol{\epsilon}_{k\mu}$ are the unit vectors parallel and perpendicular to the wave number vector \mathbf{k} , respectively. q_k and $q_{k\mu}$ are the coordinates of the longitudinal and the transverse waves, respectively. The canonical momenta conjugate to them are given by

$$p_k = \dot{q}_k, \quad p_{k\mu} = \dot{q}_{k\mu} \quad (\mu=1, 2). \quad (\text{A} \cdot 7)$$

In terms of these canonical variables an assembly of harmonic oscillators can be obtained as

$$H'_{osc} = -\frac{1}{2} \sum_{k < 2\pi/\lambda_D} (p_k p_{-k} + \omega_p^2 q_k q_{-k}) + \frac{1}{2} \sum_{k < 2\pi/\lambda_c} (p_{k\mu} p_{-k\mu} + (c^2 k^2 + \omega_p^2) q_{k\mu} q_{-k\mu}). \quad (\text{A} \cdot 8)$$

Hereafter the restrictions $k < k_D$, k_c , where $k_D = 1/\lambda_D$ and $k_c = 1/\lambda_c$, are always assumed, if not specially mentioned. The first part of the right hand side represents a set of longitudinal oscillators and the second part transverse oscillators, with angular frequencies

$$\omega_{||} = \omega_p, \quad \omega_{\perp} = (\omega_p^2 + c^2 k^2)^{1/2}, \quad (\text{A} \cdot 9)$$

respectively.

Next, \mathcal{A} defined by (A·4d) is written as

$$\mathcal{A} = (2\pi e^2/m) \left[\sum_{k, l < k_D} \epsilon_k \epsilon_l q_k q_l e^{i(k+l)x_0} + \sum_{\substack{k, l, \mu, \nu \\ k < k_c}} \epsilon_{k\mu} \epsilon_{l\nu} q_{k\mu} q_{l\nu} e^{i(k+l)x_0} \right]. \quad (\text{A} \cdot 10)$$

The restriction, $k, l < k_D$, k_c , allows us to use the random phase approximation which makes \mathcal{A} vanish, because \mathcal{A} tends to average out to zero.

Now, let us introduce, in the usual way, the annihilation and creation operators by

$$a_k = (\omega_l/2\hbar)^{1/2} (q_k - i\omega_l^{-1} p_{-k}), \quad a_k^* = -(\omega_l/2\hbar)^{1/2} (q_{-k} + i\omega_l^{-1} p_k), \quad (\text{A} \cdot 11a)$$

for longitudinal oscillators and

$$b_{k\mu} = (\omega_l/2\hbar)^{1/2} (q_{k\mu} + i\omega_l^{-1} p_{-k\mu}), \quad b_{k\mu}^* = (\omega_l/2\hbar)^{1/2} (q_{k\mu} - i\omega_l^{-1} p_{-k\mu}) \quad (\text{A} \cdot 11b)$$

for transverse oscillators. They satisfy the commutation relations

$$[a_k, a_{k'}^*] = \delta_{kk'}, \quad [b_{k\mu}, b_{k'\mu'}^*] = \delta_{kk'} \delta_{\mu\mu'}, \quad (\text{A} \cdot 12)$$

all other commutators vanishing. Unspecified frequencies ω_l and ω_t are those of longitudinal and transverse oscillators, respectively, which will be determined in what follows.

By use of (A·11) H'_{osc} is now separated into two parts:

$$H'_{osc} = H_{osc} + W. \quad (\text{A} \cdot 13)$$

H_{osc} represents sets of longitudinal and transverse oscillators, which we call collectons, with proper frequencies $\omega_l(k)$ and $\omega_t(k)$ of disperse character, respectively:

$$H_{osc} = \frac{1}{2} \sum_k \hbar \omega_l (a_k a_k^* + a_k^* a_k) + \frac{1}{2} \sum_{k\mu} \hbar \omega_t (b_{k\mu} b_{k\mu}^* + b_{k\mu}^* b_{k\mu}). \quad (\text{A} \cdot 14)$$

W is an unwanted term and will be eliminated owing to the dispersion relations; otherwise this would give rise to unharmonicity:

$$W = \sum_k \hbar (4\omega_l)^{-1} (\omega_p^2 - \omega_l^2) (a_k a_{-k}^* + a_{-k}^* a_k - a_k a_{-k} - a_{-k}^* a_k^*) + \sum_{k\mu} \hbar (4\omega_t)^{-1} (\omega_p^2 + c^2 k^2 - \omega_l^2) (b_{k\mu} b_{-k\mu}^* + b_{-k\mu}^* b_{k\mu} + b_{k\mu} b_{-k\mu} + b_{-k\mu}^* b_{k\mu}^*). \quad (\text{A} \cdot 15)$$

Beside the above two parts, the Hamiltonian contains an interaction Hamiltonian

$$H_I = (e/m) \sum_{i,k} (2\pi\hbar/\omega_i)^{1/2} \left\{ \boldsymbol{\varepsilon}_i \left(\mathbf{P}_i - \frac{\hbar\mathbf{k}}{2} \right) a_i e^{i\mathbf{k}\cdot\mathbf{x}_i} + e^{-i\mathbf{k}\cdot\mathbf{x}_i} a_k^* \boldsymbol{\varepsilon}_k \left(\mathbf{P}_i - \frac{\hbar\mathbf{k}}{2} \right) \right\} \\ + (e/m) \sum_{i,k,\mu} (2\pi\hbar/\omega_i)^{1/2} \left\{ \boldsymbol{\varepsilon}_{k\mu} \cdot \mathbf{P}_i b_{i\mu} e^{i\mathbf{k}\cdot\mathbf{x}_i} + e^{-i\mathbf{k}\cdot\mathbf{x}_i} b_{i\mu}^* \boldsymbol{\varepsilon}_{k\mu} \cdot \mathbf{P}_i \right\}. \quad (\text{A} \cdot 16)$$

In the transverse part of H_I the order of factors is unimportant, because $\boldsymbol{\varepsilon}_{k\mu} \cdot \mathbf{P}_i$ commutes with $\mathbf{k} \cdot \mathbf{x}_i$ due to the transversality condition. In the longitudinal part, on the contrary, the order of factors is essential, since $\boldsymbol{\varepsilon}_k \cdot \mathbf{P}_i$ does not commute with $\mathbf{k} \cdot \mathbf{x}_i$. This gives rise to terms containing $\hbar\mathbf{k}$ which brings about the coupling of the zero-point motion of electrons with longitudinal collectons even at zero temperature, whereas the transverse collectons couple with electrons only at finite temperatures.

The wave function of our system in the interaction representation with unperturbed Hamiltonian $H_{osc} + \sum \mathbf{P}_i^2/2m$ satisfies the wave equation

$$i\hbar\partial\psi(t)/\partial t = (H_I(t) + W(t))\psi(t), \quad (\text{A} \cdot 17)$$

where

$$H_I(t) = \left(\frac{e}{m} \right) \sum \left(\frac{2\pi\hbar}{\omega_i} \right)^{1/2} \left\{ \boldsymbol{\varepsilon}_i \cdot \left(\mathbf{P}_i - \frac{\hbar\mathbf{k}}{2} \right) a_i e^{-i\omega_i t} e^{i\mathbf{k} \cdot \left(\mathbf{x}_i + \frac{\mathbf{P}_i t}{m} \right)} \right. \\ \left. + e^{-i\mathbf{k} \cdot \left(\mathbf{x}_i + \frac{\mathbf{P}_i t}{m} \right)} a_k^* e^{i\omega_i t} \boldsymbol{\varepsilon}_k \cdot \left(\mathbf{P}_i - \frac{\hbar\mathbf{k}}{2} \right) \right\} + \left(\frac{e}{m} \right) \sum \left(\frac{2\pi\hbar}{\omega_i} \right)^{1/2} \times \\ \left\{ \boldsymbol{\varepsilon}_{k\mu} \cdot \mathbf{P}_i b_{i\mu} e^{-i\omega_i t} e^{i\mathbf{k} \cdot \left(\mathbf{x}_i + \frac{\mathbf{P}_i t}{m} \right)} + e^{-i\mathbf{k} \cdot \left(\mathbf{x}_i + \frac{\mathbf{P}_i t}{m} \right)} b_{i\mu}^* e^{i\omega_i t} \boldsymbol{\varepsilon}_{k\mu} \cdot \mathbf{P}_i \right\} \quad (\text{A} \cdot 18a)$$

and

$$W(t) = \sum \hbar(4\omega_i)^{-1} (\omega_p^2 - \omega_i^2) (a_i a_k^* + a_k^* a_i - a_i a_{-k} e^{-i(\omega_i(k) + \omega_i(-k))t} - \\ - a_k^* a_{-k} e^{i(\omega_i(k) + \omega_i(-k))t}) + \sum \hbar(4\omega_i)^{-1} (\omega_p^2 + c^2 k^2 - \omega_i^2) \times \\ \times (b_{i\mu} b_{i\mu}^* + b_{i\mu}^* b_{i\mu} + b_{i\mu} b_{-k\mu} e^{-i(\omega_i(k) + \omega_i(-k))t} + b_{i\mu}^* b_{-k\mu}^* e^{i(\omega_i(k) + \omega_i(-k))t}), \quad (\text{A} \cdot 18b)$$

\mathbf{x} and \mathbf{P} standing for $\mathbf{x}(0)$ and $\mathbf{P}(0)$ respectively.

Now, following Bohm and Pines, we pass to the representation in which the bare electron carries its accompanying cloud of virtual collectons by redefining the wave function by the relation

$$\psi = e^{iS/\hbar} \psi' \quad \text{with} \quad S = - \int_{-\infty}^t H_I(t') dt'. \quad (\text{A} \cdot 19)$$

The wave equation for ψ' now becomes

$$i\hbar\partial\psi'/\partial t = \{ W + i/2\hbar \cdot [H_I, S] \} \psi', \quad (\text{A} \cdot 20)$$

retaining only quadratic terms in the field variables in the right hand side. In the

commutator $[H_I, S] = \int_{-\infty}^{t=0} [H_I(t'), H_I(0)] dt'$ are involved terms describing the interaction of the electrons through the agency of virtual collectons, including the self-action of individual electrons. They are written in the approximation of small velocities of electrons or small $\langle (\mathbf{kP}_i/m\omega)^2 \rangle_{Av}$ as

$$-\frac{2\pi e^2}{m^2} \sum_{i,j} \frac{(\boldsymbol{\varepsilon}_k \cdot \mathbf{P}_i)(\boldsymbol{\varepsilon}_k \cdot \mathbf{P}_j)}{\omega_p^2 + k^2 \langle \mathbf{p}^2/m^2 \rangle_{Av}} e^{i\mathbf{k}(\mathbf{x}_i - \mathbf{x}_j)} \quad (\text{A} \cdot 21a)$$

$$-\frac{2\pi e^2}{m^2} \sum_{i,j} \frac{(\boldsymbol{\varepsilon}_{k_{ij}} \cdot \mathbf{P}_i)(\boldsymbol{\varepsilon}_{k_{ij}} \cdot \mathbf{P}_j)}{\omega_p^2 + c^2 k^2 - (\mathbf{k} \cdot \mathbf{P}_i/m)^2} e^{i\mathbf{k}(\mathbf{x}_i - \mathbf{x}_j)} \quad (\text{A} \cdot 21b)$$

and

$$-(1/N) \sum_{i,k} (\boldsymbol{\varepsilon}_k \cdot \mathbf{P}_i)^2 / 2m = \sum_i (\mathbf{P}_i^2 / 2m^*) - \sum_i (\mathbf{P}_i^2 / 2m), \quad (\text{A} \cdot 21c)$$

where m^* is the effective mass of the electron with its associated cloud of longitudinal collectons, i.e., quanta of collective oscillations.

A simple identity concerning the exponential function of operators enables us to easily evaluate the commutator, $[H_I, S]$. Let A and B be any two operators whose commutator commutes with either of them. Then $\exp(A+B) = \exp(-\frac{1}{2}[A, B]) \exp A \exp B$. Applying this, one has, for example,

$$\begin{aligned} & \int \left[\boldsymbol{\varepsilon}_k \left(\mathbf{P} - \frac{\hbar \mathbf{k}}{2} \right) a_k e^{-i\omega t} e^{i\mathbf{k} \left(\mathbf{x} + \frac{\mathbf{P}}{m} t \right)}, \boldsymbol{\varepsilon}_{-k} \left(\mathbf{P} + \frac{\hbar \mathbf{k}}{2} \right) a_{-k} e^{-i\mathbf{k} \cdot \mathbf{x}} \right] dt \\ &= \int \left[\boldsymbol{\varepsilon}_k \left(\mathbf{P} - \frac{\hbar \mathbf{k}}{2} \right) a_k e^{-i \left(\omega - \frac{\mathbf{kP}}{m} + \frac{\hbar k^2}{2m} \right) t} e^{i\mathbf{kx}}, e^{-i\mathbf{kx}} \left(-\boldsymbol{\varepsilon}_k \right) \cdot \left(\mathbf{P} - \frac{\hbar \mathbf{k}}{2} \right) a_{-k} \right] dt \\ &= - \int \left\{ \left(\mathbf{P} - \frac{\hbar \mathbf{k}}{2} \right)^2 e^{-i \left(\omega - \frac{\mathbf{kP}}{m} + \frac{\hbar k^2}{2m} \right) t} - \left(\mathbf{P} + \frac{\hbar \mathbf{k}}{2} \right)^2 e^{-i \left(\omega - \frac{\mathbf{kP}}{m} - \frac{\hbar k^2}{2m} \right) t} \right\} a_k a_{-k} dt \\ &= -i \left\{ \frac{(\mathbf{P} - \hbar \mathbf{k}/2)^2}{(\omega - \mathbf{kP}/m + \hbar k^2/2m)} - \frac{(\mathbf{P} + \hbar \mathbf{k}/2)^2}{(\omega - \mathbf{kP}/m - \hbar k^2/2m)} \right\} a_k a_{-k} \\ &= i m \hbar \left\{ \frac{\omega^2}{(\omega - \mathbf{kP}/m)^2 - (\hbar k^2/2m)^2} - 1 \right\} a_k a_{-k} \end{aligned}$$

In this way we have

$\frac{i}{2\hbar} \int_{-\infty}^0 [H_I(t) H_I(0)] dt = (\text{A} \cdot 21) - (\text{A} \cdot 25) -$ (terms which are quadratic in field variables multiplied by a phase factor with nonvanishing argument)

$$\begin{aligned} &= \sum_k \left(\frac{\hbar}{4\omega_i} \right) \left\{ \sum_i \frac{4\pi e^2}{m} \frac{\omega_i^2}{(\omega_i - \mathbf{kP}_i/m)^2 - (\hbar k^2/2m)^2} - \omega_p^2 \right\} \\ &\quad \times (a_k a_k^* + a_k^* a_k - a_k a_{-k} - a_{-k}^* a_k^*) \quad (\text{A} \cdot 22l) \end{aligned}$$

$$+ \sum_{k, \mu} \left(\frac{\hbar}{4\omega_l} \right) \left\{ \sum_i \frac{4\pi e^2}{m} \frac{(\mathbf{\epsilon}_{l, \mu} \cdot \mathbf{P}_i)^2 k^2 / m^2}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} \right\} \\ \times (b_{k, \mu} b_{k, \mu}^* + b_{l, \mu}^* b_{l, \mu} + b_{l, \mu} b_{-k, \mu} + b_{k, \mu}^* b_{-l, \mu}^*). \quad (\text{A} \cdot 22\text{t})$$

Combining this with W in (A.18b), we obtain the dispersion relation :

$$\frac{4\pi e^2}{m} \sum_i \frac{1}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} - 1 = 0, \quad (\text{A} \cdot 23\text{l})$$

and
$$\omega_p^2 + c^2 k^2 - \omega_l^2 + \frac{4\pi e^2}{m} \sum_{i, \mu} \frac{(\mathbf{\epsilon}_{l, \mu} \cdot \mathbf{P}_i)^2 k^2 / m^2}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} = 0. \quad (\text{A} \cdot 23\text{t})$$

The new subsidiary condition which now acts to reduce the long wave density fluctuation of the electron gas reads*)

$$\exp(iS/\hbar) \mathcal{Q} \exp(-iS/\hbar) \psi' \\ = \sum_i \sum_{k < k_D} \frac{\omega_i^2}{\omega_i^2 - (\mathbf{k} \cdot \mathbf{P}_i / m - \hbar k^2 / 2m)^2} \exp(i\mathbf{k} \cdot \mathbf{x}_i) \psi' = 0, \quad (\text{A} \cdot 24)$$

where use has been made of the dispersion relation (A.23).

The remainder of the terms in $[H, S]$ are :

$$\sum_{k, \mu} \left(\frac{\hbar}{4(\omega_l \omega_i)^{1/2}} \right) \frac{4\pi e^2}{m} \sum_i \left\{ \frac{\omega_l (\mathbf{\epsilon}_l \cdot \mathbf{k}) (\mathbf{\epsilon}_i \cdot \mathbf{P}_i) / m}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} + \frac{\omega_l (\mathbf{\epsilon}_l \cdot \mathbf{k}) (\mathbf{\epsilon}_{l, \mu} \cdot \mathbf{P}_i) / m}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} \right\} \times \\ \times (a_k^* b_{k, \mu} + b_{k, \mu}^* a_k) \quad (\text{A} \cdot 25\text{a})$$

$$+ \sum_{k, \mu} \left(\frac{\hbar}{4(\omega_l \omega_i)^{1/2}} \right) \frac{4\pi e^2}{m} \sum_i \left\{ \frac{\omega_l (\mathbf{\epsilon}_l \cdot \mathbf{k}) (\mathbf{\epsilon}_{l, \mu} \cdot \mathbf{P}_i) / m}{(\omega_l - \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} - \frac{\omega_l (\mathbf{\epsilon}_l \cdot \mathbf{k}) (\mathbf{\epsilon}_{k, \mu} \cdot \mathbf{P}_i) / m}{(\omega_l + \mathbf{k} \cdot \mathbf{P}_i / m)^2 - (\hbar k^2 / 2m)^2} \right\} \times \\ \times (a_k b_{-k, \mu} + a_k^* b_{-k, \mu}^*). \quad (\text{A} \cdot 25\text{b})$$

(A.25) describes the coupling between the longitudinal and transverse collectons through the intermediary of electrons, as exhibited, for instance, in such processes as the absorption of a quantum of plasma oscillation initially present in the plasma by an electron and the subsequent emission of a transverse quantum, and as the annihilation of a pair of transverse and longitudinal collectons, the energy thereby liberated becoming the kinetic energy of the electron.

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* The validity of the subsidiary condition is criticized by E. N. Adams (Phys. Rev. **98** (1955), 1130). As we are dealing with a large volume of a diffuse electron gas, however, the random phase approximation that makes A in (A.10) vanish may be justified, so as to allow us to assume the subsidiary condition.

Deuteron Stripping Reactions and Nuclear Shell Structure

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The purpose of this paper is to show that the mixing of the configurations can successfully be applied to deuteron stripping reactions when the angular distribution is characterized by two orbital angular momenta, and conversely that the percentage of mixing of the configurations can be determined from the observed relative heights of two peaks in the angular distribution.

The calculations are based upon the perturbation theory. The agreement between the calculated and observed angular distributions is good. For the reaction $F^{19}(d, p)F^{20}$, the unexpectedly small reduced widths are reasonably explained from the standpoint of configuration mixing. Through the investigation of $Mg^{25}(d, p)Mg^{26*}$, we found that spin assignment of the first excited state of Mg^{26} is favourable to 2.

Finally we discuss the relations between the type of mixing of the configuration used here and other types introduced to explain the quadrupole moments of odd nuclei.

§ 1. Introduction

It has been pointed out by many authors^{1),2)} that the angular distributions of certain (d, p) and (d, n) reactions should give a sensitive measure of the accuracy of the nuclear shell model in ascribing definite orbital angular momentum states to nucleons in nucleus. This possibility is due to the facts that such a reaction proceeds mainly through a stripping process and that its angular distribution is characterized by the orbital angular momentum l with which the captured particle can be accepted into an appropriate final state. The theoretical expression of the differential cross section³⁾ for the deuteron stripping reaction includes the summation over all possible values of l which are allowed by the angular momentum and parity conservations. Therefore, for each allowed value of l one may expect a peak in the angular distribution without interference with different l . The first peak at small angles, which comes from the smallest allowed l , is in general higher than the others. The height of the peak decreases rapidly and its position moves progressively towards larger angle as l increases. These features facilitate us to investigate the validity of the nuclear shell model by deuteron stripping reactions.

It is well known that many properties of nuclei can be interpreted quite successfully by the nuclear shell model with strong spin-orbit interaction. It is, however, expected that different configurations become intermingled with that predicted by the simple shell model in the presence of a short-range attractive force between nucleons. Recently the idea of mixing of the configurations has been applied with good success to magnetic moments of odd-A nuclei,⁴⁾ quadrupole moments,⁵⁾ and the beta decays and gamma transitions of forbidden types.

The aim of this paper is, first of all, to show that the mixing of configurations can successfully be applied also to a certain type of the deuteron stripping reactions. The reaction is such that its angular distribution is characterized by two values of l . This evidently shows the presence of mixing configurations in addition to the one assigned by the shell model.

The following selection rules seem to be very useful to determine uniquely the type of mixing of the configurations. Namely, the stripping process can not occur unless there is any overlapping between parent states of the product nucleus and the ground state of the target nucleus. The conservation of angular momentum and parity gives a further restriction. For example, if the target and the product nuclei have the same parity, the orbital angular momentum of the captured particle must be all even, and if different parity, all odd. Furthermore, the centrifugal force makes the heights of peaks smaller for larger value of l . Therefore it is sufficient to take account of mixing of such configurations that the orbital angular momentum corresponding to the mixed state is smaller by two units than that predicted by the simple shell model.

The general formula for the stripping reaction is given for any type of configuration interaction. Numerical values of the coefficients of mixing can be obtained by this formula from the observed relative heights of peaks in the angular distribution. On the other hand, such mixing of the configurations is considered to be due mainly to the presence of short-range attractive forces between nucleons. We assume here, for simplicity, that the radial dependence of the interaction is of δ -function type, and calculate the coefficients by making use of the perturbation method.

It is expected that the coefficients of mixing are large enough to cause the observable two peaks in the angular distributions when the numbers of proton or neutrons in the target and product nuclei lie between eight and twenty. In these nuclei the order of levels is in general $d_{3/2}$, $s_{1/2}$, $d_{5/2}$, but the $d_{3/2}$ and $s_{1/2}$ levels are only slightly separated and occasionally cross one another. These are the cases in the reactions $F^{19}(d, p)F^{20}$, $Mg^{25}(d, p)Mg^{26+}$, $P^{31}(d, p)P^{32}$ and $Cl^{35}(d, p)Cl^{36}$ in which the mixing of the configurations includes $l=0$ and $l=2$. The angular distributions calculated on the basis of the above idea are compared with experimental data, and they are found to give a good fit. Particularly, the unexpected small reduced widths found in $F^{19}(d, p)F^{20}$ are reasonably explained as well as its angular distribution. This reaction is particularly interesting because there is a vanishing contribution from the reaction matrix element between the main configurations in F^{19} and F^{20} .

It is hoped that the results which we obtained here give a support to the presence of the mixed configuration and enhance its success on the interpretation of various nuclear properties.

Finally we discuss the relation between the mixing of configurations used here and those adopted in other examples, especially, that introduced to explain the quadrupole moments of odd nuclei. It is suggestive to see that the same configurations are responsible both for the quadrupole moment and (d, n) reactions. Thus it is possible to verify the mixing by observing both quadrupole moment and angular distribution of (d, n) reaction.

§ 2. The derivation of the differential cross section

For the sake of definiteness we consider $A(d, p)B$. Before derivation of the differential cross section is attempted, it seems relevant to do some comments on the approximation adopted here, which is obtained by neglecting both Coulomb repulsive forces for charged particles and the formation of "compound" nucleus. What these neglected terms bring in the angular distribution has been investigated by many authors.⁶⁾ Their conclusions are such that these terms do not change the essential shape of the angular distribution which predicted by Butler's theory, particularly in the case of nuclei with the deuteron bombarding of large kinetic energy and with large Q value. This is the reason why the attempts to explain the angular distribution consisting of the two peaks by taking account of these correction terms have resulted in a failure. As pointed out in § 1, this should be interpreted by taking account of nuclear structure. If proton-nucleus interaction contributes to the excitation of the target nucleus, it is expected that such an excitation is negligibly rare because the proton passes far away from the target nucleus, while the neutron is captured. For the purpose of this paper, therefore, it is not necessary to include the terms above mentioned.

The wave functions of the initial state Φ_a , and of the final state Φ_b are expressed as

$$\Phi_a = \chi_A(A, j_A m_A; \hat{\xi}) \chi_d(\vec{r}_p - \vec{r}_n; \sigma_p, \sigma_n) \exp[i\vec{k}_d \cdot (\vec{r}_n + \vec{r}_p)/2], \quad (1)$$

$$\Phi_b = \chi_B(B, j_B m_B; \hat{\xi}, \vec{r}_n, \sigma_n) \exp[i\vec{k}_p \cdot \vec{r}_p'] \chi_{\mu_p}(\sigma_p), \quad (2)$$

where χ_A and χ_B are the wave functions of the ground state of A with total angular momentum j_A and its z component m_A , and of a certain state of B characterized by quantum numbers j_B and m_B : A and B denote possible other quantum numbers of the respective states, $\hat{\xi}$ stands for the coordinates of all particles in A . χ_{μ_p} is the spin wave function of the proton with component μ_p . \vec{r}_n , σ_n , \vec{r}_p and σ_p are the spatial and spin coordinates of the neutron and proton respectively, measured from the center of mass of A , while \vec{r}_p' is the spatial coordinates of the outgoing proton from the center of mass of B .

Following the assumption made by Butler,^{3),7)} the transition matrix element to be evaluated is given by

$$(\Phi_b, V_{np} \Phi_a) = \int d\hat{\xi} \, dn \, dp \exp[-i\vec{k}_p \cdot \vec{r}_p'] \chi_B^*(B, j_B m_B; \hat{\xi}, \vec{r}_n, \sigma_n) \chi_{\mu_p}^*(\sigma_p) V_{np} \\ \times \exp[i\vec{k}_d \cdot (\vec{r}_n + \vec{r}_p)/2] \chi_d(\vec{r}_p - \vec{r}_n; \sigma_p, \sigma_n) \chi_A(A, j_A m_A; \hat{\xi}), \quad (3)$$

where n and p stand for the coordinates of the neutron and the proton respectively. In order to evaluate the integral we find it convenient to introduce the new variable $\vec{r} = \vec{r}_p - \vec{r}_n$ in place of \vec{r}_p , and to make use of the fact that V_{np} is a function of \vec{r} only. Then, the integral is separated as

$$\sum_{\mu_n} (1/2 \, 1/2 \, \mu_n \mu_p | 1 \mu_d) \int d\vec{r} V_{np}(r) \chi_d(r) e^{i\vec{k}_d \cdot \vec{r}} \\ \times \int d\hat{\xi} \, dn \, \chi_B^*(B, j_B m_B; \vec{r}_n, \sigma_n, \hat{\xi}) \chi_A(A, j_A, m_A; \hat{\xi}) \chi_{\mu_n}(\sigma_n) e^{i\vec{k}_p \cdot \vec{r}_n}$$

where $\chi_d(r)$ is the internal wave function of the deuteron in the S -state*, $\chi_{\mu_n}(\sigma_n)$ the spin wave function of the neutron with component μ_n , $\vec{K} = (1/2)\vec{k}_d - \vec{k}_p$, and $\vec{k} = \vec{k}_d = (M_A/M_B)\vec{k}_p$ respectively, where M_A and M_B are the masses of A and B . If the axis of quantization of l_n , the orbital angular momentum of the participating neutron, is taken in the direction of the recoil momentum \vec{k} , we easily obtain

$$(\phi_b, V_{np}\phi_a) = G(K) \sum_i i^{l_n} [4\pi(2l_n+1)]^{1/2} \hat{O}_{m_{n,0}}(1/2, 1/2; \mu_n, \mu_p, 1; \mu_d) \langle l_n, 1/2; m_n, \mu_n | j_n M_n \rangle \\ \times \langle j_A j_n m_A M_n | j_B m_B \rangle \langle B; j_B || A; j_A j_n(l_n, 1/2) \rangle, \quad (4)$$

where $G(K) = \int d\vec{r} \chi_d(r) \exp(i\vec{K} \cdot \vec{r})$, and the last factor $\langle B; j_B || A; j_A j_n(l_n, 1/2) \rangle$ is just the reduced matrix element of the unit operator.

From this reduced matrix element, it can be seen that the stripping process can not occur unless there is any overlapping between parent states of the product nucleus and the ground state of the target nucleus. This fact very much simplifies our discussions on the mixed configurations.

There are several possible configurations for the state A of target nucleus. We write them as $|[i] j_A \rangle$ for i -th configuration. Then the wave function of the state A can be expressed as

$$|A j_A \rangle = \sum_i \alpha_i |[i] j_A \rangle,$$

where α_i is called as the coefficient of mixing of configuration $[i]$. In the similar way, the state B of product nucleus in question is,

$$|B j_B \rangle = \sum_j \beta_j |[j] j_B \rangle.$$

Each configuration of the product nucleus $|[j] j_B \rangle$ can be expanded in terms of the parent states,

$$|[j] j_B \rangle = \sum_{i, j_n' l_n'} \langle [j] j_B || [i] j_A, j_n' (l_n' 1/2) \rangle \cdot |[i] j_A, j_n' (l_n' 1/2) \rangle.$$

Here the factors $\langle [j] j_B || [i] j_A, j_n (l_n 1/2) \rangle$ are the coefficient of the expansion and are usually given by the weighted c. f. p. of the product nucleus, where the weighting factors are Racah coefficients which take account of the recoupling of the various spin involved in the transition.

Then the reduced matrix element for a pair of configurations concerned is given by

$$\langle [j] j_B || [i] j_A, j_n(l_n 1/2) \rangle = \sum_{j_n' l_n'} \langle [j] j_B || [i] j_A, j_n' (l_n' 1/2) \rangle \langle j_n' l_n' || j_n l_n \rangle. \quad (5)$$

The last factor $\langle j_n' l_n' || j_n l_n \rangle$ is called as the l -dependent dynamical factor, and is the same as derived from the standpoint of the single particle picture. As is well known, it shows the angular distribution characteristic of the deuteron stripping reactions. Its explicit form is given by

* In the stripping reaction the D -state of the deuteron does not play any essential role⁸⁾

$$\langle j_l' | j_n l_n \rangle = -\frac{\hbar^2}{2M_n^*} R_l(r_0) r_0^2 \left\{ \frac{\partial j_l(kr)}{\partial r} - \frac{1}{h_l^{(1)}(itr)} \frac{\partial h_l^{(1)}(itr)}{\partial r} \cdot j_l(kr) \right\}_{r=r_0} \delta_{jj_n} \delta_{ll_n}, \quad (6)$$

where $t^2 = 2M_n^*/\hbar^2 \cdot |E_n|$; M_n^* and E_n are the reduced mass and the binding energy of the captured neutron respectively, $j_l(r)$ and $h_l^{(1)}(r)$ are spherical Bessel and Hankel functions of the first kind, and $R_l(r_0)$ is the value of the radial wave function of the neutron at the nuclear surface, of which the explicit expression is given in Appendix I.

Now, the reduced matrix element between two states A and B can be obtained. It is

$$\langle B; j_B | A; j_A, j_n (l_n 1/2) \rangle = \sum_{j_n' l_n'} \langle j_n' l_n' | j_n l_n \rangle \cdot \sum_{i,j} \alpha_i \beta_j \langle [j] j_B [i] j_A, j_n' (l_n' 1/2) \rangle. \quad (7)$$

We define the last summation as \mathfrak{R} -factor

$$\mathfrak{R}(j_B, j_A j_n; \alpha\beta) = \sum_{i,j} \alpha_i \beta_j \langle [j] j_B [i] j_A, j_n (l_n 1/2) \rangle. \quad (8)$$

It will be seen that \mathfrak{R} -factor above defined is closely related to the intrinsic structure of the nuclear states of the target and product nuclei. In fact, it is related to the "coefficient of fractional parentage" overlapping and the coefficient of mixing of the configurations, as pointed out in our previous report.⁽⁹⁾

We are now in a position to write down the complete expression of the differential cross section for the deuteron stripping reaction. From Eq. (4) we obtain, by performing a statistical average over initial spin states and summing over final spin states,

$$\frac{d\sigma}{d\Omega} = \frac{M_p^* M_d^*}{(2\pi)^2 \hbar^4} \frac{k_p}{k_d} \sum_{m_A m_B} \sum_{\mu_p \mu_d} \frac{1}{3(2j_A + 1)} |(\Psi_b, V_{np} \Phi_a)|^2, \quad (9)$$

which is reduced by virtue of the orthogonality properties of the Clebsch-Gordan coefficients to

$$\frac{d\sigma}{d\Omega} = \frac{M_p^* M_d^*}{2\pi \hbar^4} \frac{k_p}{k_d} \frac{2j_B + 1}{2j_A + 1} G^2(K) \sum_{j_n' l_n'} \mathfrak{R}^2(j_B, j_A j_n'; \alpha\beta) |\langle j_n' l_n' | j_n l_n \rangle|^2, \quad (10)$$

where M_p^* and M_d^* is the reduced mass of the outgoing proton and the incident deuteron respectively.

We are interested in the angular distribution characterized by the presence of mixing of the configurations in nuclei concerned. Particularly it is interesting for us to determine the relative heights of two peaks: let us describe by (j_s, l_s) the peak predicted by the nuclear shell model and by (j_m, l_m) the other peak which is allowed for the selection rule alone, but essentially results from the presence of the mixed configurations. Then the relative height of the peak (j_m, l_m) to the other (j_s, l_s) is given by

$$A = \frac{\mathfrak{R}^2(j_B, j_A j_m; \alpha\beta)}{\mathfrak{R}^2(j_B, j_A j_s; \alpha\beta)} \cdot \frac{\{G(K)^2 \cdot |\langle j_m l_m | j_m l_m \rangle|^2\}_{\theta=\theta_m}}{\{G(K)^2 \cdot |\langle j_s l_s | j_s l_s \rangle|^2\}_{\theta=\theta_s}}, \quad (11)$$

where θ is the angle of the outgoing proton with respect to the direction of the incident deuteron in the *c. m.* system: θ_s and θ_m are the angles at which the peak (j_s, l_s) and the other (j_m, l_m) locate, respectively. The second factor in (11) is easily determined provided that the (j_s, l_s) and (j_m, l_m) are given, while the first depends on the configurations

adopted. When the value of l_m is smaller by two units than l_s , the second quantity is in general very large, and therefore it amplifies the first quantity very much even if it is small.

§ 3. Applications of the theory

The assignment of configuration given by the simple shell model is assumed to be correct in zero-th order. The deviation of the interaction from the average potential, due to the individual short range interaction between two nucleons, may cause the mixing between various configurations. We take into account as the first order such configurations which contain at most two nucleons in different angular momentum states from the original one of zero-th order. In this way, we can define configurations of higher order.

According to this idea, the reaction matrix element is also decomposed into that of zero-th, and of first and higher order. We shall analyse the following reactions in such terms.

a) $F^{10}(d, p)F^{20}$

The angular distribution¹⁰⁾ of the protons associated with the ground state of F^{20} indicates the admixture of $l_n=0$ and 2 transitions. Consequently, since the ground state of F^{10} has spin 1/2 and even parity, the conservation law requires that the ground state of F^{20} has spin 1 and even parity. Configurations of 0-th order are

$$(s_{1/2})_P, \{(d_{5/2})^2(0)\}_N \text{ and } (s_{1/2})_P, \{(d_{5/2})^3(3/2)\}_N$$

for the ground state of F^{10} and F^{20} respectively. It is evident that the above configurations give a vanishing reaction matrix element, since they can have no common parent state at all. Hence, in this case the stripping reaction must occur through some mixed configurations. This idea is also supported by the fact that the reduced width¹¹⁾ obtained from the experiment is unexpectedly small. We will come back to the discussion on this problem later.

The pronounced peak corresponding to $l_n=2$ transition results from configuration

$$(s_{1/2})_P, \{(d_{5/2})^2 d_{3/2}\}_N$$

of 1st order in F^{20} . While, the forward peak corresponding to $l_n=0$ transition is found to be the same magnitude as the peak $l_n=2$ in the experiment. Configuration $(s_{1/2})_P, \{(d_{5/2})^2(0)\}_{s_{1/2}N}$ of 1st order in F^{20} , however, can not mix with the configuration of 0-th order, as far as the short range force between two nucleons is concerned.

This, therefore, suggests that the configurations which contribute to the $l_n=0$ transition are not only the mixed configurations of F^{20} , but also that of F^{19} . Such configurations of 1st order in F^{19} is divided into the two types of the form:

$$(s_{1/2})_P, \{d_{5/2} d_{3/2}\}_N$$

$$(d_{3/2})_P, \{(d_{5/2})^2\}_N \text{ and } (d_{5/2})_P, \{(d_{5/2})^2\}_{N^*}$$

Since the configurations in F^{20} which contribute to the $l_n=0$ transition must have a com-

mon parent state with the configurations in F^{19} , possible configurations of 1st order in F^{20} are

$$(s_{1/2})_P, \{d_{5/2}d_{3/2}s_{1/2}\}_N$$

$$(d_{3/2})_P, \{(d_{5/2})^2s_{1/2}\}_N \text{ and } (d_{5/2})_P, \{(d_{5/2})^2s_{1/2}\}_N.$$

Even if we should take into account further configurations, the reaction matrix elements arising from such configurations are of the higher order, the order of the secondary effect on $l_n=2$ transition. Such configurations little affect the feature of the angular distribution.

Table I. Configurations of F^{19} and F^{20} which contribute to the reaction matrix elements

$F^{20}(1+)$		0-th	1st			
$F^{19}(1/2+)$		$(s_{1/2})_P, \{(d_{5/2})^2s_{1/2}\}_N$	$(s_{1/2})_P, \{(d_{5/2})^2d_{3/2}\}_N$	$(s_{1/2})_P, \{d_{5/2}d_{3/2}s_{1/2}\}_N$	$(d_{3/2})_P, \{(d_{5/2})^2s_{1/2}\}_N$	$(d_{5/2})_P, \{(d_{5/2})^2s_{1/2}\}_N$
0-th	$(s_{1/2})_P, \{(d_{5/2})^2(0)\}_N$	*	$l=2$	*	*	*
1st	$(s_{1/2})_P, \{d_{5/2}d_{3/2}\}_N$	*	$l=2^+$	$l=0$	*	*
	$(d_{3/2})_P, \{(d_{5/2})^2\}_N$	*	*	*	$l=0$	*
	$(d_{5/2})_P, \{(d_{5/2})^2\}_N$	*	*	*	*	$l=0$

* no-overlapping.

+ higher order contribution

The wave functions in which we are interested have the forms for F^{19}

$$|j, j_1^n(0); j_A\rangle + \sum_{j_P j_1} \alpha_{j_P j_1} |j_P, j_1^n(j_1); j_A\rangle + \sum_{j_2} \alpha_{j_2} |j, \{j_1^{n-1} j_2\}(j_2); j_A\rangle, \quad (12)$$

and for F^{20}

$$|j, j^{n+1}(j_0); j_B\rangle + \sum_{j_3} \beta_{j_3} |j, \{j_1^n j_2\}(j_3); j_B\rangle + \sum_{j_P j_4} \beta_{j_P j_4} |j_P, \{j_1^n j\}(j_4); j_B\rangle$$

$$+ \sum_{j_5} \beta_{j_5} |j, [\{j_1^{n-1} j_2\}, j](j_5); j_B\rangle. \quad (13)$$

The reduced reaction matrix element is given by

$$\langle B; j_B || A; j_A j_n \rangle = \beta_{j_3=j_2} \langle j_2 l_2 || j_n l_n \rangle$$

$$+ [\sum_{j_P j_1 j_4} \alpha_{j_P j_1} \beta_{j_4} \sqrt{(2j_A+1)(2j_4+1)} W(j_P j_1 j_B j; j_A j_4)$$

$$+ \sum_{j_2 j_5} \alpha_{j_2} \beta_{j_5} \sqrt{(2j_A+1)(2j_5+1)} W(j_2 j_B j; j_A j_5)] \langle j l || j_n l_n \rangle. \quad (14)$$

b) $Mg^{25}(d, p)Mg^{26*}$

The angular distribution¹²⁾ of the protons associated with the first excited state of Mg^{26} shows two pronounced peaks corresponding to the orbital angular momentum $l_n=2$ and $l_n=0$ state. From this fact, we can conclude that the first excited state of Mg^{26} is of spin 2 or 3 and of even parity.

The 0-th order configuration in the state of Mg^{26} must be expressed as

$$(d_{5/2})^2 d_{3/2}.$$

The configuration of 0-th order, however, predicts only the peak $l_n=2$ resulting from $d_{3/2}$, in spite of the presence of the clear-cut peak $l_n=0$ in the angular distribution. The experiment, therefore, enforces us to conclude that there must be configuration of 1st order which contributes to the peak $l_n=0$, in addition to one of the 0-th order. It is taken to be

$$(d_{3/2})^5 s_{1/2}$$

for such configuration in Mg^{26} .

Wave function for the ground state of Mg^{25} is denoted by

$$|j^n(j); j_A\rangle \quad (15)$$

while one for the first excited state of M^{26} is described in the form

$$|j^n(j)j_1; j_B\rangle + \beta |j^n(j)j_2; j_B\rangle. \quad (16)$$

Then the reduced reaction matrix element is given by

$$\langle B; j_B \| A; j_A j_n \rangle = \langle j_1 l_1 \| j_n l_n \rangle + \beta \langle j_2 l_2 \| j_n l_n \rangle. \quad (17)$$

The first term and the second in (17) contributes to the peak $l_n=2$ and 0 respectively.

c) $\text{P}^{31}(d, p)\text{P}^{32}$

According to the selection rules, the allowed values of l_n are 0 and 2. Configurations of 0-th order in both nuclei

$$(s_{1/2})_P \{ (s_{1/2})^2 \}_N \text{ and } (s_{1/2})_P \{ (s_{1/2})^2 d_{3/2} \}_N$$

however, predict only one peak $l_n=2$ corresponding to $d_{3/2}$.

Although the observed angular distribution¹³ can not deprive us of our hope that there might be a considerable forward scattering which results from the admixture of $l_n=0$, this forward peak is not so distinct as in the other cases. If it is the case, the reaction must be of the first or higher order.

This is particularly interesting, since the orbit $s_{1/2}$ in which the neutron is going to be captured to make a peak $l_n=0$ is already filled up in the configuration of 0-th order in P^{31} . Therefore, it is evident that the presence of the peak $l_n=0$ comes from the mixed configuration in P^{31} . It is

$$(s_{1/2})_P \{ s_{1/2} d_{3/2} \}_N.$$

Any other mixing of the configurations in P^{31} and in P^{32} , which gives rise to contributions to the respective peaks, is all of the magnitude of the higher order.

The wave function for P^{31} in which we are interested is

$$|j, \{j_1^{n_1}(0)j_2^{n_2}(0)\}(0); j_A\rangle + \sum_J \alpha_J |j, \{j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)\}(J); j_A\rangle, \quad (18)$$

where the summation over J is restricted by the relation $|j_1 - j_2| \leq J \leq j_1 + j_2$. On the other hand the wave function for P^{32} still remains as,

$$|j, \{j_1^{n_1}(0)j_2^{n_2+1}(j_2)\}(j_2); j_B\rangle. \quad (19)$$

Then the reduced reaction matrix element is given by

$$\begin{aligned}
& \langle B; j_B || A; j_A j_n \rangle \\
&= \sqrt{\frac{2j_2+1-n_2}{2j_2+1}} \langle j_2 l_2 || j_n l_n \rangle - \sum_j \alpha_j \sqrt{\frac{2n_1(2j+1)(2J+1)}{2j_1+1}} W(j j j j_1; j j_2) \langle j_1 l_1 || j_n l_n \rangle.
\end{aligned}
\tag{20}$$

The first and the second terms in (20) contribute to $l_n=2$ and 0 respectively.

d) $\text{Cl}^{35}(d, p)\text{Cl}^{36}$

The angular distribution¹⁴⁾ of the protons associated with the ground state for the reaction $\text{Cl}^{35}(d, p)\text{Cl}^{36}$ was measured at 6.90 Mev. The observed angular distribution of outgoing protons was characterized by two values of $l_n=0$ and 2.

According to the selection rules, the allowed values of l_n are 0, 2 and 4. The experiment indicates that the ground state of Cl^{35} has not only the configuration of 0-th order

$$(d_{3/2})_P, \{ (s_{1/2})^2 (d_{3/2})^2 \}_N,$$

but also that of 1st order which contributes to the peak corresponding to $l_n=0$. Such configuration in Cl^{35} must be

$$(d_{3/2})_P, \{ s_{1/2} (d_{3/2})^3 \}_N.$$

Any other mixing of the configurations in Cl^{35} and those in Cl^{36} gives rise to higher contributions to the respective peaks also in this case.

It turns out that this reaction is found in the same situation as in the case of the last reaction $\text{P}^{31}(d, p)\text{P}^{32}$, except the difference of the value of n_2 and j .

In this reaction, therefore, the wave functions and the reduced reaction matrix element are given by Eqs. (18), (19) and (20) respectively.

§ 4. Comparison with the experiments

In order to compare the theoretical angular distributions with the observed ones, it is necessary to determine the coefficients of mixing of configurations which appeared in the last section.

The coefficients are calculated with the short-range attractive forces between two nucleons. We assume, for simplicity, that the radial dependence of the interaction is of δ -function type as the short-range limit

$$\mathfrak{B}_{ij} = [V_0 + V_1(\sigma_i \cdot \sigma_j)] \cdot \delta(r_i - r_j) \cdot \delta(\cos \omega - 1) / r_i r_j.$$

Here $V_0 = (3V_t + V_s)/4$, and $V_1 = (V_t - V_s)/4$, where V_t and V_s are the interaction strengths in triplet and singlet states of two nucleons respectively. It has been assumed that the attractive force between two nucleons in the triplet state is stronger than in the singlet state so that $|V_t| = 1.5|V_s|$.

The calculation is performed by making use of perturbation method. Matrix elements of the interaction contain the factor $V_s I$ and $V_t I$, with $I = \int R_{l_1}(r) R_{l_2}(r) R_{l_1'}(r) R_{l_2'}(r) r^2 dr$. They are evaluated using the relation $V_s I_0 = -50/A$ Mev ($I_0 = \int R_{l_1}(r) r^2 dr$), which is deduced from the argument of pairing energy.¹⁵⁾ The radial wave function is approximated

as an oscillator wave function.

Another quantity which we must estimate here is the energy difference ΔE between the configurations concerned. This is assumed to be equal to the differences between the single particle levels. Possible corrections which may come from the interaction between nucleons are neglected as high order. Only in the case of $F^{19}(d, p)F^{20}$, they are taken into account, since the reaction is caused in higher order than in other cases. The energy difference between single particle levels is estimated by using empirical information with the aid of theoretical considerations. In what follows, the numerical values of the energy differences adopted here are given for respective reactions.

a) $F^{16}(d, p)F^{20}$

There have been five types of the coefficients of mixing of the configurations: two for the target nucleus and three for the product nucleus. For example, the coefficient α_{J_1} for the target nucleus is expressed in the form

$$\begin{aligned}\alpha_{J_1} &= -\langle j, j_1^n(0); j_A \mathfrak{B} | j_p, j_1^n(J_1); j_A \rangle / \Delta E_{J_1} \\ &= -\sqrt{\frac{2n(2j_1+1-n)(2J_1+1)}{(2j_1-1)(2j_1+1)(2j+1)}} \frac{V_{\pi} I b_{J_1}^{(\pi)}(jj_1; j_p j_1)}{\pi} / \Delta E_{J_1},\end{aligned}\quad (21a)$$

where

$$\begin{aligned}\sum_{\pi} V_{\pi} I b_{J_1}^{(\pi)}(jj_1; j_p j_1) &= (-)^{j_p-j_1+l+l_p-J_1} \frac{(2j_1+1) \sqrt{(2j+1)(2j_p+1)}}{4(2J_1+1)} \\ &\quad (jj_p \frac{1}{2}-1/2 | J_1 0) (j_1 j_1 \frac{1}{2}-1/2 | J_1 0) \left[\{1 + (-)^{l+l_p-J_1}\} V_0 I - \{1 - (-)^{l+l_p-J_1}\} \right. \\ &\quad \times V_1 I - (-)^{l+l_1+j-j_1} \frac{(2j_1+1)}{4J_1(J_1+1)} \{1 + (-)^{l-J_1}\} \{ (2j+1) + (-)^{l+j_p-J_1} (2j_p+1) \} 2V_1 I \Big].\end{aligned}\quad (21b)$$

The other coefficients are also given by the similar ones, and tabulated in Appendix II.

The energy difference between $d_{3/2}$ and $d_{5/2}$ is taken to be 5.08 Mev from the doublet splitting in O^{17} . The splitting energies between $s_{1/2}-d_{3/2}$ and $s_{1/2}-d_{5/2}$ are determined in the same way as 4 Mev and 1 Mev, respectively.

In F^{20} it is more difficult to estimate the energy differences between the 0-th order and the 1st order configurations than in the case of F^{19} , because the configurations of the 1st order in F^{20} contain the excitation of two particles. It is assumed that they are approximately equal to the two particle excitation. For instance, the energy difference between $(s_{1/2})_I, \{ (d_{3/2})^2 (3/2) \}_N$ and $(s_{1/2})_I, \{ d_{3/2} s_{1/2} \}_N$ is estimated as $\Delta E [(d_{3/2} \rightarrow d_{3/2}) + (d_{3/2} \rightarrow s_{1/2})]$. The excitation energy of the single particle is determined by referring to the splitting energy of O^{17} as before. According to this procedure, however, the energy difference $(s_{1/2})_I, \{ (d_{3/2})^2 (3/2) \}_N \rightarrow (d_{3/2})_I, \{ (d_{3/2})^2 s_{1/2} \}_N$ is zero. While, the difference due to the Coulomb effect is found to be about 1 Mev using the harmonic oscillator wave function. We consider the latter estimation to be more reliable and adopt the value.

The calculated values of the coefficients of mixing of the configurations are listed in Table II and III.

Table II. Coefficients of mixing of the configurations in the ground state for F^{19}

Configuration	$(s_{1/2})_P, \{ (d_{5/2})^2 (0) \}_N$	$(d_{3/2})_P, \{ (d_{5/2})^2 (J_1) \}_N$	$(d_{5/2})_P, \{ (d_{5/2})^2 (J_1) \}_N$	$(s_{1/2})_P, \{ d_{3/2} d_{5/2} \}_N (J_2)$
α		$J_1=2$	$J_1=2$	$J_2=1$
	1	-0.153	0.124	-0.0125

Table III. Coefficients of mixing of the configurations in the ground state for F^{20}

Configuration	$(s_{1/2})_P, \{ (d_{-1/2})^3 (3/2) \}_N$	$(s_{1/2})_P, \{ (d_{5/2})^2 (0) d_{3/2} \}_N (J_3)$	$(d_{3/2})_P, \{ (d_{5/2})^2 s_{1/2} \}_N (J_4)$	$(d_{5/2})_P, \{ (d_{5/2})^2 s_{1/2} \}_N (J_4)$	$(s_{1/2})_P, \{ d_{5/2} d_{3/2} (1) s_{1/2} \}_N (J_5)$	
β		$J_3=3/2$	$J_4=3/2$	$J_4=5/2$	$J_4=3/2$ $J_4=5/2$	$J_5=3/2$
	1	-0.163	0.093	-0.032	0.198 -0.053	-0.019

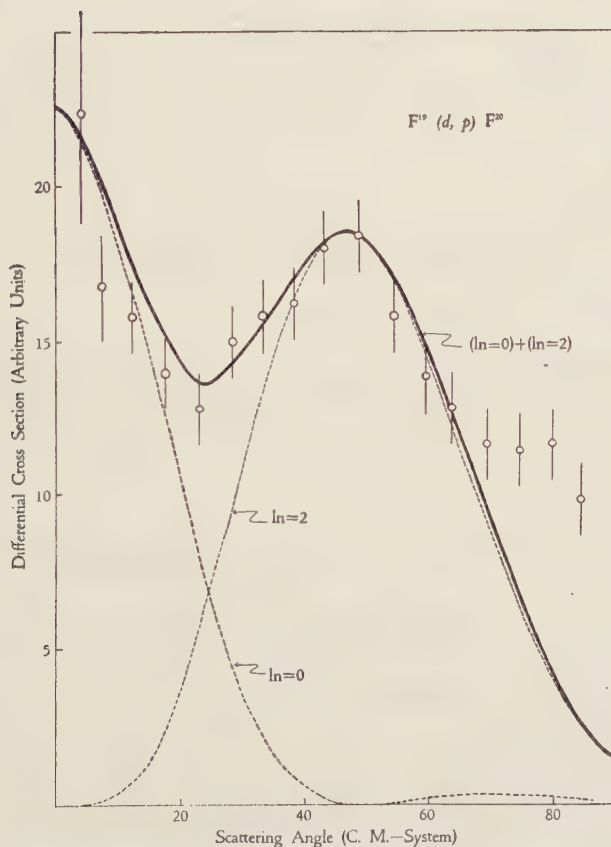


Fig. 1. The angular distribution of the protons associated with the ground state for the reaction $F^{19}(d, p)F^{20}$. The incident deuteron energy is 3.6 Mev and the Q -value of the reaction is 4.375 Mev. The theoretical curves were calculated using value of $R=5.4 \times 10^{-13}$ cm. The calculated ratio of two peaks A is 1.37.

Fig. 1 shows calculated and observed angular distributions in an arbitrary scale. Judging from the good agreement, the calculated values of the coefficients of mixing of the configurations can be concluded to be correct in relative values. Furthermore, we find that the calculated reduced widths, taking account of the coefficients of mixing, are in considerable agreement with those obtained from the experiment. Hence, it may be said that their absolute values are also of adequate order.

In unit of Wigner sum limit, the calculated reduced widths are 0.96 and 0.075 percent for $l_n=2$ and 0 transitions respectively. On the other hand, experimental ones¹¹⁾ are 0.66 and 0.042 for the respective transitions $l_n=2$ and 0.

It is interesting to note that the calculated reduced width consists of the two factors: one is the single particle reduced width, and the other is determined only by the \mathfrak{M} -factor.

$$i_{l_n=2}^2 \left/ \frac{3\hbar^2}{2r_0 M_n^*} \right. = \beta_{J_3=j_2}^2 \cdot \left[\frac{1}{3} r_0^3 R_2^2(r_0) \right],$$

$$i_{l_n=0}^2 \left/ \frac{3\hbar^2}{2r_0 M_n^*} \right. = \left[\sum_{j_p j_3 j_4} \alpha_{j_p j_3 j_4} \beta_{J_4} \sqrt{(2j_A+1)(2j_4+1)} W(j_p j_1 j_R j; j_A J_4) \right. \\ \left. + \sum_{J_2 J_5} \alpha_{J_2} \beta_{J_5} \sqrt{(2j_A+1)(2j_5+1)} W(j J_2 j_R j; j_A J_5) \right]^2 \cdot \left[\frac{1}{3} r_0^3 R_0^2(r_0) \right].$$

b) $\text{Mg}^{25}(d, p)\text{Mg}^{26*}$

The coefficient of mixing of the configuration is given by

$$\beta = -\langle j^n(j)j_1; j_R | \mathfrak{B} | j^n(j)j_2; j_R \rangle / \Delta E$$

$$= \left[\frac{2j-n}{2j-1} \sum_{\kappa} V_{\kappa} \bar{I} \bar{f}_{j_B}^{(\kappa)} + (-)^{j_1+j_2} \frac{n-1}{2j-1} \sum_s \sum_{\kappa} (2S+1) W(jj_1 j_2 j; j_R S) V_{\kappa} \bar{I} \bar{f}_s^{(\kappa)} \right] / (-JE),$$

(22a)

where

$$\sum_{\kappa} V_{\kappa} \bar{I} \bar{f}_{j_B}^{(\kappa)}(jj_1; jj_2) = (-)^{j_1-j_2+l_1-l_2} \frac{(2j+1) \sqrt{(2j_1+1)(2j_2+1)}}{4(2j_R+1)} \\ \times (jj_1 1/2-1/2 | j_R 0)(jj_2 1/2-1/2 | j_R 0) \cdot [1 + (-)^{l_1+l_2-j_B}] V_S I. \quad (22b)$$

It has not been determined whether the spin of the first excited state of Mg^{26} is 2, or 3. Hence, we shall consider the both cases. Assuming that the energy difference ΔE between the 0-th order configuration $\{(d_{5/2})^5 d_{3/2}\}_N$ and the 1st order one $\{(d_{5/2})^4 s_{1/2}\}_N$ is to be taken to be 0.4 Mev, which is the energy difference between the first excited state $s_{1/2}$ and the second one $d_{3/2}$ of Mg^{25} .

Calculated coefficient of mixing β is 0.267 and -0.638 for $j_R=2$ and 3 respectively. As one can see in Fig. 2, the agreement with the experiment is fairly good. Our results are more favourable to $j_R=2$ rather than $j_R=3$.

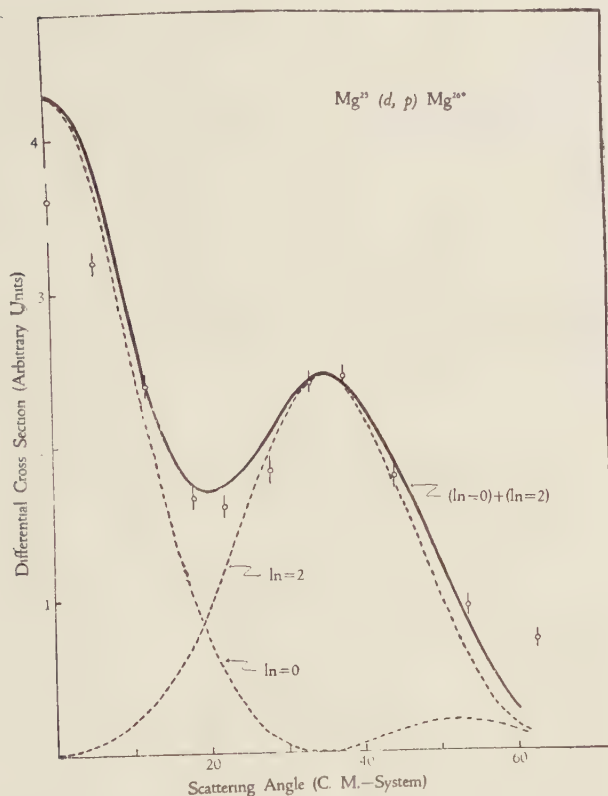


Fig. 2. The angular distribution of the protons associated with the first excited state for the reaction $\text{Mg}^{25}(d, p)\text{Mg}^{26*}$. The incident deuteron energy is 8.0 Mev and the Q -value of the reaction is 7.05 Mev. The theoretical curves were calculated using value of $R=5.27 \times 10^{-13}$ cm. The calculated ratio of two peaks A is 1.7.

c) $\text{P}^{31}(d, p)\text{P}^{32}$

The coefficient of mixing of the configuration is given by

$$\begin{aligned} \alpha_J &= -\langle j, \{j_1^{n_1}(0)j_2^{n_2}(0)\} (0) ; j_A | \mathcal{B} | j, \{j_1^{n_1-1}(j_1)j_2^{n_2+1}(j_2)\} (J) ; j_A \rangle / \Delta E_J \\ &= -(-)^J \sqrt{\frac{n_1(2j_2+1-n_2)(2J+1)}{(2j+1)(2j_1+1)(2j_2+1)}} \sum_{\kappa} V_{\kappa} I h_J^{(\kappa)}(jj_1; jj_2) / \Delta E_J, \end{aligned} \quad (23a)$$

where

$$\begin{aligned} \sum_{\kappa} V_{\kappa} I h_J^{(\kappa)}(jj_1; jj_2) &= (-)^{j-j_2-J} \frac{(2j+1)}{4(2J+1)} \frac{\sqrt{(2j+1)(2j_2+1)}}{4(2J+1)} (jj_1 \frac{1}{2} - \frac{1}{2} | J0) \\ &\times (jj_2 \frac{1}{2} - \frac{1}{2} | J0) \left[\{1 + (-)^J\} V_0 I - \{1 - (-)^J\} V_1 I - (-)^{j+j_1+j_2} \right] \end{aligned}$$

$$\times \frac{(2j+1)}{4J(J+1)} \{1 - (-)^j\} \{(2j_1+1) + (-)^{j_1+j_2-J} (2j_2+1)\} 2V_1 I \}. \quad (23b)$$

For the resultant spin J of the neutron configuration, it is not allowed for any values except $J=1$, because of the triangle relation of the angular momentum and the fact that the ground state spin of P^{32} is unity. As one can see from Eqs. (23a) and (23b), however, the coefficient corresponding to $J=1$ vanishes. This comes from the assumption of the δ -function type as the inter-nucleonic interaction. The higher order contribution to $l_n=0$ is negligibly small compared with that of the 0-th order to $l_n=2$.

It has been pointed out¹³⁾ that there must be about two percent admixture of the configurations corresponding to $l_n=0$ transition. But, our result shows that the coefficient of mixing, if any, would be very small and probably could not be revealed by the experi-

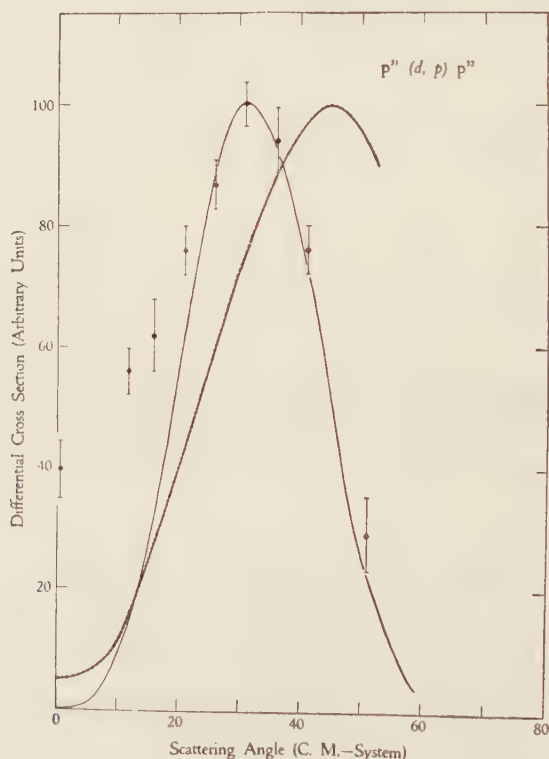


Fig. 3. The angular distribution of the protons associated with the ground state in the reaction $P^{31}(d, p)P^{32}$. The incident deuteron energy is 7.20 Mev (c. m. system) and the Q -value of the reaction is 5.5 Mev. The solid curve was calculated from the Butler theory using value of $R=5.7 \times 10^{-13}$ cm. Butler and Austern have obtained an interesting curve which is shown in the dotted line, taking account of the Coulomb force between the deuteron and target nucleus. Their case involves the capture of $l=2$ protons, an 8 Mev deuteron beam being incident on a nucleus of $Z \approx 15$, with the outgoing particles having 12 Mev, approximately corresponding to this reaction.

ment. On the other hand, the Coulomb correction estimated by Butler and Austern⁶⁾ is found to make the width of a peak wider, so that the presence of $l_n=0$ transition becomes unnecessary to account for the distribution at very small angles. Therefore this example may be regarded to ensure that our approximation on the inter-nucleonic interaction is approximately correct. In Fig. 3, the Butler curve and the curve with correction of the Coulomb effect are plotted in order to facilitate the discussion.

d) $\text{Cl}^{35}(d, p)\text{Cl}^{36}$

For the energy difference between the 0-th order configuration $(d_{3/2})_P, \{(s_{1/2})^2(d_{3/2})^2\}_N$ and the 1st order one $(d_{3/2})_P, \{s_{1/2}(d_{3/2})^3\}_N$, we refer to the energy splitting between the ground state $d_{3/2}$ and the first excited state $s_{1/2}$ of S^{33} , 0.84 Mev.

Then, calculated coefficients of mixing are $\alpha_1=0.061$ and $\alpha_2=0.389$ respectively. Comparison of the calculated angular distribution with the experimental one is shown in Fig. 4. As mentioned in the previous report⁹⁾, the agreement is very good.

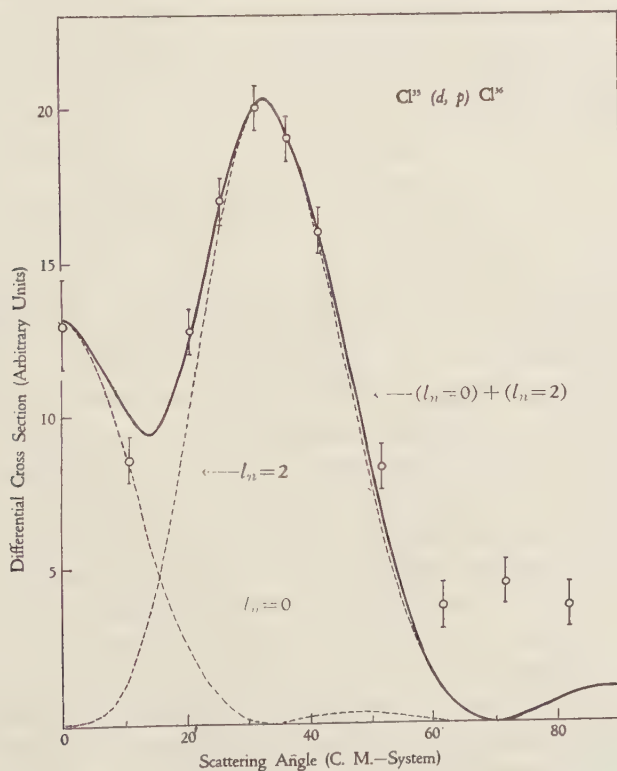


Fig. 4. The angular distribution of the protons associated with the ground state in the reaction $\text{Cl}^{35}(d, p)\text{Cl}^{36}$. The incident deuteron energy is 6.90 Mev (c. m. system) and the Q -value of the reaction is 6.3 Mev. The theoretical curves were calculated using value of $R=5.5 \times 10^{-13}$ cm. The calculated ratio of two peaks A is 0.67.

§ 5. Concluding remarks

In the deuteron stripping reaction, the Butler's formula for the differential cross section has been obtained, neglecting both the Coulomb repulsive forces for incident deuteron and outgoing proton and the effect of "compound" nucleus formation. These neglected effects have been investigated by many authors. According to Tobocman and Kalos, Coulomb force has two effects on the angular distribution: one is displacing the peak towards larger angles, and the other broadening the width of the peak. It has also been pointed out that, as a rule, compound nucleus formation plays a role to compensate the Coulomb effect, and these effects are negligibly small in the cases where the bombarding energy of the deuteron is large. Therefore, these effects do not affect the essential feature of the angular distribution which was investigated in this paper.

When the angular distribution has two peaks corresponding to different l -values due to mixing of configurations, we can classify the reaction into two types. One is the case when the contribution comes from the mixing of configurations in target nucleus. $\text{Cl}^{35}(d, p)\text{Cl}^{36}$ belongs to the former group. In general, the case occurs when the particle orbits corresponding to the mixed l value of peak are already filled up in the zero-th order configuration of the target nuclei. On the other hand, the other case is found in general when the orbit of mixed value is vacant in the zero-th order configuration of the target nuclei, such as $\text{Mg}^{25}(d, p)\text{Mg}^{26*}$.

$\text{F}^{19}(d, p)\text{F}^{20}$ is found to be rather exceptional, because the reduced reaction matrix element vanishes at zero-th order. Since the reaction occurs through the higher order terms, contribution comes from mixing of configurations both in target as well as in product nucleus.

In our investigation on $\text{Mg}^{25}(d, p)\text{Mg}^{26*}$, we find that the spin assignment of the first excited state of Mg^{26} is favourable to two rather than three. It still remains, however, not conclusive for this assignment, since the coefficient of mixing for the latter assignment is rather too large to justify the use of the first order perturbation theory.

Horie and Arima⁵⁾ have introduced a different type of the mixed configurations from ours in order to explain the quadrupole moments in odd nuclei, and have shown that the agreement between the calculated and observed values are fairly good. They have mainly concerned with mixing of the configurations for proton, while we concern mainly with those for neutron, and these two types of mixing of the configurations little affect each other on the calculated values because of the different operator for each cases. Their type is responsible to the (d, n) reaction rather than the (d, p) reaction. Therefore, it is very interesting, in this connection, to investigate the relation between the angular distribution of (d, n) reaction and quadrupole moment of nuclei participating to this reaction. As already stated in § 1, such a relation could be verified by the observation of the angular distributions in the most favourable cases. The further investigation on this point will shortly be reported.

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Appendix I

The explicit form of $R_l(r_0)$ is expressed in the form

$$R_l(r_0) = A_l h_l^{(1)}(itr_0)$$

for the square well potential in which the captured neutron is bounded. Here, A_l is the normalization constant and is given by

$$A_l^2 = \frac{2(-)^l \alpha^2}{(\alpha^2 + t^2) r_0^3 h_{l-1}^{(1)}(itr_0) \cdot h_{l+1}^{(1)}(itr_0)} \quad l > 0.$$

$$A_0^2 = \frac{2\alpha^2}{i(\alpha^2 + t^2) r_0^3 h_0^{(1)}(itr_0) \cdot h_1^{(1)}(itr_0)}.$$

where $\alpha^2 = 2M_n^*/\hbar^2(E_n + V)$: V is the well depth.

We are interested in the relative ratio of two peaks corresponding to $l=2$ and $l=0$. A useful ratio is given by

$$\left[\frac{R_0(r_0)}{R_2(r_0)} \right]^2 = \frac{ih_3^{(1)}(itr_0)}{-h_0^{(1)}(itr_0)}.$$

Even if we adopt the harmonic oscillator potential instead of the square well, this ratio is not so much varied as to affect our conclusions.

Appendix II

The mixing parameters α_J , β_J of the mixed configurations of F^{19} and F^{20} are expressed as follows:

$$\begin{aligned} \alpha_{J_2} &= -\langle j, j_1^n(0); j_A | \mathfrak{B} | j, \{j_1^{n-1} j_2\}(J_2); j_A \rangle / \Delta E_{J_2}, \\ &= -(-)^J \sqrt{\frac{n(2J_2+1)}{(2j+1)(2j_1+1)}} \sum_{\kappa} V_{\kappa} I h_{J_2}^{(\kappa)}(j j_1; j j_2) / \Delta E_{J_2}, \\ \beta_{J_3} &= -\langle j, j_1^{n+1}(J_0); j_B | \mathfrak{B} | j, \{j_1^n j_2\}(J_3); j_B \rangle / \Delta E_{J_3}, \\ &= -\sqrt{\frac{2n(n+1)(2j_1-n+1)}{(2j_1-1)(2j_1+1)(2j_2+1)}} \partial_{J_0 J_2} \sum_{\kappa} (2\kappa+1) \\ &\quad \times \langle j_1^n(\nu=2, \kappa) j_1 | j_1^{n+1}, \nu=3 J_0 \rangle \sum_{\kappa} V_{\kappa} \bar{I} \bar{h}_{J_3}^{(\kappa)}(j_1 j_1; j_1 j_2) / \Delta E_{J_3}, \\ \beta_{J_4} &= -\langle j, j_1^{n+1}(J_0); j_B | \mathfrak{B} | j, \{j_1^n j_2\}(J_4); j_B \rangle / \Delta E_{J_4}, \\ &= -\sqrt{(n+1)(2J_0+1)(2J_4+1)} (-)^{j+j_1-J_0-J_4} \end{aligned}$$

$$\begin{aligned}
& \times \sum_{\mathcal{S}} (2\mathcal{S}+1) W(j_1 j_2 J_1; \mathcal{S} J_0) W(j_1 j_2 J_1; \mathcal{S} J_4) \\
& \times \sum_{\kappa} V_{\kappa} I f_{\mathcal{S}}^{(\kappa)}(j_1; j_2 j) / \Delta E_{J_4}, \\
\beta_{J_0} = & - \langle j, j_1^{n+1}(J_0); j_2 | \mathfrak{B} | j, [\{j_1^{n-1}(j_1) j_2\} (J_2) j] (J_3); j_2 \rangle / \Delta E_{J_0}, \\
& = - \sqrt{\frac{2(n+1)(2j_1+1-n)(2J_2+1)}{2j_1-1}} \delta_{J_0 J_0} \sum_{\mathcal{S}} (2\mathcal{S}+1) \\
& \times \langle j_1^n (\nu=2, \kappa) j_1 | j_1^{n+1}, \nu=3, J_0 \rangle W(j_1 j_1 J_0 j; \kappa \mathcal{S}) W(j_1 j_2 J_0 j; J_2 \mathcal{S}) \\
& \times \sum_{\kappa} V_{\kappa} I \bar{f}_{\mathcal{S}}^{(\kappa)}(j_1 j_1; j_2 j) / \Delta E_{J_0}.
\end{aligned}$$

Here, $\langle j_1^n (\nu=2, \kappa) j_1 | j_1^{n+1}, \nu=3, J_0 \rangle$ is the coefficient of fractional parentage, and tabulated by Edmonds and Flowers.

The general expressions of $\sum_{\kappa} V_{\kappa} I b_{\lambda}^{(\kappa)}(j_1 j_2; j_1' j_2')$ etc. are expressed in the form respectively :

$$\begin{aligned}
\sum_{\kappa} V_{\kappa} I b_{\lambda}^{(\kappa)}(j_1 j_2; j_1' j_2') = & (-)^{j_1 - j_2' + l_1 + l_1' - \lambda} \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)}}{4(2\lambda+1)} \\
& \times (j_1 j_1' 1/2 - 1/2 | \lambda 0) (j_2 j_2' 1/2 - 1/2 | \lambda 0) \cdot \left[\{1 + (-)^{l_1 + l_1' - \lambda}\} V_0 I \right. \\
& - \{1 - (-)^{l_1 + l_1' - \lambda}\} V_1 I - (-)^{l_1 + l_2 + j_1 - j_2} \frac{1}{4\lambda(\lambda+1)} \{(2j_1+1) + (-)^{j_1 + j_1' - \lambda} \\
& \times (2j_1'+1)\} \{(2j_2+1) + (-)^{j_2 + j_2' - \lambda} (2j_2'+1)\} 2V_1 I \Big]. \\
\sum_{\kappa} V_{\kappa} I \bar{b}_{\lambda}^{(\kappa)}(j_1 j_2; j_1' j_2') = & (-)^{j_1 - j_2' + l_1 + l_1' - \lambda} \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)}}{4(2\lambda+1)} \\
& \times (j_1 j_1' 1/2 - 1/2 | \lambda 0) (j_2 j_2' 1/2 - 1/2 | \lambda 0) \left[1 + (-)^{l_1 + l_2 + j_1 - j_2} \frac{1}{4\lambda(\lambda+1)} \right. \\
& \times \{(2j_1+1) + (-)^{j_1 + j_1' - \lambda} (2j_1'+1)\} \{(2j_2+1) + (-)^{j_2 + j_2' - \lambda} (2j_2'+1)\} \Big] V_0 I. \\
\sum_{\kappa} V_{\kappa} I f_{\lambda}^{(\kappa)}(j_1 j_2; j_1' j_2') = & \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)}}{4(2\lambda+1)} (j_1 j_2 1/2 - 1/2 | \lambda 0) \\
& \times (j_1' j_2' 1/2 - 1/2 | \lambda 0) \left[(-)^{l_2 + l_2' - j_2 - j_2'} (V_1 I - V_0 I) - \{(-)^{l_1 + l_2 - \lambda} \right. \\
& + (-)^{l_1 + l_2' - \lambda}\} V_1 I + \frac{1}{4\lambda(\lambda+1)} \{(-)^{j_1 + j_2 - \lambda} (2j_1+1) + (2j_2+1)\} \\
& \times \{(-)^{j_1' + j_2' - \lambda} (2j_1'+1) + (2j_2'+1)\} (V_0 I + V_1 I) \Big]. \\
\sum_{\kappa} V_{\kappa} I \bar{f}_{\lambda}^{(\kappa)}(j_1 j_2; j_1' j_2') = & (-)^{j_2 - j_2' + l_2 + l_2' - \lambda} \frac{\sqrt{(2j_1+1)(2j_2+1)(2j_1'+1)(2j_2'+1)}}{4(2\lambda+1)}
\end{aligned}$$

$$\times (j_1 j_2 \ 1/2 - 1/2 | \lambda 0) (j_1' j_2' \ 1/2 - 1/2 | \lambda 0) [1 + (-)^{l_1 + l_2 - \lambda}] V_s I.$$

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Effect of Hard Core on the Binding Energies of H^3 and He^3 , I*Takashi KIKUTA,[†] Masato MORITA* and Masami YAMADA*Department of Physics, Faculty of Science, University of Tokyo*
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The binding energies of H^3 and He^3 are calculated assuming two-body forces of exponential type with hard cores, radii of which are taken equal for both the spin singlet and the triplet states. The outside potentials are supposed charge-independent, and are chosen so as to fit the nuclear two-body data in the low energy region. The following results have been obtained concerning the effects of hard core: 1) it diminishes the binding energies of H^3 and He^3 considerably, and 2) the wave function is pushed out so that the Coulomb energy of He^3 , which is too large without hard core, is reasonably reduced to the experimental value for the difference between the binding energies of H^3 and He^3 .

§ 1. Introduction

A large number of works have been performed about the deuteron in order to get the knowledge concerning the properties of nuclear forces. H^3 and He^3 , which are the simplest nuclei next to the deuteron, are also investigated by many researchers¹⁾ for that purpose. However, the work before 1949 is hardly reliable for quantitative results, because the accurate ranges of nuclear forces were not known at that period.

When we try to study the nuclear forces with the data of H^3 and He^3 , the most important experimental values will be their binding energies,^{2)**}

$$\text{B.E. exp}(H^3) = 8.49 \text{ Mev}, \quad \text{B.E. exp}(He^3) = 7.72 \text{ Mev},$$

$$\Delta \text{B.E. exp} = \text{B.E. exp}(H^3) - \text{B.E. exp}(He^3) = 0.764 \text{ Mev}.$$

On the other hand, the results of the theoretical work until now show that: (i) if we assume two-body central forces of ordinary type (without hard core) whose depths and ranges are adjusted to fit the low energy two-body data, the theoretical binding energy of H^3 is too large; e.g., B.E. (H^3) is more than 10 Mev for the exponential potential and more than 12 Mev for the Yukawa potential³⁾; (ii) if we take the same $p-p$ and $n-n$ forces as usual, so that the difference between B.E. (H^3) and B.E. (He^3) is due mostly to the Coulomb energy of He^3 , the theoretical values for this difference are too large for both the exponential and the Yukawa wells. Pease and Feshbach⁴⁾ took into account the tensor

* The preliminary report of this work was published in Phys. Rev. **101** (1955), 509.

^{††} Abbreviated as B.E., hereafter.

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forces, and could choose a reasonable range of the tensor force so as to give the experimental value of B.E. (H^3); yet the Coulomb energy of He^3 still remains too large (about 36%).

According to the recent investigation on nucleon-nucleon scattering by Jastrow,⁵⁾ it is highly probable that the nuclear force in the spin singlet even state is strongly repulsive at short distances within 0.6×10^{-13} cm. The repulsive hard core will be favourable for the nuclear saturation,⁶⁾ though it seems difficult to discuss this effect on quantitative bases owing to the complicated nature of many-body problems.

Recently B.E. (H^3) has been calculated by Feshbach and Rubinow⁷⁾ using a hard core interaction. However, their computation is performed only with a special type of nuclear force, the central part of the Lévy potential, so that the effect of hard core is not clear. The purpose of the present work is to investigate the effect of hard core on B.E. (H^3) and B.E. (He^3) and on the difference Δ B.E. between them. It is expected that the theoretical Δ B.E. will be reasonably reduced to the experimental value by the existence of hard core which prevents the nucleons from approaching each other too closely. The following calculation will show that this expectation is correct, so that one of the difficulties in the theory of three-body problem is removed by the introduction of hard core.

In the present paper, the calculation is performed assuming only two-body central forces of exponential type with hard core, which are charge-independent. The radii of hard cores are taken equal both for the spin singlet and triplet states. The ranges and the depths of these forces are chosen so as to fit the nuclear two-body data in the low energy region (§ 2). Tensor forces are neglected to avoid too complicated calculations, expecting that this neglect will not lead to too serious trouble for the effects of hard core. We calculate the binding energy by the standard variational method for various preassigned values of the radius of hard core (§ 3). The numerical results are summarized in § 4. Precision of the results is discussed in § 5. Conclusions are given in § 6. We give in Appendix 1 a method of integration whose region is complicated by the introduction of hard core, and give in Appendix 2 the explicit formulas of normalization, kinetic energy and potential energy involved in the variational expressions (5) and (6). In Appendix 3 the detailed numerical results are given, which may be useful for the calculations employing force parameters other than used in the text. In Appendix 4 we give the relations between force parameters with and without hard core, which enables us to adjust the parameters of hard core potential from the results of Blatt and Jackson⁸⁾ without laborious calculation.

§ 2. Adjustment of the force parameters

As the nuclear force we assume only two-body central force which is expressed by the potential

$$V_{ij} = (3 + \sigma_i \sigma_j) V_t(r_{ij})/4 + (1 - \sigma_i \sigma_j) V_s(r_{ij})/4, \quad (1)$$

which is charge-independent. $V_t(r_{ij})$ and $V_s(r_{ij})$ are potentials for spin triplet and singlet states, respectively, and contain in general the Majorana exchange operators. In the approximation used in this paper, we need $V_t(r_{ij})$ and $V_s(r_{ij})$ for even states only, which are assumed to have the following forms:

$$V_t^{(\text{even})}(r) = \begin{cases} -A_t e^{-\alpha_t(r-D)} & : r > D \\ \infty & : r < D, \end{cases}$$
$$V_s^{(\text{even})}(r) = \begin{cases} -A_s e^{-\alpha_s(r-D)} & : r > D \\ \infty & : r < D. \end{cases}$$

(2)

The force parameters $A_t, A_s, \alpha_t, \alpha_s$ are adjusted so as to fit the following low energy data of two-body system :

- Binding energy of deuteron=2.226 Mev,
- Triplet scattering length of $n-p$ system= 0.5378×10^{-12} cm,
- Singlet scattering length of $n-p$ system= -2.369×10^{-12} cm,
- Singlet effective range of $n-p$ system (r_{0s})
= 0.27×10^{-12} cm or 0.24×10^{-12} cm.

It is well known that the nuclear forces are not strictly charge-independent. We use the potential parameters determined by the $n-p$ data throughout this paper. Since the singlet effective range of $n-p$ system is not yet determined exactly, the calculation will be carried out for the above two different values of r_{0s} . As for the radius of the hard core, we take four values $D=0, 0.2, 0.4, 0.6 \times 10^{-13}$ cm. The force parameters are shown in Table I, together with the depth parameters s_t and s_s defined in Reference 8.

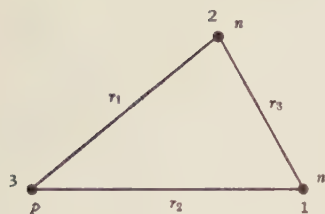
Table I. Force parameters appearing in (2).
 D : in 10^{-13} cm, A : in Mev, α : in 10^{12} cm $^{-1}$.

	D	0	0.2	0.4	0.6
	A_t	192.606	286.246	475.044	947.023
	α_t	15.060	18.946	25.214	36.765
	s_t	1.41707	1.33061	1.24683	1.16907
$r_{0s}=0.27 \times 10^{-12}$ cm	A_s	109.466	155.046	235.414	397.307
	α_s	14.0385	16.608	20.3435	26.272
	s_s	0.92682	0.93791	0.94913	0.96048
$r_{0s}=0.24 \times 10^{-12}$ cm	A_s	137.727	203.895	330.794	623.411
	α_s	15.686	18.972	24.021	32.780
	s_s	0.93399	0.94523	0.95659	0.96808

§ 3. Variational approach

The best way to solve the three-body problem will be the Rayleigh-Ritz variational method. The potential (2) is rather complicated, so we calculate with a simple trial function. The variation-iteration method used by Svartholm⁹⁾ seems too complicated to be applied in the present case.

We use the same coordinate (Fig. 1) and notation as Pease and Feshbach⁴⁾ except for r_3 which is written as ρ in their paper. As the potential used here is of the central type, both the total spin and the total orbital angular momentum are conserved, and they

Fig. 1. Coordinate system for H^3 .

are one half and zero respectively in the ground state. The total τ -spin is also conserved for H^3 , but we do not use the τ -spin formalism here. Then, the wave function must be antisymmetric between the like nucleons. Therefore, the wave function is a linear combination of two wave functions, one of which is spin-symmetric and the other is spin-antisymmetric with respect to like nucleons. The wave function which has $1/2$ for the z -

component of the total spin can be written as

$$\psi = \chi_a \psi_s + \chi_s \psi_a,$$

where

$$\chi_a = [\alpha(1) \beta(2) - \alpha(2) \beta(1)] \alpha(3) / \sqrt{2},$$

and

$$\chi_s = [\alpha(1) \beta(2) + \alpha(2) \beta(1)] \alpha(3) / \sqrt{6} - \sqrt{2/3} \alpha(1) \alpha(2) \beta(3).$$

ψ_s and ψ_a are the space wave functions which are symmetric and anti-symmetric respectively in regard to the like nucleons. They must satisfy the following simultaneous equations for H^3 . (For He^3 it is necessary to include the extra Coulomb energy terms.)

$$\begin{aligned} \{K + V_s(r_3) + V_s(r_1)/4 + V_s(r_2)/4 + 3V_t(r_1)/4 + 3V_t(r_2)/4 - E\} \psi_s \\ + (\sqrt{3}/4) \{V_s(r_1) - V_s(r_2) - V_t(r_1) + V_t(r_2)\} \psi_a = 0, \\ \{K + V_t(r_3) + V_t(r_1)/4 + V_t(r_2)/4 + 3V_s(r_1)/4 + 3V_s(r_2)/4 - E\} \psi_a \\ + (\sqrt{3}/4) \{V_s(r_1) - V_s(r_2) - V_t(r_1) + V_t(r_2)\} \psi_s = 0, \end{aligned} \quad (3)$$

where

$$K \equiv -\frac{\hbar^2}{M} \sum_{\text{cyclic}} \left\{ \frac{1}{r_1^2} \frac{\partial}{\partial r_1} \left(r_1^2 \frac{\partial}{\partial r_1} \right) + \frac{(r_2^2 + r_3^2 - r_1^2)}{2r_2 r_3} \frac{\partial^2}{\partial r_2 \partial r_3} \right\}.$$

If the nuclear forces were spin-independent, i.e., $V_s(r) = V_t(r)$, ψ_a should be zero, and ψ_s should be not only symmetric between the like nucleons but also totally symmetric in three nucleons. In the actual case, $V_s(r)$ is somewhat different from $V_t(r)$ as shown in Table I, so that, strictly speaking, ψ_a is not zero and ψ_s is not totally symmetric.

In this paper, however, we take a totally symmetric function $\psi_s^{(0)}$ as the trial function and ψ_a as zero, which, therefore, should be regarded as the first approximation. This approximation will not cause much error (see, also § 5). So we take the trial function as

$$\begin{aligned} \psi_s^{(0)} = \begin{cases} \prod_i [e^{-\mu(r_i - D)} - e^{-\nu(r_i - D)}] & : \text{all } r_i > D, \\ 0 & : \text{otherwise,} \end{cases} \\ \psi_a^{(0)} = 0. \end{aligned} \quad (4)$$

Then, the binding energy can be written as

$$\text{B.E.} = -E = -(K + U_i + U_s), \quad (5)$$

where

$$U_i = (3/2) (\phi_s^{(i)}, V_i^{(\text{even})}(r_1) \phi_s^{(i)}) / N, \quad (6)$$

$$U_s = (3/2) (\phi_s^{(i)}, V_s^{(\text{even})}(r_1) \phi_s^{(i)}) / N,$$

$$K = (\phi_s^{(i)}, \mathbf{K} \phi_s^{(i)}) / N,$$

with

$$N = (\phi_s^{(i)}, \phi_s^{(i)}).$$

μ and ν in (4) are variational parameters, and they are adjusted so as to give the minimum value of E .

§ 4. Results

The binding energies of H^3 calculated by the method in § 3 are summarized in Table II, and the adjusted values of μ and ν are given in Table III. From Table II we can see how much the binding energy is reduced by the introduction of the hard core.

Table II. Variationally computed binding energy of H^3 .

Hard core radius D in 10^{-13} cm	B.E. (H^3) in Mev	
	$r_{0s} = 2.7 \times 10^{-13}$ cm	$r_{0s} = 2.4 \times 10^{-13}$ cm
0.0	10.26	11.38
0.2	7.86	8.88
0.4	6.49	7.65
0.6	4.78	6.19
Experimental value	8.49	

Table III. Adjusted values of μ and ν in 10^{13} cm^{-1} .

D in 10^{-13} cm	$r_{0s} = 2.7 \times 10^{-13}$ cm		$r_{0s} = 2.4 \times 10^{-13}$ cm	
	μ	ν	μ	ν
0.0	0.4787	∞	0.5033	∞
0.2	0.4444	5.004	0.4672	5.128
0.4	0.4415	4.100	0.4811	4.230
0.6	0.4385	4.279	0.4846	4.407

He^3 is the mirror nucleus of H^3 . The binding energy of He^3 is somewhat smaller than that of H^3 , and this difference will be caused mainly by the Coulomb force between two protons in He^3 . This Coulomb energy is calculated by the perturbation method, which is given in Table IV. It is very remarkable that the hard core of rather short range reduces the Coulomb energy to the experimental value.

For very small values of hard core radius, the binding energy of H^3 , the adjusted

Table IV. Coulomb energy of He^3 .

Hard core radius D in 10^{-13} cm	Coulomb energy of He^3 in Mev	
	$r_{0s} = 2.7 \times 10^{-13}$ cm	$r_{0s} = 2.4 \times 10^{-13}$ cm
0.0	0.986	1.037
0.2	0.810	0.845
0.4	0.729	0.777
0.6	0.676	0.723
Experimental value	0.764	

values of μ and $1/\nu$, and the Coulomb energy of He^3 can be expanded into power series in D (in 10^{-13} cm).

$$\begin{aligned}
 \text{B.E.}(H^3) \text{ (in Mev)} &= \begin{cases} 10.26 - 80.9D + 2012D^2 + \dots : r_{0s} = 2.7 \times 10^{-13} \text{ cm,} \\ 11.38 - 93.5D + 2451D^2 + \dots : r_{0s} = 2.4 \times 10^{-13} \text{ cm,} \end{cases} \\
 \mu \text{ (in } 10^{13} \text{ cm}^{-1}) &= \begin{cases} 0.4787 - 2.097D + \dots : r_{0s} = 2.7 \times 10^{-13} \text{ cm,} \\ 0.5033 - 2.298D + \dots : r_{0s} = 2.4 \times 10^{-13} \text{ cm,} \end{cases} \quad (7) \\
 (1/\nu) \text{ (in } 10^{-13} \text{ cm)} &= \sqrt{2}D + \dots : r_{0s} = 2.7 \times 10^{-13} \text{ cm, } 2.4 \times 10^{-13} \text{ cm,} \\
 E_c(He^3) \text{ (in Mev)} &= \begin{cases} 0.986 - 4.318D + \dots : r_{0s} = 2.7 \times 10^{-13} \text{ cm,} \\ 1.037 - 4.734D + \dots : r_{0s} = 2.4 \times 10^{-13} \text{ cm.} \end{cases}
 \end{aligned}$$

Convergency of these power series is not good except for very small D ($D < 0.01 \times 10^{-13}$ cm). The series, however, show the rapid decrease of B.E. (H^3) as the range of hard core increases.

§ 5. Discussion about the precision of the results

i) Errors in B.E. (H^3)

Since we have used the simple trial functions, some errors will be included in the results described in § 4. In order to estimate these errors, let us compare our results with the accurate calculation by Rarita and Present.¹⁰⁾ The potential used by them is of the form of (1), (2) with

$$\begin{aligned}
 D=0, \quad \alpha_t = \alpha_s = 11.56 \times 10^{12} \text{ cm}^{-1}, \\
 A_t = 123.61 \text{ Mev, } A_s = 70.31 \text{ Mev,}
 \end{aligned} \quad (8)$$

and the odd parts, $V_t^{(\text{odd})}(r) = -V_t^{(\text{even})}(r)$ and $V_s^{(\text{odd})}(r) = -V_s^{(\text{even})}(r)$. Their potential gives the correct binding energy of deuteron, but its range is about 20% larger than ours. If we apply the trial function (4) with $D=0$ to the potential (8), we get B.E. (H^3) = 7.49 Mev, while the accurate value by Rarita and Present is B.E. (H^3) = 8.25 Mev. As the odd part of our potential is not specified, we can not say for certain but the effect of the odd potential will be small in this case, and then the error in B.E. (H^3) for $D=0$ will be about 1 Mev.

One of the origins of the errors is the omission of the anti-symmetric state of the orbital wave function ψ_a and the non-totally-symmetric part of ψ_s which are nearly proportional to the difference between $V_t^{(even)}(r)$ and $V_s^{(even)}(r)$.^{*} As the hard core radius increases, this difference does not seem to become appreciably large. While the difference between their ranges gets larger, the difference between the depth parameters becomes smaller (cf. Table I). Therefore, the error in $D \neq 0$ from this source will be of the same order of magnitude as that in $D=0$.

Another source of the errors is the too simple form (4) of totally symmetric space wave function. In this connection, it may be interesting to solve the deuteron problem with the trial function whose space part is

$$\psi = \begin{cases} e^{-\mu(r-D)} - e^{-\nu(r-D)} & : r > D \\ 0 & : r < D, \end{cases} \quad (9)$$

where r is the distance between two nucleons. The potential of § 2 is used, and the results are given in Table V.

Table V. The binding energies of deuteron with the trial function (9). Accurate calculation should give the correct binding energy 2.226 Mev.

Hard core radius D in 10^{-13}	B.E. (H^2) in Mev
0.0	1.947
0.2	1.972
0.4	1.905
0.6	1.649

If we expand the binding energy of the deuteron in the power series in D (in 10^{-13} cm),

$$\text{B.E. } (H^2) \text{ (in Mev)} = 1.947 - 11.97D + 387D^2 + \dots$$

According to Table V and the above equation, the trial function for $D \neq 0$ (4) will have comparable accuracy to that for $D=0$, except in large D .

ii) Errors in the Coulomb energy of He^3

The Coulomb energy of He^3 has been calculated by the perturbation method in § 4. If we compute B.E. (He^3) by the variational method, (trial function is of the form (4)), including the Coulomb energy from the beginning, we get the value of B.E. (H^3) - B.E. (He^3) about 1% smaller than that in § 4.

Larger errors than the above will arise from the omission of the anti-symmetric state of space wave function ψ_a and the non-totally-symmetric part of ψ_s , and also from the too

* Considering from the view point of permutation group, our trial function (4) has the symmetry character $S(3)$, (e.g., see, F. Hund, Zeits. f. Phys. 43 (1927), 788). Generally ψ_s includes not only $S(3)$ but $S(2+1)$, and ψ_a includes $S(2+1)$ and $S(1+1+1)$. If we use the space function of character $S(3)$ or $S(1+1+1)$ only, the total τ -spin is conserved automatically, but a symmetry relation is necessary between the functions of character $S(2+1)$ in ψ_s and ψ_a . This relation will be useful if we want to improve the approximation.

simplified form of the totally symmetric trial function (4). The latter effect is difficult to estimate, but the former effect will certainly diminish the Coulomb energy as will be seen from the following argument. When we use only the totally symmetric trial function, the potential between the like nucleons is $V_s^{(\text{even})}$, and the one between the unlike nucleons is $3/4 V_t^{(\text{even})} + 1/4 V_s^{(\text{even})}$. As is known from Table I, the former is weaker and has longer range than the latter. Therefore, the like nucleons have a tendency to separate from each other, and the unlike nucleons to approach each other, this fact is the main reason to mix ψ_n and the non-totally-symmetric part of ψ_s . From the above discussion, it is obvious that these omitted parts of the wave function reduce the Coulomb energy.

In the case of $D=0$, we can discuss more definitely by comparing our method with the accurate calculation by Rarita and Present.⁽¹⁾ Their Coulomb energy is 0.756 Mev, while our trial function (4) gives 0.805 Mev when it is applied to their potential (8).

From the above arguments, it is inferred that our Coulomb energy is over-estimated by several per cent.

The arguments in i) and ii) of this section are consistent with the results obtained by Svartholm⁽⁹⁾ in the case of the Gauss and the Yukawa potentials.

iii) Errors caused by assuming the interaction (1)

Our arguments have so far been based on potential (1). This potential, however, deviates certainly from the actual one, though the latter is not fully determined yet.

The largest difference will be due to our omission of tensor forces. They are very important in the triplet state of two-body system. As pointed out first by Gerjuoy and Schwinger⁽¹¹⁾ and recently by Pease and Feshbach,⁽¹⁾ the tensor forces reduce the binding energy of H^3 , and the magnitude of reduction depends sensitively on the range of the tensor force. For this reason, the absolute values for B.E. (H^3) in § 4 have no serious meaning, only their relative values are useful to know the effect of the hard core. Effect of the tensor force on the Coulomb energy of He^3 will be smaller than that on the B.E. (H^3) (cf. Reference 4).

The radial shape of our potential is of the exponential type. Generally a potential which is deep near the origin gives a large value B.E. (H^3). A calculation with the Yukawa potential whose short-distance part is replaced by a hard core is now in progress. The qualitative results will be probably as follows. If D is small, the Yukawa potential plus hard core will give a considerably larger B.E. (H^3) than the potential (1). As D increases, this difference will become smaller.

Our potential is charge-independent, but, strictly speaking, the actual potential is apparently not charge-independent. We have used the singlet potential determined from the $n-p$ data. From the $p-p$ data the following values have been obtained for $D=0$,⁽¹²⁾

$$s_8 = 0.907, \quad r_{08} = 2.67 \times 10^{-13} \text{ cm.}$$

As this potential is somewhat smaller than ours, if we replace the $n-p$ data by the $p-p$ one only for $n-n$ pair, the diminuation of B.E. (H^3) will be about 0.2 Mev.

There may be various other effects; e.g., those due to three-body forces, relativistic corrections, etc., may be small though their estimation is difficult at the present stage, and hence they are omitted in the discussion in this paper.

§ 6. Summary

The binding energy of H^3 (B.E.(H^3)) has been calculated by the standard variational method assuming the central two-body interaction (1) for various radii of hard core. This interaction is adjusted so as to give correctly the two-body data in the low energy region: the scattering lengths of the triplet and the singlet states in the $n-p$ system, the binding energy of the deuteron, and the effective range r_{0s} of the singlet state, which is chosen as 2.7×10^{-13} cm or 2.4×10^{-13} cm. The charge independence of nuclear forces is also assumed. We can see from Table II that *a rather small hard core diminishes the binding energy of H^3 considerably*. Tensor forces are not taken into account to avoid too laborious calculations. However, we believe that the relative decrease of the binding energy of H^3 shows the effect of the hard core clearly, even if tensor forces are neglected. We expect that this investigation will also serve for the explanation of the nuclear saturation.

The Coulomb energy of He^3 has also been calculated as a small perturbation using the adjusted trial function of H^3 . The hard core interaction pushes out the wave function so that the Coulomb energy decreases to the experimental difference between B.E.(H^3) and B.E.(He^3), (See, Table IV). Owing to the crudeness of our variational treatment, the exact values of B.E.(H^3) will be larger than the values in Table II by the order of magnitude 1 Mev, and the Coulomb energy in Table IV may be several per cent larger than the exact value. Taking into consideration these errors we can adjust the radius of hard core so as to explain both B.E.(H^3) and the Coulomb energy. Such a radius is about 0.2×10^{-13} cm for $r_{0s} = 2.7 \times 10^{-13}$ cm and is about 0.4×10^{-13} cm for $r_{0s} = 2.4 \times 10^{-13}$ cm. From this fact it is inferred that the diminuation of the Coulomb energy is not due to the behavior of the wave function at large distances, which is mainly specified by the binding energy, but due to the vanishing of the wave function at short distances. Thus we are able to attain to the following conclusion: *The hitherto inexplicable discrepancy between the experimental and the theoretical values of the difference between B.E.(H^3) and B.E.(He^3) may be attributed to the existence of the hard core.*

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Appendix I. Method of integration

When hard cores are introduced, the domain of integration becomes very complicated. We have used the following separation of the domain of integration.

$$\begin{aligned} \iiint dr_1 dr_2 dr_3 &= \int_D^\infty dr_1 \int_D^\infty dr_2 \int_D^\infty dr_3 \\ &- \int_D^\infty dr_2 \int_D^\infty dr_3 \int_0^\infty du - \int_D^\infty dr_3 \int_D^\infty dr_1 \int_0^\infty dv - \int_D^\infty dr_1 \int_D^\infty dr_2 \int_0^\infty dw \end{aligned} \quad (A1)$$

with

$$u=r_1-r_2-r_3, \quad v=r_2-r_3-r_1, \quad w=r_3-r_1-r_2.$$

When the integrand is symmetric with respect to r_1 , r_2 and r_3 , (A1) is reduced to

$$\iiint dr_1 dr_2 dr_3 = \int_D^\infty dr_1 \int_D^\infty dr_2 \int_D^\infty dr_3 - 3 \int_D^\infty dr_1 \int_D^\infty dr_2 \int_0^\infty dw, \quad (A1')$$

with

$$w=r_3-r_1-r_2.$$

Appendix 2. Explicit formulas for normalization, kinetic energy, potential energy and Coulomb energy

We use the following abbreviations :

$$A_n(x) \equiv \frac{1}{(x+2\mu)^n} - \frac{2}{(x+\mu+\nu)^n} + \frac{1}{(x+2\nu)^n},$$

$$B_n(x) \equiv \frac{\mu^2}{(x+2\mu)^n} - \frac{\mu^2+\nu^2}{(x+\mu+\nu)^n} + \frac{\nu^2}{(x+2\nu)^n},$$

$$C_n(x) \equiv \frac{\mu}{(x+2\mu)^n} - \frac{\mu+\nu}{(x+\mu+\nu)^n} + \frac{\nu}{(x+2\nu)^n}.$$

i) Normalization

$$\begin{aligned} N &= \iiint \psi_s^{(t)*} \psi_s^{(t)} r_1 r_2 r_3 dr_1 dr_2 dr_3 \\ &= \{DA_1(0) + A_2(0)\}^3 - N'(2\mu) + 2N'(\mu+\nu) - N'(2\nu), \end{aligned}$$

where

$$\begin{aligned} N'(x) &= 3 \{4A_3(x) + (4D+1/x)A_2(x) + (2D^2+D/x)A_1(x)\} \\ &\quad \times \{A_2(x) + DA_1(x)\} e^{-x^D/x}. \end{aligned}$$

ii) Kinetic energy K

$$\begin{aligned} NK &= -\frac{\hbar^2}{M} \iiint \psi_s^{(t)*} \sum_{\text{cyclic}} \left\{ \frac{1}{r_1^2} \frac{\partial}{\partial r_1} \left(r_1^2 \frac{\partial \psi_s^{(t)}}{\partial r_1} \right) + \frac{(r_2^2+r_3^2-r_1^2)}{2r_2 r_3} \frac{\partial^2 \psi_s^{(t)}}{\partial r_2 \partial r_3} \right\} r_1 r_2 r_3 dr_1 dr_2 dr_3 \\ &= -(3\hbar^2/M) [\{A_2(0) + DA_1(0)\}^2 \{B_2(0) + DB_1(0)\} \\ &\quad - \mu^2 K'(2\mu) + (\mu^2 + \nu^2) K'(\mu+\nu) - \nu^2 K'(2\nu) \\ &\quad - K''(2\mu) + 2K''(\mu+\nu) - K''(2\nu)], \end{aligned}$$

where

$$\begin{aligned} K'(x) &= \{4A_3(x) + (4D+1/x)A_2(x) + (2D^2+D/x)A_1(x)\} \\ &\quad \times \{A_2(x) + DA_1(x)\} e^{-x^D/x}, \\ K''(x) &= -\{A_2(x) + DA_1(x)\}^2 e^{-x^D} \\ &\quad + \{2B_3(x) + (3D+1/x)B_2(x) + (2D^2+D/x)B_1(x)\} \\ &\quad \times \{A_2(x) + DA_1(x)\} 2e^{-x^D/x} \end{aligned}$$

$$\begin{aligned}
& + \{B_2(x) + DB_1(x)\} \{2A_3(x) + DA_2(x)\} 2e^{-x^D}/x \\
& + \{4C_3(x) + (6D-6/x)C_2(x) + (D^2-16D/x-8/x^2)C_1(x)\} \\
& \times \{A_2(x) + DA_1(x)\} e^{-x^D}/2 \\
& + \{(4D-12/x)A_3(x) + (5D^2-4D/x+2/x^2)A_2(x) \\
& + (3D^3+2D^2/x+2D/x^2)A_1(x)\} C_1(x) e^{-x^D}/2 \\
& + \{2A_3(x) + DA_2(x)\} C_2(x) e^{-x^D} \\
& - \{(4D+4/x)C_3(x) + (7D^2+10D/x+6/x^2)C_2(x) \\
& + (2D^3+5D^2/x+6D/x^2+3/x^3)C_1(x)\} C_1(x) e^{-x^D}/x \\
& - \{4C_3(x) + (4D+3/x)C_2(x) - D^2C_1(x)\} C_2(x) e^{-x^D}/x.
\end{aligned}$$

iii) Potential energy $U_i (i=t, s)$

$$\begin{aligned}
NU_i &= -(3/2)A_i \iiint \phi_s^{(t)} * \phi_s^{(t)} e^{-\alpha_i(r_1-D)} r_1 r_2 r_3 dr_1 dr_2 dr_3 \\
&= -(3/2)A_i [\{DA_1(0) + A_2(0)\}^2 \{DA_1(\alpha_i) + A_2(\alpha_i)\} - U_i'(2\mu) \\
&\quad + 2U_i'(\mu+\nu) - U_i'(2\nu)],
\end{aligned}$$

where

$$\begin{aligned}
U_i'(x) &= 2[\{2A_3(x+\alpha_i) + (3D+1/x)A_2(x+\alpha_i) \\
&\quad + (2D^2+D/x)A_1(x+\alpha_i)\} \{A_2(x) + DA_1(x)\} \\
&\quad + \{A_2(x+\alpha_i) + DA_1(x+\alpha_i)\} \{2A_3(x) + DA_2(x)\}] e^{-x^D}/x \\
&\quad + \left\{ 4A_3(x+\alpha_i) + \left(4D + \frac{1}{x+\alpha_i}\right)A_2(x+\alpha_i) + \left(2D^2 + \frac{D}{x+\alpha_i}\right)A_1(x+\alpha_i) \right\} \\
&\quad \times \{A_2(x+\alpha_i) + DA_1(x+\alpha_i)\} \frac{e^{-(x+\alpha_i)D}}{x+\alpha_i}.
\end{aligned}$$

iv) Coulomb energy of He^3 , E_c

$$\begin{aligned}
NE_c &= e^2 \iiint \phi_s^{(t)} * \phi_s^{(t)} r_2 r_3 dr_1 dr_2 dr_3 \\
&= e^2 [A_1(0) \{A_2(0) + DA_1(0)\}^2 - E_c'(2\mu) + 2E_c'(\mu+\nu) - E_c'(2\nu)],
\end{aligned}$$

where

$$\begin{aligned}
E_c'(x) &= 2 \{2A_3(x) + DA_2(x)\} A_1(x) e^{-x^D}/x \\
&\quad + \{3A_2(x) + (5D+2/x)A_1(x)\} \{A_2(x) + DA_1(x)\} e^{-x^D}/x.
\end{aligned}$$

Appendix 3. Details of the results of calculation

The adjustable parameters μ and ν in (4) can be determined in principle by

$$\partial E / \partial \mu = 0, \quad \partial E / \partial \nu = 0.$$

However, this procedure is so complicated that we avoided its direct application by using the following technique. We calculated N , U_t , U_s and K in Appendix 2 at more than ten sets (μ, ν) 's near the stationary value of E . They are given in the following tables. Near the stationary point we assume E as the quadratic function in μ and ν ,

$$E = a\mu^2 + 2b\mu\nu + c\nu^2 + 2d\mu + 2e\nu + f, \quad (A2)$$

whose coefficients are determined from the combination of six sets of (μ, ν) 's. If we take various combinations of (μ, ν) 's, we find that there exist only small differences less than 0.1 Mev for B.E.(H^3) and about 1% for μ and ν . The values in Tables II, III and IV has been determined by the combination of sets, for which the approximation (A2) seems best (indicated by asterisks in the following tables). When the hard core does not exist, we only have to put $\nu = \infty$, because the following inequality can be obtained,

$$\left(-\frac{\partial E}{\partial (1/\nu)} \right)_{\nu \rightarrow \infty} > 0.$$

Table VI is expected to be useful for the calculations using somewhat different force parameters from ours.

Table VIa. No hard core.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B.E.(H^3) in Mev	
			$r_{0s} =$ $2.7 \times 10^{-13} \text{ cm}$	$r_{0s} =$ $2.4 \times 10^{-13} \text{ cm}$		$r_{0s} =$ $2.7 \times 10^{-13} \text{ cm}$	$r_{0s} =$ $2.4 \times 10^{-13} \text{ cm}$
0.5	∞	33.42	21.19	22.39	44.42	*10.18	*11.38
0.4835	∞	31.67	20.12	21.19	41.54	*10.25	*11.33
0.45	∞	28.15	17.96	18.79	35.98	*10.12	*10.96

Table VIb. Hard core radius $D = 0.2 \times 10^{-13} \text{ cm}$.

μ in 10^{13} cm^{-1}	ν in 10^{13} cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B.E.(H^3) in Mev		N
			$r_{0s} =$ $2.7 \times 10^{-13} \text{ cm}$	$r_{0s} =$ $2.4 \times 10^{-13} \text{ cm}$		$r_{0s} =$ $2.7 \times 10^{-13} \text{ cm}$	$r_{0s} =$ $2.4 \times 10^{-13} \text{ cm}$	
0.55	4.5	41.92	28.12	29.79	64.04	5.99	7.66	0.005945
0.5	6.5	40.51	27.04	28.79	61.44	6.11	7.87	0.011327
0.5	5.5	38.77	26.02	27.55	57.69	*7.09	*8.62	0.010975
0.5	5	37.63	25.35	26.74	55.60	*7.37	*8.77	0.010723
0.5	4.5	36.26	24.53	25.77	53.33	*7.47	*8.71	0.010393
0.5	3.5	32.53	22.27	23.11	47.93	6.87	7.72	0.009344
0.45	5.5	32.61	22.09	23.17	46.93	7.77	8.85	0.020033
0.45	5	31.70	21.55	22.53	45.40	*7.85	*8.83	0.019672
0.45	4.5	30.60	20.89	21.74	43.73	*7.76	*8.62	0.019194
0.45	4	29.25	20.07	20.78	41.86	7.47	8.18	0.018546
0.4	4.5	25.01	17.11	17.77	34.87	*7.26	*7.92	0.037981

Table VIc. Hard core radius $D=0.4 \times 10^{-13}$ cm.

μ in 10^{13}cm^{-1}	ν in 10^{13}cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B.E.(H ³) in Mev		N
			$r_{0,s} =$ 2.7×10^{-13} cm	$r_{0,s} =$ 2.4×10^{-13} cm		$r_{0,s} =$ 2.7×10^{-13} cm	$r_{0,s} =$ 2.4×10^{-13} cm	
0.5	5.5	48.04	34.06	36.46	78.82	3.28	5.68	0.017722
0.5	4.5	43.73	31.42	33.21	69.44	*5.71	*7.50	0.016515
0.5	3.5	37.64	27.70	28.75	59.17	*6.17	*7.22	0.014507
0.5	3	33.77	25.25	25.89	53.53	5.49	6.13	0.012975
0.45	3.5	31.65	23.58	24.24	49.02	*6.21	*6.87	0.026619
0.4	4.5	29.72	21.93	22.70	45.31	*6.33	*7.11	0.055483
0.4	4	27.92	20.82	21.38	42.49	*6.24	*6.80	0.05462
0.4	3.5	25.73	19.47	19.80	39.50	*5.71	*6.03	0.05208
0.4	3	23.22	17.82	17.90	36.24	4.81	4.89	0.04851
0.35	4.5	23.09	17.30	17.70	34.75	5.64	6.04	0.11764

Table VIId. Hard core radius $D=0.6 \times 10^{-13}$ cm.

μ in 10^{13}cm^{-1}	ν in 10^{13}cm^{-1}	$-U_t$ in Mev	$-U_s$ in Mev		K in Mev	B.E.(H ³) in Mev		N
			$r_{0,s} =$ 2.7×10^{-13} cm	$r_{0,s} =$ 2.4×10^{-13} cm		$r_{0,s} =$ 2.7×10^{-13} cm	$r_{0,s} =$ 2.4×10^{-13} cm	
0.5	4.5	51.18	39.70	41.89	86.93	*3.94	*6.14	0.025025
0.5	4	46.75	36.98	38.53	79.32	*4.40	*5.96	0.023497
0.5	3.5	42.03	34.03	34.93	71.95	*4.10	*5.00	0.021423
0.5	3	36.42	30.32	30.57	63.75	2.99	3.25	0.018909
0.45	4.5	42.60	33.56	35.04	71.56	*4.60	*6.08	0.04371
0.4	5.5	39.17	30.43	32.04	65.97	3.64	5.25	0.08576
0.4	5	36.88	29.07	30.33	61.62	4.33	5.59	0.08382
0.4	4.5	34.32	27.50	28.38	57.19	*4.63	*5.51	0.08132
0.4	4	31.42	25.68	26.16	52.69	*4.41	*4.89	0.07815
0.35	4.5	26.49	21.62	22.03	44.01	4.08	4.51	0.16429
0.3	5.5	21.88	17.65	18.11	36.42	3.12	3.58	0.38152
0.3	5	20.67	16.92	17.20	34.34	3.25	3.53	0.37685
0.3	4.5	19.30	16.06	16.15	32.22	3.14	3.23	0.37085
0.3	3.5	15.97	13.88	13.58	27.71	2.15	1.84	0.35203

Appendix 4. The effective range theory with hard cores

The definite informations which can be obtained from the experiments for two-nucleon system at low energies are the values of scattering length a and effective range r_0 of nuclear force. We must adjust the intrinsic range b and the depth parameter s of nuclear potential for every shape so that this potential fits these experimental data. This can be done according to the tables and graphs in Reference 8, when the potentials are Yukawa, exponential, Gauss and square well types with no hard core. Here we shall show that it is

also possible to adjust the force parameters s and b easily by the aid of Reference 8 even when the hard core interactions are taken into consideration. For this, we have only to notice that the wave equation with hard core is reduced to the equation without hard core by translating the coordinate r by the radius of hard core D . Let the index $^\circ$ denote the quantities after translation. The effective range r_0 is expressed by r_0° as follows* (for notation, see Reference 8)

$$\begin{aligned} r_0 &= 2 \left\{ \int_0^D \varphi^2(r) dr + (1 - \alpha D)^2 \int_0^\infty [(\varphi^\circ(r))^2 - (u^\circ(r))^2] dr \right\} \\ &= 2D - 2\alpha D^2 + 2\alpha^2 D^3/3 + (1 - \alpha D)^2 r_0^\circ, \end{aligned} \quad (A3)$$

where

$$\begin{aligned} \varphi(r) &= 1 - \alpha r, \quad \varphi^\circ(r) = 1 - \alpha^\circ r, \quad \alpha^\circ = 1/(a - D), \\ u(r) &\rightarrow \varphi(r), \quad u^\circ(r) \rightarrow \varphi^\circ(r), \quad r > (\text{force range}), \\ r_0^\circ &= 2 \int [(\varphi^\circ(r))^2 - (u^\circ(r))^2] dr. \end{aligned}$$

when $\alpha = 1/a$ tends to zero, (A3) is reduced to the intrinsic range,

$$b = 2D + b^\circ. \quad (A4)$$

The depth parameter s is common to both cases with and without hard core. Then in order to express the potential by b and s , we have only to replace b and r in the Blatt-Jackson's formula (4.5) by b° and $r - D$ respectively. To obtain the values of s and b° from α and r_0 , we first calculate α° and r_0° by (A3). These α° and r_0° should be considered as α and r_0 in Blatt-Jackson's paper. Thus we can get the values of s from their Table V, and get b° from their Table I.

In the spin triplet S -state, we should adjust the force parameters to the binding energy of the deuteron and the scattering length rather than α and r_0 . To do this the third "shape dependent" term P is necessary, if we want more accuracy than that of "shape independent" approximation. In this approximation the effective range is written as,

$$r_0 = 2(\gamma - \alpha)/\gamma^2 - 2Pr_0^3\gamma^2,$$

$$\gamma^2 = (\text{Nucleon mass}) \times (\text{Binding energy of deuteron})/\hbar^2.$$

Then P must be expressed by the quantities with $^\circ$. Generally the relation between quantities with $^\circ$ and without $^\circ$ is expressed in the following way,

$$\delta = \delta^\circ - kD, \quad (A5)$$

where

$$\begin{aligned} k \cot \delta &= -\alpha + r_0 k^2/2 - Pr_0^3 k^4 + \dots, \\ k \cot \delta^\circ &= -\alpha^\circ + r_0^\circ k^2/2 - P^\circ r_0^{\circ 3} k^4 + \dots. \end{aligned} \quad (A6)$$

* If we notice the effective range ρ of the deuteron rather than r_0 , we get a simpler relation:

$$\rho = [1 - \exp(-2\gamma D)]/\gamma + \exp(-2\gamma D)\rho^\circ.$$

Comparing (A5) with (A6), we get, e.g.,

$$Pr_0^3 = \{P^3 r_0^{\circ 3} - r_0^{\circ 2} D/4 - r_0^{\circ} D^2/2 - (1 - \alpha^{\circ} r_0^{\circ}) D^3/3 - 2\alpha^{\circ 2} D^5/15 \\ + \alpha^{\circ} D^2 (r_0^{\circ}/2 - \alpha^{\circ} D^2/3)^2 / (1 + \alpha^{\circ} D)\} / (1 + \alpha^{\circ} D)^2.$$

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On the Use of Feynman Amplitudes in the Quantum Field Theory

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A set of Feynman amplitudes (connected Feynman amplitudes) with no irrelevant dependences due to vacuum effects is introduced systematically in conjunction with the definition of connected parts of Green's functions. Coupled differential equations of these amplitudes are derived, and as an illustration of the method, equations for the anharmonic oscillator are studied. Detailed investigations on the asymptotic behaviour of Feynman amplitudes show that only the connected F. A. provides S -matrices for arbitrarily complicated processes in unambiguous way.

§ 1. Introduction and summary

Various theories on the description of interacting quantized fields based on the over-all space-time point of view have been developed by several authors. Most of these investigations, especially as to the theory of Feynman amplitude, seem to have been motivated by two inherent problems in present approach to the theory of elementary particles.

One is to explore such a new method of relativistic approach that would be powerful even in the deeper analysis of, e.g., pion-nucleon system. Thus, starting from this point of view, Matthews and Salam¹⁾ developed a theory of so called "Covariant Fock Equations" which was also proposed independently by Freese and Zimmermann.^{2), 3)} The other problem is to construct a general theory of quantized fields capable of treating the full problem of many particle system, that is, to construct the S -matrix for arbitrary reaction processes including bound states on the one hand, to evaluate the expectation values of physical quantities in any quantum states on the other. This program has been pushed forwards by Nishijima^{4), 5), 6)} in his excellent theory which skillfully avoids the explicit use of adiabatic switching hypothesis between particles and surrounding fields.

In any of these theories, relativistic wave functions of the system (Feynman amplitudes) play an essential role as the basic concept of the theory. There remains, however, some arbitrariness in the definition of Feynman amplitudes (F.A.). In fact, three types of F.A.'s have been introduced; (a) Feynman amplitudes in original sense, making use of which Matthews, Salam et al. have formulated the system of covariant Fock equations, (b) "covariant components" in Nishijima's theory, and (c) F.A. recently suggested by Freese and Polkinghorne.^{7) 8)} The last type of F.A. differs from above two in vanishing property of "wave functions of the vacuum."

Accordingly, it seems to be desirable to investigate interrelations among these F.A.'s

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in order to clarify merits or demerits of each F.A. as the basic tool of the theory. Throughout this paper, our main interest shall be concentrated upon the properties of the F.A. of the last type (c), since few are known about this amplitudes. In the next section, we introduce the connected F.A. in somewhat different manner from those of Freese and Polkinghorne. Our intuitive definition gives compact and explicit expressions for the connected F.A. together with the introduction of the generalized Wick's contraction. § 3 is devoted mainly to compare the asymptotic behaviours of the F.A.'s, and we shall show that only the connected F.A. provides unambiguous asymptotic forms (or S -matrices) for most complicated processes such as pair-wise productions of composite particles. In the last section, we give an elementary application of our theory to the anharmonic oscillator and discuss in detail the properties of connected F.A. and generalized contractions.

§ 2. Definition of connected F.A. and equations of motion

We begin our discussions with the survey of usual definitions of F.A. Let $|\rangle$ be the state vector for given stationary states, specifically $|0\rangle$ denote the actual vacuum, and $A(x_1)$, $B(x_2)$ etc., (or briefly $A(1)$, $B(2)$, etc.) be field operators in the Heisenberg representation, then

F.A. of type (a) is expressed by the wave function

$$f(1, 2, \dots, n) = \langle 0 | N_a(A(1)B(2) \dots Z(n)) | \rangle, \quad (2.1)$$

where N_a is the normal-product defined as

$$\begin{aligned} N_a(AB \dots Z) &= T(AB \dots Z) - \sum \langle 0 | T(AB) | 0 \rangle T(C \dots Z) \\ &+ \sum \sum \langle 0 | T(AB) | 0 \rangle \langle 0 | T(CD) | 0 \rangle T(E \dots Z) \\ &- \dots \dots \dots \end{aligned} \quad (2.2)^*$$

Signs of terms in the right hand side of (2.2) are to be properly chosen, when A , B , etc., contain fermion field operators. If we use simply T -products instead of N_a -products in (2.1), we have Bethe-Salpeter's amplitude :

$$\chi(1, 2, \dots, n) = \langle 0 | T(A(1)B(2) \dots Z(n)) | \rangle, \quad (2.3)$$

this, however, may hardly be regarded as a kind of F.A. because it does not reduce to the Schroedinger-Fock's wave function in the limit of non interacting cases.

F.A. of type (b) is obtained by introducing $A^*B^* = \langle 0 | T(AB) | 0 \rangle$ instead of Wick's contraction⁹⁾ $A^*B^* = \langle 0 | T(AB) | 0 \rangle$ in (2.2) :

$$g(1, 2, \dots, n) = \langle 0 | N_b(A(1)B(2) \dots Z(n)) | \rangle, \quad (2.4)$$

$$\begin{aligned} N_b(AB \dots Z) &= T(AB \dots Z) \\ &- \sum \langle 0 | T(AB) | 0 \rangle T(C \dots Z) \\ &+ \dots \dots \dots \end{aligned} \quad (2.5)$$

* Thin letters $A(1)$, $B(2)$, etc., are free field operators and $|0\rangle$ denotes the free particle vacuum.

Now, we may represent the B - S amplitudes $\chi(1, 2, \dots)$ in the interaction representation by the block diagram such as Fig. 1, for example, in the case of one nucleon problems of interacting meson-nucleon system. Then, it is easy to see that the wave functions of the type (a) correspond to the diagram obtained by removing those parts of Fig. 1 in which bare particle lines (Fig. 2) are disconnected. In the same way, the "covariant-components" $g(1, 2, \dots)$ are represented by the diagrams in which all graphs including disconnected propagators (Fig. 3) are removed.

We will now turn to define the F.A. of the type (c) as the connected part of original Feynman graphs which is obtained by the subtraction of all diagrams including not only such as Figs. 2 and 3 but also such as Fig. 4 as *isolated islands*. Fig. 5 indicates one of the simplest connected parts of Fig. 1.

F.A. of this type may be most conveniently formulated as follows. First, let us define "contractions" of arbitrary number of operators by

$$A \cdot B \cdots K \equiv \langle 0 | T_c(AB \cdots K) | 0 \rangle, \quad (2.6)$$

where $\langle 0 | T_c(\cdots) | 0 \rangle$ is the connected part of the ordinary Green's function* $\langle 0 | T(\cdots) | 0 \rangle$. Then our F.A. is represented by the following wave function :

$$w(1, 2, \dots, n) = \langle 0 | N_c(A(1)B(2) \cdots Z(n)) | \rangle, \quad (2.7)$$

where the normal product in this case means

$$\begin{aligned} N_c(AB \cdots Z) &= T(AB \cdots Z) \\ &- \sum A \cdot B \cdots K \cdot T(L \cdots Z) \\ &+ \sum \sum A \cdot B \cdots K \cdot L \cdots P \cdots T(Q \cdots Z) \end{aligned} \quad (2.8)$$

which is quite a natural generalization of Wick's S -product into the case of interacting fields.

It should be noted that the T_c -product in (2.6) as an operator obeys the identity

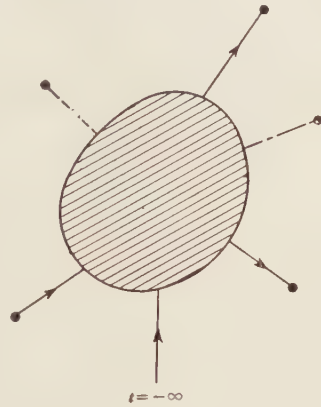


Fig. 1



Fig. 2

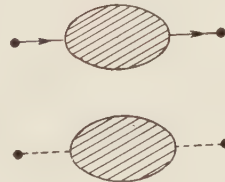


Fig. 3

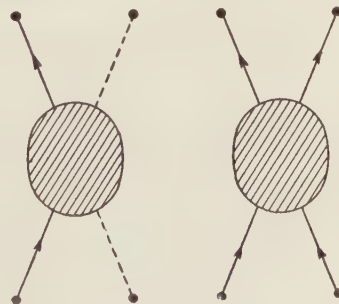


Fig. 4

* The notion of connected Green's functions is of course not new ; several authors have already employed these functions as a basis of e.g., covariant Tamm-Dancoff approximation.^{10),11),12)}

$$T_c(AB \cdots K) = N_c(AB \cdots K) + A \cdot B \cdots K \quad (2.9)$$

which completely determines connected Green's functions in cooperation with (2.7) and (2.8). Eqs. (2.7-9) give an explicit formula for our F.A. We may call this "connected wave functions."

It is possible, conversely, to express all T -products by suitable combinations of N_c -products and contractions:

$$\begin{aligned} T(AB \cdots Z) = & N_c(AB \cdots Z) \\ & + \sum A \cdot B \cdots K T(L \cdots Z) \\ & + \cdots \end{aligned} \quad (2.10)$$

from which we see the number of basic vectors $\langle 0 | N_c(\cdots)$ is sufficiently large to represent arbitrary state $|\rangle$, in so far as we are concerned with canonical systems. (2.10) is expressed equivalently in the form

$$\begin{aligned} T(AB \cdots Z) = & N_c(AB \cdots Z) + \sum \langle 0 | T(AB \cdots K) | 0 \rangle \\ & \times N_c(L \cdots Z), \end{aligned} \quad (2.11)$$

by which Polkinghorne has defined his N' -products. An essential defect of this form may be, however, in that it is hard to obtain, in general, explicit formulae for N_c -products without introducing intractable complications.

By (2.9), we have

$$\langle 0 | N_c(AB \cdots Z) | 0 \rangle = 0 \quad (2.12)$$

as it should be, or, in other words, wave functions of the vacuum always vanish. Vacuum wave functions of type (a) and type (b), on the contrary, do not always vanish; these amplitudes still contain the irrelevant dependence due to the vacuum fluctuation effects that have better to be separated.

Equations of motion.

In contrast to the ordinary F.A., coupled equations of motion involve, in this case, not only connected F.A.'s themselves but also contractions as unknown functions to be determined.

For definiteness we confine ourselves to the case of the meson-nucleon system with neutral pseudo-scalar interaction. Our wave functions $w(x_1 \cdots x_i | y_1 \cdots y_j | z_1 \cdots z_k)$ and contractions $c(x_1 \cdots x_i | y_1 \cdots y_j | z_1 \cdots z_k)$ are then represented as matrix elements of corresponding N_c - or T_c -products of nucleon- and meson-field operators $\phi(x)$, $\bar{\phi}(y)$ and $\phi(z)$:

$$\begin{aligned} w(x_1 \cdots x_i | y_1 \cdots y_j | z_1 \cdots z_k) &= \langle 0 | N_c(\phi(x_1) \cdots \phi(x_i); \bar{\phi}(y_1) \cdots \bar{\phi}(y_j); \phi(z_1) \cdots \phi(z_k)) | \rangle, \\ c(x_1 \cdots x_i | y_1 \cdots y_j | z_1 \cdots z_k) &= \langle 0 | T_c(\phi(x_1) \cdots \phi(x_i); \bar{\phi}(y_1) \cdots \bar{\phi}(y_j); \phi(z_1) \cdots \phi(z_k)) | \rangle. \end{aligned}$$

From the field equations for the Heisenberg operators



Fig. 5

$$\left. \begin{aligned} i(\gamma_\mu \partial_\mu + M)_x \phi(x) &= g\gamma_5 \phi(x) \phi(x), \\ i(-\gamma_\mu^T \partial_\mu + M)_y \bar{\phi}(y) &= g\bar{\phi}(y) \gamma_5 \phi(y), \\ -i(\square - \mu^2)_z \phi(z) &= (g/2)[\bar{\phi}(z), \gamma_5 \phi(z)] \end{aligned} \right\} \quad (I)$$

we obtain two sets of coupled differential equations of motion for the wave functions and for the contractions respectively. The former set of equations for the wave functions depends on the state in concern. For instance, the one nucleon Feynman amplitudes obey

$$\left. \begin{aligned} i(\gamma_\mu \partial_\mu + M)_x w_\alpha(x||) &= g\gamma_5(x) w_\alpha(x|x) \\ i(\gamma_\mu \partial_\mu + M)_x w_\alpha(x|z) &= g\gamma_5(x) [w_\alpha(x|xz) + c(|xz) w_\alpha(x||)] \\ -i(\square - \mu^2)_z w_\alpha(x|z) &= g\gamma_5(x) [w_\alpha(xz|z) + c(x|z) w_\alpha(z||) + c(z|z) w_\alpha(x||)] \\ i(\gamma_\mu \partial_\mu + M)_x w_\alpha(xx'|y) &= g\gamma_5(x) [w_\alpha(xx'|y|x) - c(x|y) w_\alpha(x'|x) \\ &\quad + c(x'|y|x) w_\alpha(x||)] \\ &\dots\dots\dots, \end{aligned} \right\} \quad (2 \cdot 13)$$

making use of the usual convention, i.e., $\gamma_5(x)[\dots\phi(x)\dots] = [\dots\gamma_5\phi(x)\dots]$ and $\gamma_5(y)[\dots\bar{\phi}(y)\dots] = [\dots\bar{\phi}(y)\gamma_5\dots]$. The coupled equations for the contractions, on the other hand, are independent of the required states:

$$\left. \begin{aligned} i(\gamma_\mu \partial_\mu + M)_x c(x|y) &= \delta(x-y) + g\gamma_5(x) c(x|y|x) \\ i(-\gamma_\mu^T \partial_\mu + M)_y c(x|y) &= \delta(x-y) + g\gamma_5(y) c(x|y|y) \\ -i(\square - \mu^2)_z c(|zz') &= \delta(z-z') + g\gamma_5(z) c(z|z|z') \\ i(\gamma_\mu \partial_\mu + M)_x c(x|y|z) &= g\gamma_5(x) [c(x|y|zx) + c(x|y) c(|zx)] \\ &\dots\dots\dots, \end{aligned} \right\} \quad (II)$$

as was given also by Freese.¹²⁾

§ 3. Properties of connected F.A.

In this section, we compare general properties of our amplitudes with those of usual amplitudes. Solutions are obtained, if necessary, in terms of corresponding $B-S$ amplitudes.

A. Evaluation of the connected F.A. in terms of corresponding $B-S$ amplitude.

It would be instructive as a first step to investigate formal solutions of this amplitudes by simple examples in which differences between our amplitude and conventional one could be most clearly manifested.

Example. The wave function with two virtual nucleon-pairs produced in the one nucleonic proper field.

Our wave function

$$w_\alpha(123|45) = \langle 0 | N_c(\phi(1)\phi(2)\phi(3); \bar{\phi}(5)\bar{\phi}(4)) | a \rangle, \quad (3 \cdot 1)$$

where $|a\rangle$ denotes a single nucleon state, can be obtained most easily by the graphical

investigations on the integral equation for $B-S$ amplitude $\chi_a(123|45)$.

We start with the integral equation for the three body Feynman kernel

$$K(123|456) = \langle 0 | T(\psi(1)\psi(2)\psi(3); \bar{\psi}(6)\bar{\psi}(5)\bar{\psi}(4) | 0 \rangle, \quad (3.2)$$

$$K(123|456) = S_F'(123|456) + \left(\sum_{\text{cycl.}} \mathcal{J}(1, 2) + \mathcal{J}(1, 2, 3) \right) \times K(123|456), \quad (3.3)$$

where $S_F'(123|456)$ denotes a "Slater determinant"

$$S_F'(123|456) = \begin{vmatrix} S_F'(1, 4) & S_F'(1, 5) & S_F'(1, 6) \\ S_F'(2, 4) & S_F'(2, 5) & S_F'(2, 6) \\ S_F'(3, 4) & S_F'(3, 5) & S_F'(3, 6) \end{vmatrix}, \quad (3.4)^*$$

and $\mathcal{J}(1, 2)$, $\mathcal{J}(1, 2, 3)$ are integral operators which introduce two-body and three-body forces between nucleons:

$$\mathcal{J}(1, 2) \equiv \int dx_1' \cdots dx_i' S_F'(1, 1') S_F'(2, 2') G_2(1'2'|3'4') \times, \quad (3.5)$$

$$\mathcal{J}(1, 2, 3) \equiv \int dx_1' \cdots dx_i' S_F'(1, 1') S_F'(2, 2') S_F'(3, 3') \times G_3(1'2'3'|4'5'6') \times. \quad (3.6)$$

Now, taking the limit $\bar{\psi}(6) \rightarrow \bar{\psi}^{in}(6)$,**

$$S_F'(123|456) \rightarrow S_F'(123|456^{in}) = \sum_a \chi_a^0(123|45) \bar{g}_a(6), \quad (3.7)$$

where

$$\chi_a^0 \equiv \sum_{\text{cycl.}} S_F'(12|45) g_a(3), \quad (3.8)$$

$$\left. \begin{aligned} g_a(1) &= \langle 0 | \psi(1) | a \rangle \\ \bar{g}_a(1) &= \langle a | \bar{\psi}(1) | 0 \rangle \end{aligned} \right\}, \quad (3.9)$$

and \sum_a means the summation over single nucleon states. The other states in the original summation do vanish by the requirement that (3.7) should satisfy the free nucleon equation in x_6 . Similarly,

$$\begin{aligned} K(123|456) &\rightarrow K(123|456^{in}) \\ &= \sum_a \langle 0 | T(\psi(1)\psi(2)\psi(3); \bar{\psi}(5)\bar{\psi}(4)) | a \rangle \bar{g}_a(6) \\ &= \sum_a \chi_a(123|45) \bar{g}_a(6). \end{aligned} \quad (3.10)$$

Thus, we find the integral equation for the amplitude χ_a :

* $S_F'(1, 2) = \langle 0 | T(\psi(1)\bar{\psi}(2)) | 0 \rangle$.

** The chronological ordering operators "in" and "out" are those of Nishijima, see references (5) and (6).

$$\begin{aligned} \chi_a(123|45) = & \chi_a^0(123|45) + [\sum_{\text{cycl.}} \mathcal{F}(1, 2) + \mathcal{F}(1, 2, 3)] \\ & \times \chi_a(123|45). \end{aligned} \quad (3 \cdot 11)$$

Since χ_a and χ_a^0 correspond to diagrams Fig. 6a and Fig. 6b respectively, we see immediately that those terms which should be subtracted from χ_a in order to obtain required $w_a(123|45)$ are, besides χ_a^0 itself, terms which correspond to the diagrams indicated by Figs. 6c and 6d. The amplitude with one virtual nucleon pair produced in the one nucleon cloud is obtained in much simpler way, that is,

$$\begin{aligned} w_a(12|3) = & g_a(12|3) = \chi_a(12|3) - \chi_a^0(12|3) \\ = & \mathcal{F}(1, 2) \chi_a(12|3) \end{aligned} \quad (3 \cdot 12)$$

thereby, however, no distinction occurs between w_a and the covariant component g_a .

Now, the terms which correspond to the diagrams Figs. 6c and 6d are evidently

$$\begin{aligned} & \sum_{\text{cycl.}} (\mathcal{F}(1, 2) + \mathcal{F}(1, 2) \mathcal{F}(1, 2) \\ & + \dots) \chi_a^0(123, 45) \\ = & \sum_{\text{cycl.}} \mathcal{F}(1, 2) (1 - \mathcal{F}(1, 2))^{-1} \\ & \times \chi_a^0(123|45). \end{aligned} \quad (3 \cdot 13)$$

Therefore, the required wave function reads

$$\begin{aligned} w_a(123|45) = & \chi_a(123|45) \\ & + [1 + \sum_{\text{cycl.}} \mathcal{F}(1, 2) (1 - \mathcal{F}(1, 2))^{-1}] \\ & \times \chi_a^0(123|45), \end{aligned} \quad (3 \cdot 14)$$

or, after some calculations making use of the integral equation (3.11) we finally obtain

$$\begin{aligned} w_a(123|45) = & \sum_{\text{cycl.}} \mathcal{F}(1, 2) (1 - \mathcal{F}(1, 2))^{-1} (\mathcal{F}(2, 3) + \mathcal{F}(3, 1)) \\ & \times \chi_a(123|45) \\ & + \sum_{\text{cycl.}} \mathcal{F}(1, 2) (1 - \mathcal{F}(1, 2))^{-1} \mathcal{F}(1, 2, 3) \\ & \times \chi_a(123|45) \\ & + \mathcal{F}(1, 2, 3) \chi_a(123|45). \end{aligned} \quad (3 \cdot 15)$$

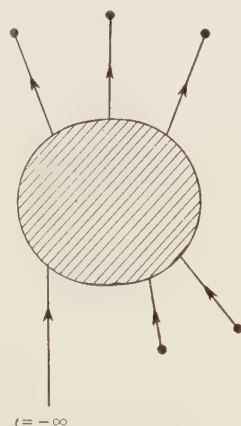


Fig. 6a

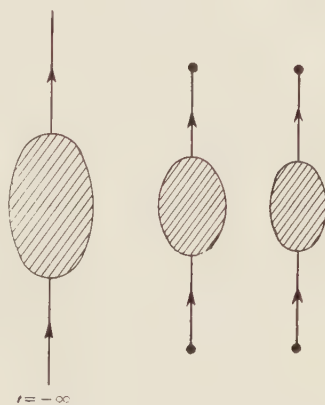


Fig. 6b

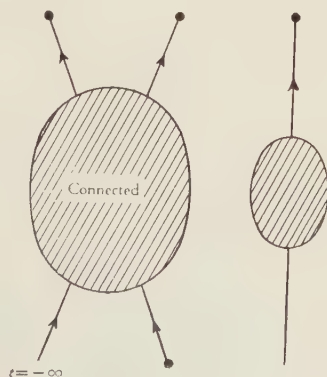


Fig. 6c

The first line in r.h.s. of (3.15) represents contributions arising from the existence of two-body forces, the second line is due to three-body forces, and the rest is explained as the crossing effects.

On the other hand, in order to find the covariant component g_a from the χ_a , one must be careful not to subtract those terms represented by the Fig. 6d. Then, after similar prescriptions, we get

$$\begin{aligned}
 g_a(123|45) &= \chi_a(123|45) \\
 &\quad - \sum_{\text{cycl.}} [1 + \mathcal{J}(1, 2)(1 - \mathcal{J}(1, 2))^{-1} \\
 &\quad + \mathcal{J}(3, 1)(1 - \mathcal{J}(3, 1))^{-1}] \\
 &\quad \times S_F'(23|45)g_a(1).
 \end{aligned}
 \tag{3.16}$$

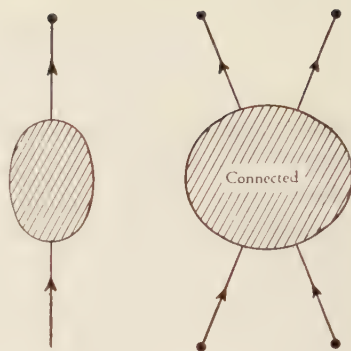


Fig. 6d

The covariant component seems to have rather intricate structure than that of the connected wave function in that we could not find a compact relation to the original χ_a nor a simple interpretation such as for (3.15).

B. Asymptotic behaviours of wave functions.

Once F.A.'s were settled, we could construct the S -matrix for various reaction processes by means of taking appropriate asymptotic forms of wave functions which are responsible to the process in question. As was stressed by Nishijima, the present theory can be applied, in principle, to deal with even the most general reaction processes involving bound states, since our method of constructing S matrices does not involve the adiabatic switching hypothesis between particles and surrounding fields. It is, however, the important requirement for the consistent application of our S matrix theory that each member of the F.A. for the given process should be *unambiguously* determined in their asymptotic forms.

According to Nishijima, asymptotic forms of a wave function

$$f(1, 2, \dots, n) = \langle 0 | N(A(1)B(2) \dots Z(n)) | \rangle$$

are given by $\sim \langle 0 | N(A^{out}B^{out} \dots Z^{out}) | \rangle$ for the final state in which all particles are freely outgoing, and $\sim \langle 0 | N((AB)^{out} \dots Z^{out}) | \rangle$ for the final state in which some of particles, say, A and B , compose a stable composite particle, and so on. We shall investigate these two cases separately.

1. Cases having no composite particles in the final states.

Theorem. The asymptotic forms of every kind of F.A. are identical and uniquely determined, if no composite particles (stable or metastable) be involved in the final states of the process, or

$$f(1^{out}2^{out} \dots n^{out}) = g(1^{out}2^{out} \dots n^{out}) = w(1^{out}2^{out} \dots n^{out}). \tag{3.17}$$

To prove the statement we assume the adiabatic switching off theorem for compactness

* Rigorously speaking, this expression is indefinite without fixing the relative time. c.f. (5).

of the argument. Making use of Gell-Mann and Low's relation :

$$\begin{aligned} \langle 0|T(A(1)B(2)\dots)|0\rangle &= \lim_{\tau \rightarrow +\infty} \langle 0|T[U(\tau, -\tau)A(1)B(2)\dots]|0\rangle \\ &\times \langle 0|U(\tau, -\tau)|0\rangle^{-1} \end{aligned} \quad (3.18)$$

we obtain, in virtue of the assumed theorem,

$$\begin{aligned} \langle 0|T(A^{out}B^{out}\dots)|0\rangle \\ = \lim_{\tau \rightarrow \infty} \lim_{t's \rightarrow \infty} \langle 0|T(A(1)B(2)\dots)U(\tau, -\tau)|0\rangle \langle 0|U(\tau, -\tau)|0\rangle^{-1}. \end{aligned} \quad (3.19)$$

Corresponding relation for the N_α -product is

$$\begin{aligned} \langle 0|T(A^{out}B^{out}\dots)|0\rangle &= \lim_{\tau \rightarrow \infty} \lim_{t's \rightarrow \infty} \langle 0|:AB\dots:U(\tau, -\tau)|0\rangle \\ &\times \langle 0|U(\tau, -\tau)|0\rangle^{-1} \end{aligned} \quad (3.20)$$

which yields

$$f_0(1^{out}2^{out}\dots) = 0. \quad (3.21)$$

The subscript 0 means f_0 is the wave function of the vacuum.

If we *renormalize* all field operators, then we get

$$\lim_{t_2 \rightarrow \infty} \lim_{t_1 \rightarrow \infty} \langle 0|T(A(1)B(2))|0\rangle \rightarrow \lim_{t_2 \rightarrow \infty} \lim_{t_1 \rightarrow \infty} \langle 0|T(AB)|0\rangle \quad (3.22)$$

and the same for the reversed order of limiting. This gives

$$g_0(1^{out}, 2^{out}\dots) = 0 \quad (3.23)$$

together with the verification of the first equality in (3.17).

(3.23) means

$$\langle 0|T_c(A^{out}(1)B^{out}(2)\dots Z^{out}(n))|0\rangle = 0, \quad (n \geq 3) \quad (3.24)$$

because T_c -products are always decomposed into the sum of N_b -products and their vacuum expectation values by the successive use of (2.6-9) :

$$\left. \begin{aligned} T_c(ABC) &= T(ABC) - \sum \langle 0|T(AB)|0\rangle C = N_b(ABC), \\ T_c(ABCD) &= N_b(ABCD) - \sum \langle 0|T_c(ABC)|0\rangle D \\ &= N_b(ABCD) - \sum \langle 0|N_b(ABC)|0\rangle D, \\ &\dots\dots\dots \end{aligned} \right\} \quad (3.25)$$

Hence we have the second equality of the theorem :

$$w(1^{out}, 2^{out}\dots) = g(1^{out}, 2^{out}, \dots). \quad (3.26)$$

Although our proof is based on the adiabatic theorem in some parts, our conclusion is valid without recourse to this assumption as exemplified in the next subsection.

2. Cases involving one or several number of stable composite particles in the final states.

In this subsection, we show that the asymptotic forms of F.A. of type (a) and (b) are not always unambiguously determined contrary to the F.A. of type (c) through a

typical example of the complicated process.

Example. $\langle 0|T_c(\psi(1)\psi(2); \bar{\psi}(4)\bar{\psi}(3))|0\rangle$

Let $t_1, t_2 > t_3, t_4$, then

$$\begin{aligned}\langle 0|T_c(12|34)|0\rangle &= \langle 0|T(12|34)|0\rangle - S_F'(12|34) \\ &= \sum_n \langle 0|T(\psi(1)\psi(2))|n\rangle \langle n|T(\bar{\psi}(4)\bar{\psi}(3))|0\rangle \\ &\quad - \sum_n g_n^0(1, 2)\bar{g}_n^0(3, 4)\end{aligned}\quad (3.27)$$

where $\sum_n \dots |n\rangle \langle n| \dots$ means the summation over the complete set of stationary states of the system and

$$\left. \begin{aligned}g_n^0(1, 2) &= g_{n'}(1)g_{n''}(2) - g_{n''}(1)g_{n'}(2) \\ g_{n'}(1) &= \langle 0|\psi(1)|n'\rangle.\end{aligned}\right\} \quad (3.28)$$

It would be instructive at this stage to check (3.24) by this example. Taking $1 \rightarrow 1^{out}, 2 \rightarrow 2^{out}$, we see from (3.27)

$$\begin{aligned}\langle 0|T_c(1^{out}, 2^{out}|3, 4)|0\rangle &= \sum_s \langle 0|T(\psi^{out}(1)\psi^{out}(2))|s\rangle \langle s|T(\bar{\psi}(4)\bar{\psi}(3))|0\rangle \\ &\quad - \sum_s g_s^0(1^{out}, 2^{out})\bar{g}_s^0(3^{out}, 4^{out}),\end{aligned}\quad (3.29)$$

where $|s\rangle$ denotes the two nucleonic scattering states. The right hand side of (3.24) is re-expressed by virtue of the integral equation for $K(1, 2|3, 4) = \langle 0|T(\psi(1)\psi(2)\psi(4)\bar{\psi}(3))|0\rangle$ as follows:

$$\begin{aligned}(3.29) &= (1/2) \sum_s g_s(1^{out}, 2^{out}) \int \bar{g}_s(1', 2') G(1', 2'|3', 4') S_F'(3', 4'|3, 4) dx_1 \dots dx_4 \\ &= (1/2) \sum_s g_s(1^{out}, 2^{out}) \int \bar{g}_s(1', 2') G(1', 2'|3', 4') \\ &\quad \times [K(3', 4'|3, 4) - (1/2) \int S_F'(3', 4'|5, 6) G(5, 6|5', 6') K(5', 6'|3, 4) dx_5 \dots dx_6] \\ &\quad \times dx_1 \dots dx_4.\end{aligned}\quad (3.30)$$

Then, further application of the limiting $3 \rightarrow 3^{out}, 4 \rightarrow 4^{out}$ in (3.30) yields

$$\begin{aligned}\lim_{\substack{3 \rightarrow 3^{out} \\ 4 \rightarrow 4^{out}}} \lim_{\substack{1 \rightarrow 1^{out} \\ 2 \rightarrow 2^{out}}} \langle 0|T_c(1, 2|3, 4)|0\rangle \\ = (1/2) \sum_{s, s'} g_s(1^{out}, 2^{out}) \mathcal{R}_{ss'} g_{s'}(3^{out}, 4^{out}),\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_{ss'} &= \int \bar{g}_s(1', 2') G(1', 2'|3', 4') \bar{g}_{s'}(3', 4') dx_1 \dots dx_4 \\ &\quad - (1/2) \int \bar{g}_s(1', 2') G(1', 2'|3', 4') S_F'(3', 4'|5, 6) G(5, 6|5', 6') \bar{g}_{s'}(5', 6') dx_1 \dots dx_6.\end{aligned}$$

It is, however, easily seen that $\mathcal{R}_{ss'} = 0$ from the conservation of energy since both

$\bar{g}_s(1', 2')$ and $\bar{g}_{s'}(5', 6')$ have the negative frequency dependence e^{iEt} ($E > 0$). Since above arguments are also valid by reversing the role of t_1, t_2 and t_3, t_4 , we thus conclude

$$\langle 0 | T_c(1^{out}, 2^{out} | 3^{out}, 4^{out}) | 0 \rangle = 0, \quad (3.31)$$

which verifies (3.24).

Next, in the cases in which particles 1 and 2 compose a stable composite particle e.g., a deuteron, the asymptotic behaviour of (3.27) in 1 and 2 reads

$$\begin{aligned} \langle 0 | T_c((1, 2)^{out} | 3, 4) | 0 \rangle &= \sum_b \langle 0 | T[(\phi(1)\phi(2))^{out}] | b \rangle \\ &\times \langle b | T(\bar{\phi}(4)\bar{\phi}(3)) | 0 \rangle, \end{aligned} \quad (3.32)$$

where $|b\rangle$ denotes a two nucleonic bound state. It is worth while noticing that the asymptotic function $S_F'((1, 2)^{out} | 3, 4)$ never exists. If we apply the limit $(3, 4) \rightarrow (3, 4)^{out}$ in (3.32), we find two different results reflecting the internal inconsistency of the formal application of the Gell-Mann and Low's limiting procedure or Nishijima's rule for the construction of asymptotic forms of wave functions which have been successfully used for rather simple problems.⁵⁾ That is, when we remember the steady character of stable composite particles :

$$\left. \begin{aligned} g_b((1, 2)^{out}) &= g_b((1, 2)^{in}) = g_b(1, 2) \\ \text{and the same relations for the reciprocal amplitudes,} \end{aligned} \right\} \quad (3.33)$$

we obtain from (3.32)

$$\begin{aligned} \lim_{(3,4) \rightarrow (3,4)^{out}} \lim_{(1,2) \rightarrow (1,2)^{out}} \langle 0 | T_c(1, 2 | 3, 4) | 0 \rangle \\ = \sum_b g_b(1, 2) \bar{g}_b(3, 4) \neq 0, \end{aligned} \quad (3.34)$$

which is evidently not identical with $\lim_{(1,2) \rightarrow (1,2)^{out}} \lim_{(3,4) \rightarrow (3,4)^{out}} \langle 0 | T_c(1, 2 | 3, 4) | 0 \rangle$. It should be noticed that the steady condition (3.33) is a basic computational rule of constructing the asymptotic form of wave functions including stable bound states and can be checked also by a direct application of the limit $(1, 2) \rightarrow (1, 2)^{out}$, say, to the bound state equation of the two nucleon system

$$g_b(1, 2) = (1/2) \int S_F'(1, 2 | 1', 2') G(1', 2' | 3, 4) g_b(3, 4) dx_{1'} \cdots dx_4$$

bearing in mind that $(1, 2) \rightarrow (1, 2)^{out}$ means the limiting $T_{1,2} = (t_1 + t_2)/2 \rightarrow +\infty$, leaving the relative time $t_{1,2} = t_1 - t_2$ fixed. On the other hand, when we adopt the formal factorization procedure such as (3.30) in this case, we obtain

$$\langle 0 | T_c((1, 2)^{out} | (3, 4)^{out}) | 0 \rangle = 0. \quad (3.34')$$

This limiting procedure, however, corresponds to the introduction of some kind of adiabatic switching hypothesis in that the same procedure applied to the bound state equation yields the results $g_b((1, 2)^{in}) = 0$ and apparently contradict with the steady requirement (3.33). We must note here that the factorization procedure (3.30) applied to the scattering states

is certainly correct since the same procedure can be utilized to prove the important requirement

$$g_s(1^{in}, 2^{in}) = g_s^0(1^{in}, 2^{in}).$$

It will be easily understood in this example that the asymptotic behaviours of wave functions such as

$$f(\dots(1, 2)^{out} \dots (3, 4)^{out} \dots) \text{ or } g(\dots(1, 2)^{out} \dots (3, 4)^{out} \dots)$$

may depend upon the way of taking "out" operations by which bound states are composed in the outgoing waves. The pair-wise production of a deuteron and an anti-deuteron is one of the simplest example. In general, pair-wise productions of N composite particles and N anti-composite-particles are such the case. The asymptotic forms of the connected wave functions are, however, always determined uniquely because the ambiguous factors such as $\langle 0|T_c(12|34)|0\rangle$ are already subtracted in the very beginning.

As a matter of fact, the subtraction of the "generalized contractions" may be considered as a generalized version of the procedure subtracting $A'B'$ or $A'B$ from the $B-S$ amplitudes in order to secure the uniqueness of asymptotic forms in the cases of the scattering or production process of individual particles.

§ 4. An elementary application

In order to give a simple illustration of our theory so far developed, we treat here the anharmonic oscillator

$$\left. \begin{aligned} H &= 1/2m \cdot p^2 + m\lambda/4 \cdot q^4, \\ \text{with } qp - pq &= i. \end{aligned} \right\} \quad (4.1)$$

Defining the T -product of q 's and p 's (all of which refer to the same time t) as the symmetrized product of q 's and p 's, for example,

$$T(q(t)p(t)) = (1/2)(q(t)p(t) + p(t)q(t)), \quad (4.2)$$

we obtain a set of simultaneous linear equations for the $B-S$ amplitudes $\chi(k|l) = \langle 0|T(q(t_1) \dots q(t_k), p(t_1') \dots p(t_l'))|a\rangle|_{t_1=\dots=t_l'=t}$:

$$\left. \begin{aligned} i\omega\chi(1|) &= (1/m)\chi(|1), \\ i\omega\chi(|1) &= -\lambda m\chi(3|), \\ i\omega\chi(3|) &= (3/m)\chi(2|1), \\ i\omega\chi(2|1) &= (2/m)\chi(1|2) - m\lambda\chi(5|), \\ i\omega\chi(1|2) &= (1/m)\chi(|3) - 2m\lambda\chi(4|1), \\ i\omega\chi(|3) &= -3m\lambda\chi(3|2) + (3/2)m\lambda\chi(1|), \\ &\dots\dots\dots, \end{aligned} \right\} \quad (4.3)$$

where $\omega = \omega_0 - \omega_a$, ω_0 and ω_a are the frequencies of the vacuum $|0\rangle$ and the stationary state $|a\rangle$ respectively. Of course, we may substitute any other comparison state ϕ for the vacuum $|0\rangle$ in so far as we are concerned with the equations (4.3). In fact, the identification of ϕ with a specified comparison state is established posteriorly by considering physical consequences of the solutions.

According to the general scheme of the present theory, we are to construct the wave function $\varphi(k|l)$ from $\chi(k|l)$'s and generalized contractions $\Delta_c(m|n)$:

$$\left. \begin{aligned} \Delta_c(2|) &= \langle 0|qq|0\rangle \equiv \Delta, \\ \Delta_c(|2) &= \langle 0|pp|0\rangle \equiv \Gamma, \\ \Delta_c(1|1) &= \langle 0|T(qp)|0\rangle = 0, \\ \Delta_c(m|n) &= 0 \quad (\text{for } m+n=\text{odd}), \\ \text{etc.} \end{aligned} \right\} \quad (4.4)$$

As $\Delta_c(m|n)$'s are unknown quantities, they should be determined properly in company with the determination of ω and χ^* . In order to solve (4.3), we start with the convergence assumption of the "new Tamm-Dancoff approximation," which may be stated as:

"If we take some large number N , (4.3) has sufficiently accurate solutions under the ansatz

$$\varphi(k|l) = 0 \quad \text{for } k+l > N,$$

and the solutions thus obtained converge to the exact ones when N tends to ∞ ."

A. The 1st order approximation ($N=1$).

This is just identical with Heisenberg's 1st order approximation. Summarizing briefly, the ansatz $\varphi(k|l) = 0$ ($k+l > 1$) reduces the system of equations (4.3) to

$$\left. \begin{aligned} i\omega\chi(1|) &= (1/m)\chi(|1), \\ i\omega\chi(|1) &= 3\lambda\Delta m\chi(1|) \end{aligned} \right\} \quad (4.5)$$

which immediately gives

$$\omega^2 = 3\lambda\Delta. \quad (4.6)$$

We get here only one energy level 1, hence $\omega = \omega_{01}$ and $\langle 0|$ may be supposed to be the vacuum. Then we may approximate Δ by

$$\Delta = \sum_k |\chi_{0k}(1|)|^2 \cong |\chi_{01}(1|)|^2. \quad (4.7)$$

The normalization of $\chi_{01}(1|)$ is easily determined by the sum rule:

$$1 = 2m \sum \omega_{k0} |\chi_{0k}(1|)|^2 \cong 2m\omega_{10} |\chi_{01}(1|)|^2. \quad (4.8)$$

Thus, we obtain $\Delta \cong (2m\omega_{10})^{-1}$ and $\Gamma \cong m^2\omega_{10}^2\Delta$.

* If one neglects all Δ_c 's other than Δ and Γ , and substitutes the values for the harmonic oscillator into Δ and Γ ; our treatment below would reduce to Dyson's NTD method.

B. The 2nd order approximation ($N=3$)

In this case, the assumption $\varphi(k|l)=0$ ($k+l \geq 5$) yields the reduction formulae (see (2.10))

$$\left. \begin{aligned} \chi(5|) &= 10J\chi(3|) - 15J^2\chi(1|) + 5J_c(4|)\chi(1|), \\ \chi(4|1) &= 6J\chi(2|1) - 3J^2\chi(1|) + 4J_c(3|1)\chi(1|), \\ \chi(3|2) &= 3J\chi(1|2) + \Gamma\chi(3|) + 3(J_c(2|2) - J\Gamma)\chi(1|) \end{aligned} \right\} \quad (4.9)$$

which should be substituted into the last three equations in (4.3). Making use of the identities :

$$\left. \begin{aligned} J_c(4|) &= (m^2\lambda)^{-1} \Gamma - 3J^2 \\ J_c(3|1) &= 0 \end{aligned} \right\}, \quad (4.10)$$

energy eigenvalues ω are determined by

$$\omega^6 - 63J\omega^4 + 3\lambda(132J^2 + (1/m^2)\Gamma)\omega^2 - 9\lambda^2\{90J^3 - (1/m^2)(9J\Gamma + 6J_c(2|2) - 1)\} = 0. \quad (4.11)$$

In (4.11) we use those J , Γ and $J_c(2,2)$ which should be found in the 1st order approximation. $J_c(2|2)$ are then evaluated as follows :

$$\begin{aligned} J_c(2|2) &= (1/2)\langle 0|qT(qpp) + T(qpp)q|0\rangle - J\Gamma \\ &= (1/2)\sum_k (\chi_{0k}(1|)\chi_{k0}(1|2) + \text{c.c.}) - J\Gamma \\ &= (1/2)\sum_k \{\chi_{0k}(1|)(\varphi_{k0}(1|2) + \Gamma\chi_{k0}(1|)) + \text{c.c.}\} - J\Gamma \cong J\Gamma - J\Gamma = 0. \end{aligned} \quad (4.12)$$

Three $|\omega|$'s determined from (4.11) are shown to correspond to the frequencies ω_{10} , ω_{21} , and ω_{30} ($\omega_{10} < \omega_{21} < \omega_{30}$) in the physical considerations on numerical results. Normalizations of χ 's are again performed taking into account the sum rule

$$2m(\omega_{10}|\chi_{01}(1)|^2 + \omega_{30}|\chi_{03}(1)|^2) \cong 1 \quad (4.13)$$

and the similar formula

$$\left(1 - \frac{2m}{3\lambda}\omega_{10}^2\right)|\chi_{01}(1)|^2 + \left(1 - \frac{2m}{3\lambda}\omega_{30}^2\right)|\chi_{03}(1)|^2 \cong 0, \quad (4.14)$$

which is derived from

$$\langle 0|pq^3 - q^3p|0\rangle = -3iJ. \quad (4.15)$$

We are now able to evaluate the corrections to the original J_c 's and obtain J_c 's in the 2nd order approximation by using these ω 's and χ 's. In general, if we know $J_c^{[n-1]*}$, then we can find $\omega^{[n]}$, $\chi^{[n]}$, and $J_c^{[n]}$.

Numerical results for $\lambda=m=1$ are given in the table.

(1°) The 2nd order approximation of our method fairly improves the results of

* The superscript $[n]$ denotes the quantities which relate the n -th order approximation.

the 1st order one. Comparing with Heisenberg's treatment, our energy levels and \mathcal{A}_c 's are more or less close to the exact solution.

(2°) The wave functions φ in several configurations are also evaluated;

	The 1st order approx.	The 2nd approximation		exact solutions*
		Heisenberg treatment	ours	
ω_{10}	1.14	1.15	1.13	1.087
ω_{21}	—	1.31	1.33	1.450
ω_{30}	—	4.89	4.95	4.200
\mathcal{A}	0.436	0.431	0.440	0.4561
Γ	0.572	0.582	0.567	0.5611
$\mathcal{A}_c(4)$	0	—	0.013	?
$\mathcal{A}_c(2 2)$	0	—	0.217	?

$$\left. \begin{aligned} \varphi_{01}(3|) &\cong -0.03\chi_{01}(1|), \\ \varphi_{01}(2|1) &\cong -i0.01\chi_{01}(1|), \\ \varphi_{01}(1|2) &\cong 0.50\chi_{01}(1|), \\ \varphi_{01}(|3) &\cong i2.16\chi_{01}(1|. \end{aligned} \right\} \quad (4.16)$$

Although the first expression in (4.16) and

$$|\chi_{03}(1|)|^2 \approx 0.0002|\chi_{01}(1|)|^2 \quad (4.17)$$

seem to justify the 1st order evaluation, it is to be realized from the last two relations in (4.16) and the fact

$$\varphi_{03}(3|) \cong 23.17\chi_{03}(1|)$$

that the straightforward application of the ansatz $\varphi(k|l) = 0 (k+l > N)$ is somewhat dangerous especially for small N . As to our 2nd order calculation, neglect of $\varphi(5|)$, $\varphi(4|)$, $\varphi(3|2)$, and $\varphi(2|3)**$ should be justified in the next (3rd) order approximation.

(3°) The final remark. It should be noted that $\mathcal{A}_c(2|2)$ in our treatment is just the same order quantity as \mathcal{J}^2 , Γ^2 , $\mathcal{J}\Gamma'$ (~ 0.2), therefore, $\mathcal{A}_c(2|2)$ may not be neglected in the next order approximation. In the higher order calculation, too, it would be very likely that some of the generalized contractions $\mathcal{A}_c(m|n)$ remain finite, and that they compete with products of the lower contractions, say, \mathcal{A}' to affect the consistency of our approximation scheme.

§ 5. Discussions

We may conclude from the investigations so far developed in this work that the use

* See reference (13).

** We have assumed in (4.12) that $\varphi(2|3)$ is smaller than $\chi(1|)$.

of the connected F.A. in the theory of Feynman amplitude is, in the first place, profitable for the theoretical completeness. Among all, the uniqueness of the asymptotic forms of this amplitude seems to be an important character in that it assures us the consistency of our S -matrix theory.

It is worthy to notice here that, even in practical applications of the theory, our wave functions might serve to secure the consistency of approximations. For example, the "new Tamm-Dancoff" method, if it works, would be valid only when we use this wave function, since the repeated occurrence of generalized contractions, e.g., $c(12|34)$ might prevent the higher configurations of the ordinary amplitudes $f(1, 2, \dots, N)$ and $g(1, 2, \dots, N)$, where N means some large number, from vanishing as well as due to the effect of ordinary contractions S_F' and J_F' . These vacuum effects probably have an importance in the non-linear self-interactions such as $i\phi^4(x)$ or $i(\bar{\psi}(x)\psi(x))^2$. Symanzik's criticism⁴⁾ to the NTD method applied to the anharmonic oscillator may well be understood as a disproof to the use of ordinary F.A. In fact, it is easy to show that Symanzik's argument against the validity of NTD method is no longer held once we take the connected F.A.,* although the rigorous proof of the convergence assumption of our " NTD " method still remains as an open problem.

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Appendix

Functional Derivation of Connected F.A.

For completeness, we give here brief accounts for the method of generating functions. As is well known,¹⁴⁾ the generating function of the $B-S$ amplitudes is given by

$$T[\bar{\eta}, \eta, J] = T \exp \left[i \int_{-\infty}^{+\infty} dx (\bar{\eta} \phi(x) + \bar{\psi}(x) \eta + J \phi(x)) \right], \quad (\text{A} \cdot 1)$$

where $\bar{\eta}(x)$, $\eta(x)$ and $J(x)$ are familiar external source functions for nucleon- and meson-fields. T generates $B-S$ amplitudes by the formulae:

$$\left. \begin{aligned} \langle 0 | T[\bar{\eta}, \eta, J] | a \rangle &= \sum_{l, m, n} i^{l+m+n} (\chi_{lmn}'[\bar{\eta}, \eta, J] / l! m! n!), \\ \chi_{lmn}^{lr}[\bar{\eta}, \eta, J] &= \int \bar{\eta}(x_l) \cdots \bar{\eta}(x_1) \chi_{lr}(x_1 \cdots x_l, \gamma_1 \cdots \gamma_n | z_1 \cdots z_m) \\ &\quad \times \eta(\gamma_n) \cdots \eta(\gamma_1) J(z_1) \cdots J(z_m) dx_1 \cdots dz_m. \end{aligned} \right\} \quad (\text{A} \cdot 2)$$

* The author is much indebted to Mr. K. Yamazaki for valuable discussions on this point during his stay at the Institute for Fundamental Physics, Kyoto University (December 1955).

Then the generating function $W[\bar{\eta}, \eta, J]$ for the connected F.A. can be constructed from T as

$$W = T/T_{00}, \quad T_{00} \equiv \langle 0|T|0 \rangle. \quad (A.3)$$

To prove this formula is trivial if one notice the vanishing property of vacuum functions. (A.3), being essentially identical with the Freese's one, also indicates a close analogy between the subtraction procedure of disconnected diagrams in the F.A. and the separation of bubbling terms in the S -matrix theory of Feynman and Dyson.

The concrete expression for W is found to be

$$W[\bar{\eta}, \eta, J] = \exp \left[-\bar{\eta}(\psi \bar{\psi})\eta + (1/2)J(\phi \phi)J - i\bar{\eta}(\psi \bar{\psi} \phi)\eta J - (1/4)\bar{\eta}\eta(\psi \phi \bar{\psi} \bar{\phi})\eta J + \dots \right] T[\bar{\eta}, \eta, J], \quad (A.4)$$

where

$$\begin{aligned} \bar{\eta}(\psi \phi)\eta &= \int \bar{\eta}(x)\psi(x)\phi(y)\eta(y)dx dy \\ &= \int \bar{\eta}(x)S_F'(x, y)\eta(y)dx dy \end{aligned} \quad (A.5)$$

and so on.

We then obtain the generating function of F.A. of the type (b) by omitting the third and subsequent terms in the exponent appeared in (A.5).

Suppose, now, the operators $\bar{\psi}'$, ψ' and ϕ' defined by the relations

$$\begin{aligned} \bar{\psi}'(x) &= \bar{\psi}(x) - \int dy' S_F'(x, y') \frac{\delta}{\delta \bar{\psi}(y')} \\ &\quad - \int dy' dz' c(x|y'|z') \frac{\delta}{\delta \bar{\psi}(y')} \frac{\delta}{\delta \phi(z')} \\ &\quad + \frac{1}{2} \int dx' dy' dy'' \frac{\delta}{\delta \psi(x')} c(xx'|y'y'') \frac{\delta}{\delta \psi(y'')} \frac{\delta}{\delta \psi(y')} \\ &\quad - \dots, \end{aligned} \quad (A.6a)$$

$$\begin{aligned} \bar{\psi}'(y) &= \bar{\psi}(y) + \int dx' \frac{\delta}{\delta \psi(x')} S_F'(x', y) \\ &\quad + \int dx' dz' c(x'|y|z') \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \phi(z')} \\ &\quad + \frac{1}{2} \int dx' dx'' dy' \frac{\delta}{\delta \psi(x')} \frac{\delta}{\delta \psi(x'')} c(x''x'|y'y) \frac{\delta}{\delta \psi(y')} \\ &\quad + \dots \end{aligned} \quad (A.6b)$$

and

$$\begin{aligned} \phi'(z) &= \phi(z) - \int dz' \Delta_F'(z, z') \frac{\delta}{\delta \phi(z')} \\ &\quad + \int dx' dy' \frac{\delta}{\delta \psi(x')} c(x'|y'|z) \frac{\delta}{\delta \bar{\psi}(y')} + \dots \end{aligned} \quad (A.6c)$$

Then one finds

$$T(A'B'\cdots Z')|\rangle = N_c(AB\cdots Z)|\rangle. \quad (\text{A}\cdot 7)$$

This is just the generalized formulae of Nambu-Kinoshita¹³⁾ and Anderson¹⁴⁾ which convert the T -products into the N -products by a transformation of field operators.

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A Collective Description of the Surface Oscillation of Atomic Nuclei

— *Extension of Tomonaga's Method to Three Dimensional Nucleus* —

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Tomonaga's ingenious method of the description of collective surface motion is extended to the case of three dimensional nuclei. In this extension some difficulty appears which was not the case with Tomonaga's two dimensional example, and a method to keep out from such a difficulty is given in this paper. Applying this method a Hamiltonian is derived which contains terms standing in one to one correspondence with those of Bohr's together with other correction terms. Brief calculation of the nuclear surface energy is also performed. The method of collective description mentioned here may be useful for many other problems where the collective motion takes place.

§ 1. Introduction

The so-called molecular model of atomic nuclei proposed by Bohr¹⁾ and extended by Bohr and Mottelson^{2,3)} has had spectacular successes in explaining many properties of heavy nuclei at low energies. We know, however, that the shell model which stands, in a certain sense, on the opposite extreme to the Bohr model is also useful in explaining many experimental results, and indeed the Bohr model, inspite of its success in the explanation of general features of nuclei, is not enough in discussing in detail the properties which would have been possible if we had stood on the shell model, although this model, of course, has its own limit in its scope.

It seems to us that the reason why the Bohr model is limited in its applicability lies mainly in the fact that in that model we cannot say definitely which particles lie outside the core and which lie inside it; or, in other words, this situation may be said to mean that the Bohr model does not leave sufficient room to take into account the Pauli principle.

Thus it is now clear that our urgent necessity is to try to describe the core of the Bohr model not as phenomenological liquid, but as collection of individual particles. If we succeed in making such a description it would allow us not only to make calculation in more detail based on the Bohr model, but also to investigate the intimate relation between the shell- and the Bohr models starting from first principles.

Many attempts have already been made by several authors³⁻⁶⁾ along this line of thinking. They, however, introduce^{3,4)} a number of redundant coordinates which might change the nature of the problem itself, or otherwise^{5,6)} forcibly separate the constituent particles into

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those which constitute the core and the others which constitute the outer particles.

The most comprehensive answer to this question, so far presented, would be the one proposed recently by Tomonaga⁷. Indeed his method does not commit any deficiency which appeared in the above cited papers³⁻⁶. Tomonaga, however, has presented his theory taking a schematic two-dimensional nucleus as an example, and so there remains necessity of extending his reasoning to the actual three dimensional nucleus, in order to investigate how far his method has succeeded in refining the Bohr model, or further to apply his reasoning to the discussion of the relation between the shell model and the Bohr model.

The purpose of this paper is thus to extend Tomonaga's method to the actual three dimensional nucleus. In doing this, however, we at once meet with a difficulty, which we will explain in some detail in § 3. In short the difficulty is that of finding out appropriate (mutually independent) momentum operators π_m which are canonically conjugate to the collective coordinates ξ_m , the latter being taken as velocity potentials. Such a situation makes it difficult to extend Tomonaga's method directly to our case.

Such a difficulty, however, is not characteristic only of the three dimensional case. In fact, even in the case of the two dimensional nucleus, if we take one more degree of freedom into account, in addition to the one considered by Tomonaga, the two momentum operators π_1 and π_2 , which are taken appropriately to these two kinds of freedom ξ_1 and ξ_2 , do not commute with one another, and a similar difficulty also appears here.

We have, however, found a method which permits us to get around this difficulty. Namely, we first introduce some new redundant coordinates α_m and their conjugate momenta β_m , but at the same time impose an appropriate number of subsidiary conditions to the system, so as to cancel out the increase of the degrees of freedom. Next we perform appropriate canonical transformations, so that the newly introduced coordinates have the meaning of the collective coordinates now in concern. Then the Hamiltonian is rewritten in a form of a sum of three parts, one describing the collective surface oscillation, another the internal motion and the remaining one the interaction between them.

This method seems to be novel and somewhat queer, but in fact a similar method to this has already been used by Bohm and Pines⁸, and Tomonaga⁷ has also briefly discussed the relation of their theory to his one. Bohm and Pines start with an original Hamiltonian depending only on variables of electrons, but afterwards introduce (redundant) variables of electromagnetic field, together with appropriate subsidiary conditions, and then, performing canonical transformations, change the Hamiltonian into a form in which is contained a term describing the collective plasma oscillation.

Thus it will be easy to see that our method is not so unfamiliar. The application of this method to our case, however, is much farther reaching than in the case of Bohm and Pines, because the effect of noncommutativity of π_m and $\pi_{m'}$ ($m \neq m'$), and the non-canonical nature of π_m against ξ_m can be taken into account in our final Hamiltonian in the form of correction terms; or, in other words, we can go without being forced to find operators π_m , mutually independent and canonically conjugate to ξ_m .

In § 2 we will illustrate our method applying it to the two dimensional nucleus, taking into account two degrees of freedom of the collective motion mentioned above, and

in § 3 we apply the method to the actual three dimensional nucleus. There we also compare our final Hamiltonian with that of Bohr, which has been obtained phenomenologically.

To illustrate the application of our result to actual problems, we calculate the nuclear surface energy, and the result is briefly accounted in § 4. Some discussions of our result, so far obtained in this paper, are given in § 5.

§ 2. Collective description of a two dimensional nucleus

In order to illustrate our method of collective description, let us first take a simple example of the two-dimensional nucleus. This example is rather too simple in many respects, but still very useful since it demonstrates general features of the method and will give us a clue to deal with more realistic problems.

Let us denote by x_n, y_n the Cartesian coordinates of the n -th nucleon and by p_{xn}, p_{yn} their conjugate momenta, respectively. The total number N of the nucleons is supposed to be very large. Provided that there act ordinary nuclear potentials among them, the total Hamiltonian H_0 is given by

$$\begin{aligned} H_0 &= T + V, \\ T &= \sum_n (1/2m) (p_{xn}^2 + p_{yn}^2), \\ V &= V(x_1, y_1; x_2, y_2; \dots; x_n, y_n), \end{aligned} \quad (1)$$

where V is the potential depending only on relative coordinates of the particles.

In accordance with ordinary concept of the collective motion, we take up only those modes of the collective motion which are most likely to occur in the nucleus. From physical reasons they may consist of surface waves accompanied with an irrotational, incompressible flow inside the nucleus. Among various modes of the flow those of low excitation may be the ones which are derived from velocity potentials given by harmonic functions of second order of the particle coordinates. There are two independent modes whose potentials are given by

$$\begin{aligned} \xi_1 &= (Nr_0^2)^{-1} \sum_n (1/2) (x_n^2 - y_n^2), \\ \xi_2 &= (Nr_0^2)^{-1} \sum_n x_n y_n, \end{aligned} \quad (2)$$

where r_0 is defined conveniently $r_0^2 = (1/2) R_0^2$, R_0 being the nuclear radius. As the volume of the nucleus must be kept constant under the condition above mentioned, the surface necessarily changes itself into ellipse whose principal axes lie on the coordinate axes or make 45° with them. The collective motions described by (2) are, therefore, nothing but the surface oscillations expressed by second order spherical harmonics. It goes without saying that a travelling wave is also obtained by suitably superposing these two modes of oscillation.

In order to separate the Hamiltonian into collective and internal parts, we have to transform the set of coordinates into a new set in which ξ_1 and ξ_2 are contained and its

members. In choosing new coordinates it is desirable to impose some conditions in such a manner that the collective motion should become dynamically as little dependent on the internal motion as possible and, further, the separation should be performed explicitly without making use of any concrete form of the internal coordinates. Canonical momenta to $\hat{\xi}_1, \hat{\xi}_2$ must be symmetric functions of the particle coordinates, but they cannot be fixed by merely requiring that they should satisfy canonical commutation relations with $\hat{\xi}_1, \hat{\xi}_2$. Instead of trying to find required canonical variables, we will take another way round and we suppose for the moment that we have already found such variables and denoted them as α_1, α_2 and β_1, β_2 . We look upon them as dynamically independent of particle variables and add them to the variables describing the system. As we have thus two extra variables, we must restrict the degree of freedom by imposing two subsidiary conditions so that the system remains essentially unchanged. It may easily be seen that these auxiliary variables play the same role as the longitudinal electric field in the theory of plasma oscillation of Bohm and Pines⁸⁾.

We begin with a trivial extension of the original formalism in the following way: The wave function $\Psi_0(x, \alpha)$ is now a function not only of x but also of α_i and satisfies the subsidiary condition

$$\alpha_j \Psi_0(x, \alpha) = 0 \quad (j=1, 2), \quad (3)$$

and the Hamiltonian remains unchanged. It is evident that this formalism has exactly the same content as the original one.

From this formalism we proceed now to a little less trivial representation by the following canonical transformation

$$\begin{aligned} Q &\rightarrow U_0^{-1} Q U_0, \\ \Psi &\rightarrow \Psi' = U_0^{-1} \Psi_0, \end{aligned} \quad (4)$$

where the transformation operator is defined by

$$U_0 = \exp(i S_0), \quad S_0 = \hat{\xi}_1 \beta_1 + \hat{\xi}_2 \beta_2. \quad (5)$$

The transformed Hamiltonian and the subsidiary condition become

$$H = \sum_n (1/2m) (\mathbf{p}_n + \beta_1 \nabla_n \hat{\xi}_1 + \beta_2 \nabla_n \hat{\xi}_2)^2 + V(x) \quad (6)$$

and

$$(\alpha_j - \hat{\xi}_j) \Psi' = 0. \quad (7)$$

By comparing these expressions with those of Bohm and Pines, we see that roles of coordinates and momenta are reversed, but this difference is trivial.

The Hamiltonian (6) can be rewritten in the following form

$$\begin{aligned} H &= (1/2m) \dot{p}^2 + (1/m N r_0^2) (\pi_1 \beta_1 + \pi_2 \beta_2) \\ &\quad + (r^2/2m N r_0^4) (\beta_1^2 + \beta_2^2) + V(x), \end{aligned} \quad (8)$$

where we used the notation

$$\begin{aligned}
 p^2 &= \sum_n (p_{xn}^2 + p_{yn}^2) = \sum_n p_n^2, \\
 \pi_j &= (Nr_0^2/2) \sum_n (\mathbf{p}_n \cdot \nabla_n \hat{\xi}_j + \nabla_n \hat{\xi}_j \cdot \mathbf{p}_n) \\
 &= (Nr_0^2/2) (\mathbf{p} \cdot \nabla \hat{\xi}_j + \nabla \hat{\xi}_j \cdot \mathbf{p}).
 \end{aligned} \tag{9}$$

The quantities π_1, π_2 introduced there are proportional to the canonical momenta to $\hat{\xi}_1, \hat{\xi}_2$ used by Tomonaga⁽⁷⁾, and they satisfy the commutation relations

$$\begin{aligned}
 [\hat{\xi}_1, \pi_1] &= [\hat{\xi}_2, \pi_2] = i(r^2/r_0^2), \\
 [\hat{\xi}_1, \pi_2] &= [\hat{\xi}_2, \pi_1] = 0, \\
 [\pi_1, \pi_2] &= -2il, \\
 (l &= \sum_n (x_n p_{yn} - y_n p_{xn})),
 \end{aligned} \tag{10}$$

where r defined by

$$r^2 = (1/N) \sum_n (x_n^2 + y_n^2) \tag{11}$$

is a mean value of particle distances from the centre of the nucleus and its value is approximately equal to r_0^2 for a spherical nucleus of constant density. The total angular momentum operator $\sum_n (x_n p_{yn} - y_n p_{xn})$ in the original representation is transformed under the transformation (4) into

$$l + 2(\hat{\xi}_1 \beta_2 - \hat{\xi}_2 \beta_1). \tag{12}$$

The second term in (12) is the angular momentum due to the collective motion. Furthermore, we used the following relations on deriving the result (8):

$$(\Gamma \hat{\xi}_j \cdot \Gamma \hat{\xi}_j) = (r^2/r_0^2) \partial_{jj}.$$

This orthogonality holds only for the special types of flow under consideration and does not always hold for general cases, especially in three dimensional problems.

Next problem of the collective description is to separate the kinetic energy in the Hamiltonian (8) into terms describing collective and internal motions. Moreover it may be convenient to reduce the subsidiary condition into a form that restricts only the internal degrees of freedom^(8,9). These requirements can be fulfilled by the canonical transformation

$$\begin{aligned}
 \Psi &\rightarrow \phi = U^{-1} \Psi, \\
 U &= \exp(iS), \\
 S &= -(\gamma_1 \alpha_1 + \gamma_2 \alpha_2),
 \end{aligned} \tag{13}$$

where γ_j are defined by

$$\gamma_j = (1/2) \{ (r_0^2/r^2) \pi_j + \pi_j (r_0^2/r^2) \} \tag{14}$$

which satisfy the commutation relations

$$[\hat{\xi}_j, \gamma_k] = i \partial_{jk},$$

$$[\eta_1, \eta_2] = -2i(r_0^2/r^2)^2 \{l - 2(\hat{\xi}_1 \eta_2 - \hat{\xi}_2 \eta_1)\}. \quad (15)$$

The subsidiary conditions become, as we expected, just

$$\hat{\xi}_j \Phi = 0. \quad (16)$$

In order to express the Hamiltonian in the new representation, we first rewrite (8) in a more convenient form :

$$H = (p^2/2m) + V(x) - (1/2m N r_0^2) \eta_1 (r^2/r_0^2) \eta_1 - (1/2m N r_0^2) \eta_2 (r^2/r_0^2) \eta_2 \\ + (1/2m N r_0^2) (\beta_1 + \gamma_1) (r^2/r_0^2) (\beta_1 + \gamma_1) + (1/2m N r_0^2) (\beta_2 + \gamma_2) (r^2/r_0^2) (\beta_2 + \gamma_2). \quad (17)$$

Various quantities appearing in this expression have to be transformed into the new representation. If our expectation is correct, however, the amplitudes of the collective motions will be small in practical cases, and it will be allowed to calculate the transformed quantities in the form of power series with respect to the amplitudes. Usual procedures of calculating the transformed quantities give the following results :

$$\begin{aligned} \hat{\xi}_j &\rightarrow \hat{\xi}_j + \alpha_j, \\ \eta_1 &\rightarrow \eta_1 - i[\eta_1, \eta_2] \alpha_2 + \cdots, \\ \eta_2 &\rightarrow \eta_2 + i[\eta_1, \eta_2] \alpha_1 + \cdots, \\ \alpha_j &\rightarrow \alpha_j, \\ \beta_1 &\rightarrow \beta_1 - \eta_1 + (i/2)[\eta_1, \eta_2] \alpha_2 + \cdots, \\ \beta_2 &\rightarrow \beta_2 - \eta_2 - (i/2)[\eta_1, \eta_2] \alpha_1 + \cdots. \end{aligned} \quad (18)$$

The commutator in the above equations can be obtained from (15), in which $\hat{\xi}_1$ and $\hat{\xi}_2$ can be put equal to zero when they operate directly on the wave function. The meaning of the total angular momentum can be seen by transforming it into the new representation ; the result is given by

$$\begin{aligned} \mathcal{Q} &= l + 2(\alpha_1 \beta_2 - \alpha_2 \beta_1) \\ &\quad + 2(\beta_2 - \eta_2 - (i/2)[\eta_1, \eta_2] \alpha_1 + \cdots) \hat{\xi}_1 \\ &\quad - 2(\beta_1 - \eta_1 + (i/2)[\eta_1, \eta_2] \alpha_2 + \cdots) \hat{\xi}_2. \end{aligned} \quad (19)$$

The first term l is the angular momentum of the internal motion, the second term that of the collective motion with the eigen values 0, ± 2 , ± 4 , \cdots , and the remaining terms vanish when they are operated on the wave function.

Taking account of the transformation (18) we see that the transformed Hamiltonian becomes

$$\begin{aligned} U^{-1}[(p^2/2m) + V(x) - (1/2m N r_0^2) \eta_1 (r^2/r_0^2) \eta_1 - (1/2m N r_0^2) \eta_2 (r^2/r_0^2) \eta_2] U \\ + (1/2m N r_0^2) (\beta_1 - (i/2)[\eta_1, \eta_2] \alpha_2 + \cdots) U^{-1} (r^2/r_0^2) U (\beta_1 - (i/2)[\eta_1, \eta_2] \alpha_2 + \cdots) \\ + (1/2m N r_0^2) (\beta_2 + (i/2)[\eta_1, \eta_2] \alpha_1 + \cdots) U^{-1} (r^2/r_0^2) U (\beta_2 + (i/2)[\eta_1, \eta_2] \alpha_1 + \cdots). \end{aligned} \quad (20)$$

Notice that

$$\begin{aligned} r^2/r_0^2 &\rightarrow (r^2/r_0^2) + 2(r_0^2/r^2)(\alpha_1^2 + \alpha_2^2) \\ &+ 4(r_0^2/r^2)(\alpha_1 \xi_1 + \alpha_2 \xi_2) \\ &- 8(r_0^2/r^2)^3(\alpha_1 \xi_1 + \alpha_2 \xi_2)^2 + \dots \end{aligned} \quad (21)$$

The change in r^2/r_0^2 due to the collective motion is a small quantity of order $(r_0^2/r^2)\alpha^2$ which can be neglected when it is included in the terms of collective kinetic energy, the second and third terms of (20). Thus the kinetic energy of the collective motion is given by

$$T_c = (1/2m N r_0^2) \{ (r^2/r_0^2) (\beta_1^2 + \beta_2^2) + 2l(r_0^2/r^2)(\alpha_1\beta_2 - \alpha_2\beta_1) + \dots \}, \quad (22)$$

where terms that vanish by the subsidiary condition have already been put aside. The second term of (22) is due to coupling between the internal and collective angular momenta as a result of the non-commutativity of π_1 and π_2 , as is mentioned at the end of the preceding section.

If we replace r^2 and η_j by r_0^2 and π_j respectively in the first term of (20), the operator in the square bracket just corresponds to H_{int} (internal Hamiltonian) defined by Tomonaga⁷⁾. Expanding in powers of α_1 and α_2 , we obtain

$$\begin{aligned} U^{-1} H_{\text{int}} U &= H_{\text{int}} + \sum_j i[\eta_j, H_{\text{int}}] \alpha_j \\ &+ \sum_j \sum_k (i^2/2) [\eta_j, [\eta_k, H_{\text{int}}]] \alpha_j \alpha_k \\ &+ \dots \end{aligned} \quad (23)$$

which corresponds to Tomonaga's expansion (4.2) and (4.9) when π_j are substituted for η_j . The condition $\xi=0$ on the coefficients of α is also satisfied in our case because of the subsidiary condition (16). The second term on the right-hand side of (23) gives the coupling between the internal and the collective motions, and the third term gives the effective potential for the surface deformation α . As this section is devoted, however, only for illustration of our method and no further use is attempted, we will not give the explicit form of these terms.

§ 3. Realistic case—Three dimensional nucleus

In the case of the three dimensional nucleus the total Hamiltonian H_0 is given by

$$\begin{aligned} H_0 &= T + V, \\ T &= (1/2\mu) \sum_n \sum_v p_v^n p_v^{n*}, \\ V &= V(x) = V(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N), \end{aligned} \quad (24)$$

where μ is the nucleon mass and as usual¹⁰⁾

$$p_0 = p_z, \quad p_{\pm} = \mp \frac{1}{\sqrt{2}} (p_x \pm i p_y).$$

Now the collective motion which can be considered to occur most easily at low energies will be, as was stated in the preceding section, those which are derived from velocity potentials given as solid harmonics of second order of the particle coordinates. We will define the velocity potential as follows* so as to make comparison with Bohr and Motelson's work easy,

$$\xi_m = (4\pi/3NR_0^2) \sum_n Y_{2m}(\mathbf{r}_n). \quad (25)$$

There are, of course, five ξ_m 's as m ranges from -2 to 2 . These five velocity potentials are orthogonal in the sense that they satisfy the following relations

$$\int \xi_m^* \xi_{m'} d\tau = \text{const. } \delta_{mm'}, \quad (26)$$

the integration being performed over spatial coordinates of all the constituent particles. The motions induced by these velocity potentials, however, are not independent with each other, if we take π_m defined by

$$\pi_m = (\sqrt{5}/\sqrt{2}i) \sum_{m_1 m_2} (1m_1 1m_2 | 2-m) \sum_n Y_{1m_1}(\mathbf{r}_n) \mathbf{r}_{m_2}^n \quad (27)$$

as the momentum operator conjugate to ξ_m in the sense of Tomonaga⁷⁾. Indeed the commutation relations between π_m and ξ_m are

$$[\pi_{m'}, \xi_m] = (R^2/iR_0^2) - (a/i) \sum_M (-1)^{m'} (2-m' 2m | 2M) \xi_M; \quad a = \sqrt{35/8\pi}, \quad (28)$$

where

$$R^2 = \frac{5}{3N} \sum_n \mathbf{r}_n^2 = \frac{5}{3} \langle r^2 \rangle \approx R_0^2. \quad (29)$$

The appearance of the second term in the right hand side of (28) shows that π_m and ξ_m cannot be considered as sets of (independent) canonical variables, even if we follow the convention to conceive R^2 of (29) as a constant, as the last version of (29) shows, in the sense of Tomonaga⁷⁾.

Moreover π_m and $\pi_{m'}$ with $m \neq m'$ do not commute with each other as in § 2, and we obtain

$$[\pi_{m'}, \pi_m] = (15\sqrt{5}/8\sqrt{2}\pi) \sum_M (2m' 2m | 1M) L_M, \quad (30)$$

where

$$L_M = -\sqrt{2} \sum_{m_1 m_2} (1m_1 1m_2 | 1M) \sum_n \mathbf{r}_{m_1}^n \mathbf{r}_{m_2}^n = \sum_n (\mathbf{r}_n \times \mathbf{p}_n)_M \quad (31)$$

is the angular momentum operator.

Because of these difficulties it is impossible to apply the procedure of Tomonaga directly to our case. The method developed in last section, however, which introduces new (redundant) variables, together with appropriate subsidiary conditions, can be applied in this case

* Here and in the following the symbol $Y_{lm}(\)$ with argument written by Gothic or Italic letters should be considered as solid or surface harmonics, respectively.

too, and the remainder of this section will be devoted to the development of such procedures.

Taking $\alpha_m (m = -2, \dots, 2)$ as new variables and $\beta_m (m = -2, \dots, 2)$ as momentum operators canonically conjugate to the formers, we impose the following subsidiary conditions to the wave function Ψ_0 , an eigenfunction corresponding to the Hamiltonian H_0 of (24),

$$\alpha_m \Psi_0 = 0. \quad (32)$$

Now we perform the following canonical transformation

$$\begin{aligned} Q &\rightarrow U_1^{-1} Q U_1, \\ \Psi_0 &\rightarrow \Psi_1 = U_1^{-1} \Psi_0, \end{aligned} \quad (33)$$

where

$$U_1 = \exp(i S_1); \quad S_1 = \sum_m \hat{\xi}_m \beta_m.$$

Then our original Hamiltonian (24) and subsidiary conditions (32) are transformed into

$$H_1 = (1/2\mu) \sum_n (p_n^2 + \sum_m \beta_m (F_n^2 \hat{\xi}_m)) + V(x), \quad (34)$$

$$(\alpha_m - \hat{\xi}_m) \Psi_1 = 0. \quad (35)$$

Now using π_m defined in (27), and further introducing the operators η_m defined by

$$\eta_m = R_0^{-2} (R^{-2} \pi_m + \pi_m R^{-2})/2, \quad (36)$$

we rewrite (34) into the following form

$$\begin{aligned} H_1 &= (1/2\mu) \sum_n p_n^2 + V(x) + (4\pi/3\mu NR_0^4) \sum_m (\beta_m + \eta_m)^* R^2 (\beta_m + \eta_m) \\ &+ (25 R_0^4 / 4\pi\mu N) \sum_m \hat{\xi}_m^* R^{-6} \hat{\xi}_m - (4\pi/3\mu N) \sum_m \pi_m^* R^{-2} \pi_m - 25/3\mu NR^2. \end{aligned} \quad (37)$$

To this Hamiltonian (37) and subsidiary conditions (35), we further perform a canonical transformation which corresponds to that given in (13) of the preceding section. It is

$$\begin{aligned} \Psi_1 &\rightarrow \Psi_2 = U_2^{-1} \Psi_1, \\ U_2 &= \exp(-i S_2), \quad S_2 = \sum_m \eta_m \alpha_m. \end{aligned} \quad (38)$$

Although our calculation is exact up to the derivation of (35) and (37), the unitary transformation (38) cannot be performed in a closed form and we will express the result in the form of series expansion in powers of α_m or $\hat{\xi}_m$, following the same reasoning as was stated below (17) of last section.

First we derive the subsidiary condition in this new representation, and the result up to second order in α_m or $\hat{\xi}_m$ is

$$\{\hat{\xi}_m - (a R_0^2 / 2 R^2) \sum_{m_1 m_2} (2m_1 2m_2 | 2m) (2\alpha_{m_1} \hat{\xi}_{m_2} + \alpha_{m_1} \alpha_{m_2})\} \Psi_2 = 0. \quad (39)$$

We see that this subsidiary condition contains variables α_m of collective motion, but we know^{8,9)} that the subsidiary condition which depends only on variables of the internal motion

is much more convenient and so we hope to find another representation in which such a requirement is fulfilled. It is easy, however, to see that such a representation is obtained, so long as we content ourselves in first order approximation in α_m or $\hat{\xi}_m$, by performing a further canonical transformation to Ψ_2 such as given below :

$$\begin{aligned}\Psi_2 &\rightarrow \Psi_3 = U_3^{-1} \Psi_2, \\ U_3 &= \exp(-i S_3), \\ S_3 &= (a R_0^2 / 2 R^2) \sum_{m_1 m_2} (2m_1 \ 2m_2 | 2m) (2\alpha_{m_1} \hat{\xi}_{m_2} + \alpha_{m_1} \alpha_{m_2}) \eta_m. \quad (40)\end{aligned}$$

Then our subsidiary condition, correct up to second order in α_m or $\hat{\xi}_m$, takes the following simple form

$$\hat{\xi}_m \Psi_3 = 0. \quad (41)$$

Our task in this section now has become to obtain an explicit form of the Hamiltonian appropriate to this representation. This can, of course, be achieved by performing two canonical transformations U_2 and U_3 successively to H_1 of (37). We will give the result again in the form of power series expansion in α_m and up to order of α_m^2 :

$$\begin{aligned}H_3 &= U_3^{-1} U_2^{-1} H_1 U_2 U_3 \\ &= H_1 + i \sum_m [\alpha_m \eta_m, H_1] + (i^2/2) \sum_{mm'} [\alpha_{m'} \eta_{m'}, [\alpha_m \eta_m, H_1]] \\ &\quad + (ai R_0^2/2) \sum_{m_1 m_2 m} (2m_1 \ 2m_2 | 2m) [(2\alpha_{m_1} \hat{\xi}_{m_2} + \alpha_{m_1} \alpha_{m_2}) R^{-2} \eta_m, H_1]. \quad (42)\end{aligned}$$

In (42) the terms which are proportional to a are those which have originated from the presence of the second term in the right hand side of (28). Thus we see that, in our procedure, we have succeeded in taking into account the correction which is caused by the presence of unwanted terms in the commutator, by transcribing it into correction terms in the Hamiltonian. The correction which originates from the non-commutativity of π_m and $\pi_{m'}$ with $m \neq m'$, shown in (30), can also be taken into account in calculating the commutator in the expansion coefficients which appear in (42).

The actual calculation of these commutators can be very much facilitated by the fact that we are considering in the representation in which $\hat{\xi}_m$ is a vanishing operator, because in this case we can put any term equal to zero which contains $\hat{\xi}_m$ as a factor at the right most.

To write down (42) in a more intelligible form we first split H_1 into two parts ;

$$H_1 = H_{\text{int}} + H_{1,c} \quad (43)$$

where

$$H_{\text{int}} = \frac{1}{2\mu} \sum_{n\nu} p_\nu^{n*} p_\nu^n + V(x) - \frac{4\pi}{3\mu N} \sum_m \pi_m^* R^{-2} \pi_m + \frac{25 R_0^4}{4\pi \mu N} \sum_m \hat{\xi}_m^* R^{-6} \hat{\xi}_m - \frac{25}{3\pi N R^2} \quad (44)$$

is the part of the Hamiltonian which describes only the internal motion, while

$$H_{1,c} = \frac{4\pi}{3\mu N R_0^4} \sum_m (\beta_m + \eta_m)^* R^2 (\beta_m + \eta_m) - \frac{4\pi a}{3\mu N R_0^2} \sum_{m_1 m_2 m} (2m_1 \ 2m_2 | 2m) \beta_{m_1} \beta_{m_2} \hat{\xi}_m \quad (45)$$

may well be interpreted as the kinetic energy part of the Hamiltonian for the collective motion, in the representation Ψ_1 .

The result of operation of successive canonical transformations U_2 and U_3 to $H_{1,c}$ now becomes as

$$H_{3,c} = U_3^{-1} U_2^{-1} H_{1,c} U_2 U_3$$

$$= \frac{4\pi}{3\mu NR_0^2} \sum_m \beta_m^* R^2 \beta_m - \frac{4\pi a}{3\mu NR_0^2} \sum_{m_1 m_2 m} (2m_1 \ 2m_2 | 2m) (\beta_{m_1} \beta_{m_2} + \eta_{m_1} \eta_{m_2}) \alpha_m, \quad (46)$$

and after this calculation we find it more convenient to rewrite H_3 of (42) in the following form

$$H_3 = H_{int} + i \sum \{ [\eta_m, H_{int}] + a R_0^2 \sum_{m_1 m_2} (2m_1 \ 2m_2 | 2m) [R^{-2} \xi_{m_1} \eta_{m_2}, H_{int}] \} \alpha_m$$

$$- \frac{4\pi a}{3\mu NR_0^2} \sum_{m_1 m_2 m} (2m_1 \ 2m_2 | 2m) \eta_{m_1} \eta_{m_2} \alpha_m + \frac{4\pi}{3\mu NR_0^2} \sum_m \beta_m^* R^2 \beta_m - (1/2)$$

$$\times \sum_m [\eta_m^* [\eta_m, H_{int}]] \alpha_m^* \alpha_m + \sum_{\substack{m m' \\ (m \neq m')}} \left\{ -\frac{1}{2} [\eta_{m'} [\eta_m, H_{int}]] + \frac{ia R_0^2}{2} \sum_{m_1} (2m \ 2m' | 2m_1) \right.$$

$$\left. [R^{-2} \eta_{m_1}, H_{int}] \right\} \alpha_m \alpha_{m'} - \frac{4\pi a}{3\mu NR_0^2} \sum_{m_1 m_2 m} (2m_1 \ 2m_2 | 2m) \beta_{m_2} \beta_{m_1} \alpha_m. \quad (47)$$

In (47) the fourth and the fifth terms are those which may be considered, respectively as kinetic and potential energy parts of the collective motion, and indeed have the form of a set of five independent harmonic oscillators*, while the last two are correction terms arising from the fact that these harmonic oscillators are not independent of each other, in the higher approximation.

The meaning of the first term is already explained as the Hamiltonian of the internal motion, and finally the second and the third terms are interaction terms between these internal and the collective motions.

The reason that we have stopped the expansion of (46) by the term linear in α_m and not in α_m^2 is now clear, because the leading term, i.e., the first term in the right hand side of (46) is already expected to be of the order of α_m^2 . This is easily seen from the well-known nature of harmonic oscillators that the expectation values of the kinetic and the potential terms coincide.

Our derivation of the Hamiltonian will be completed if we calculate explicitly the commutators which appear in (47). Among five terms of (44), however, the last three terms are expected to be small, being of the order of $1/N$ compared with the first two, and it may be permissible to neglect them in calculating these commutators, although they may play important roles, in some cases, in discussing internal motions, as was exemplified

* If we take the fourth term in the form $(1/2I) \sum \beta_m^* \beta_m$, then $I = 3\mu NR_0^4 / 8\pi \langle R^2 \rangle$. If we further put $\langle R^2 \rangle = R_0^2$, then $I = 3\pi NR_0^2 / 8\pi$, and this exactly coincides with the moment of inertia obtained by Bohr¹⁾.

by Tomonaga⁷⁾. We will give here the result in the following simple form, using the first two terms of (44) in the calculation of these commutators.

$$\left[\eta_m, \frac{1}{2\mu} \sum_{n\nu} p_\nu^{n*} p_\nu^n \right] = \frac{1}{2\mu} \left\{ \frac{\sqrt{15}}{\sqrt{2\pi} R^2 i} \sum_{m_1 m_2} (1m_1 \ 1m_2 | 2-m) (-1)^m p_{m_1}^n p_{m_2}^n - \frac{20R_0^2}{3NR^4} \right. \\ \left. \times \sum_{n\nu} r_\nu^n p_\nu^{n*} \pi_m - \frac{10R_0^2}{R^4} \pi_m + \frac{25 R_0^4}{3\pi i NR^6} \xi_m^* \sum_{n\nu} r_\nu^n p_\nu^{n*} \right\}, \quad (48-a)$$

$$[\eta_m, V(x)] = [\eta_m, \sum_{nn'} V(r_{nn'})] \\ = \sqrt{\frac{10\pi}{3}} \frac{R_0^2}{R^2 i} \sum_{m_1 m_2} (-1)^m (1m_1 \ 1m_2 | 2-m) \sum_{nn'} V'(r_{nn'}) Y_{1m_1}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{1m_2}(\mathbf{r}_n - \mathbf{r}_{n'}),$$

$$[\eta_{m'} [\eta_m, \frac{1}{2\mu} \sum_{n\nu} p_\nu^n p_\nu^{n*}]] \quad (48-b)$$

$$= -\frac{15R_0^4}{4\pi\mu R^4} \sum_{m_1 m_2 m_3} (-1)^{m+m'+m_1} (1m_1 \ 1m_2 | 2-m) (1-m_1 \ 1m_3 | 2-m') p_{m_2}^n p_{m_3}^n \\ - \frac{75R_0^4}{8\pi\mu R^6} (-1)^m \delta_{m, -m'} + \frac{20R_0^4}{3\mu NR^6} \pi_m \pi_{m'}, \quad (49-a)$$

$$[\eta_{m'} [\eta_m, V]]$$

$$= -\frac{10\pi}{3} \frac{R_0^4}{R^4} \sum_{m_1 m_2 m_1' m_2'} (-1)^{m+m'} (1m_1 \ 1m_2 | 2-m) (1m_1' \ 1m_2' | 2-m') \\ \cdot \sum_{nn'} V''(r_{nn'}) Y_{m_1}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{m_2}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{1m_1'}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{1m_2'}(\mathbf{r}_n - \mathbf{r}_{n'}) \\ - 5 \frac{R_0^4}{R^4} \sum_{m_1 m_2 m_1'} (-1)^{m+m'+m_1} (1m_1 \ 1m_2 | 2-m) (1m_1' \ 1-m_1' | 2-m') \\ \cdot \sum_{nn'} V'(r_{nn'}) Y_{1m_1}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{1-m_1'}(\mathbf{r}_n - \mathbf{r}_{n'}), \quad (49-b)$$

where

$$V'(r_{nn'}) = (1/r_{nn'}) \partial V(r_{nn'}) / \partial r_{nn'}, \quad V''(r_{nn'}) = \frac{1}{r_{nn'}} \frac{\partial}{\partial r_{nn'}} \left(\frac{1}{r_{nn'}} \frac{\partial V(r_{nn'})}{\partial r_{nn'}} \right), \quad (50)$$

and

$$[R^{-2} \xi_{m_1} \eta_{m_2}, \frac{1}{2\mu} \sum_{n\nu} p_\nu^n p_\nu^{n*}] = -\frac{8\pi i}{3\mu NR_0^2 R^2} \pi_{m_1}^* \eta_{m_2},$$

$$[R^{-2} \xi_{m_1} \eta_{m_2}, V(x)] = 0,$$

$$[R^{-2} \eta_{m_1}, (1/2\mu) \sum_{n\nu} p_\nu^n p_\nu^{n*}]$$

$$= \frac{\sqrt{15}}{2\mu \sqrt{2\pi} R^4 i} \sum_{m_1' m_2'} (1m_1' \ 1m_2' | 2-m_1) (-1)^{m_1} p_{m_1'}^n p_{m_2'}^n - \frac{20R_0^2}{3\mu NR^6} \sum_{n\nu} r_\nu^n p_\nu^{n*} \pi_{m_1} - \frac{10R_0^2}{\mu R^6} \pi_{m_1},$$

$$[R^{-2} \eta_{m_1}, V(x)] \quad (51)$$

$$= \sqrt{\frac{10\pi}{3}} \frac{R_0^2}{R^4 i} \sum_{m_1' m_2'} (-1)^{m_1} (1m_1' 1m_2' | 2-m_1) \sum_{nn'} V'(r_{nn'}) Y_{m_1'}(\mathbf{r}_n - \mathbf{r}_{n'}) Y_{1m_2'}(\mathbf{r}_n - \mathbf{r}_{n'}).$$

In these formulae, terms which are almost clear to be of the order of $1/N$ have already been put equal to zero, as well as terms which contain ξ_m as a factor.

This section will be closed by giving the explicit form of angular momentum operators in various representations.

The angular momentum operator L_m in the representation Ψ_0 is, of course, as is given in (31),

$$L_m = -\sqrt{2} \sum_{m_1 m_2} (1m_1 1m_2 | 1m) r_{m_1}^n r_{m_2}^n.$$

This is transformed, in the representation Ψ_1 , into

$$L_{m,1} = L_m - \sqrt{10} i \sum_{m' m''} (2m' 2m'' | 1m) \beta_{m'}^* \xi_{m''}. \quad (52)$$

Finally its form in the representation Ψ_3 , up to the lowest order in α_m , is as follows;

$$L_{m,3} = L_m + \sqrt{10} i \sum_{m_1 m_2} (2m_1 2m_2 | 1m) \alpha_{m_1} \beta_{m_2}^* + \sqrt{10} i \sum_{m_1 m_2} (2m_1 2m_2 | 1m) \alpha_{m_1} \eta_{m_2}. \quad (53)$$

The second term of (53) may be interpreted as the collective part of the angular momentum*, while the third term is the coupling term between the internal and the collective motions.

Similar transformations applied to the electric and magnetic multipole moments may be useful in discussing nuclear properties.

§ 4. A numerical calculation. The surface energy

In the preceding section we have succeeded in deriving a Hamiltonian appropriate to describe the coexistence of, or the correlation between the internal and the collective motions. Thus it is now possible, at least in principle, to investigate the intimate relation between the individual particle model and the Bohr model. We will discuss these problems, however,

* In BI the angular momentum operator of surface oscillation is given by

$$(\mathfrak{M})_m = iB \sum_{m_1 m_2} \alpha_{m_1} \alpha_{m_2}^* (M_{m_2 m_1})_m. \quad (\text{BI-33})$$

Using the relation $\dot{\alpha}_{m_2}^* = \pi_{m_2}/B$ of (BI-7), or in our notation $\dot{\alpha}_{m_2}^* = \beta_{m_2}/B$, this is rewritten as

$$(\mathfrak{M})_m = i \sum_{m_1 m_2} \alpha_{m_1} \beta_{m_2} (M_{m_2 m_1})_m.$$

But

$$(M_{m_2 m_1})_m = \int Y_{2m_1}^* L_m Y_{2m_2} d\Omega = \sqrt{10} (-1)^{m_2} (2m_1 2-m_2 | 1m),$$

and so

$$(\mathfrak{M})_m = \sqrt{10} i \sum_{m_1 m_2} (2m_1 2m_2 | 1m) \alpha_{m_1} \beta_{m_2}^*.$$

Thus we see that Bohr's $(\mathfrak{M})_m$ coincides with the second term of our equation (53).

in other separated papers, and in this section we will try to calculate the surface energy of the nucleus from two body interaction, as an example of application of our result to actual problems.

As is noted by Tomonaga⁹⁾ the surface energy, which should correspond to that used by Bohr, is to be obtained as the expectation value, concerning the internal motion in its ground state, of the coefficient of $\alpha_m^* \alpha_m$ of the fifth term in our final Hamiltonian (47). Thus the surface energy, O is given by

$$O = \langle - (1/2) [\eta_m^* [\eta_m, H_{int}]] \rangle_0 \quad (54)$$

where H_{int} is given in (44).

In calculating this expectation value, we must first solve the equation of motion of the internal motion, the Hamiltonian corresponding to it being H_{int} of (44): In solving this equation we must further take the subsidiary condition (41) into account, and this makes our task rather difficult. It is also noted by Tomonaga⁹⁾ that if the usual individual particle model is a good approximation to the correct solution corresponding to the original Hamiltonian H_0 of (24), then that model is also a good approximation to our present case. If this is the case, the modification caused by the difference between H_0 and H_{int} , and by the presence of the subsidiary condition can be neglected in the zeroth order approximation. Therefore the calculation of (54) is similar to the usual calculation of the binding energy or so.

As we have already stated in the preceding section, we leave only the first two terms in H_{int} , so the operator of (54) is nothing but the sum of (49-a) and (49-b), multiplied by $-1/2$.

We first begin with the evaluation of the potential energy part of the surface tension. For such a purpose we find it convenient to express the two body interaction operator in the form of Fourier integral:

$$V(x) = \sum_{nn'} V(r_{nn'}); \quad V(r_{nn'}) = \int f(k) e^{ik \cdot r_{nn'}} dk, \quad (55)$$

where $r_{nn'} = r_n - r_{n'}$.

Then the expression of (49-b) is rewritten in a somewhat different form and we obtain for the potential energy part $O_{P.E.}$ of the surface tension,

$$\begin{aligned} O_{P.E.} = & - (1/2) \langle [\eta_m^* [\eta_m, V(x)]] \rangle_0 \\ = & - (10\pi/3) \sum (-1)^m (1m_1 \ 1m_2 | 2m) (1m_1' \ 1m_2' | 2-m) \\ & \times \langle \sum_{nn'} \int f(k) Y_{1m_2}(k) Y_{1m_2'}(k) e^{ik \cdot r_{nn'}} Y_{1m_1}(r_n) Y_{1m_1'}(r_n) dk \rangle_0 \\ = & - (10\pi/3) \sum (-1)^m (1m_1 \ 1m_2 | 2m) (1m_1' \ 1m_2' | 2-m) \\ & \times \langle \sum_{nn'} \int f(k) Y_{1m_2}(k) Y_{1m_2'}(k) e^{ik \cdot r_{nn'}} Y_{1m_1}(r_n) Y_{1m_1'}(r_{n'}) dk \rangle_0 \\ & + (5\pi/3i) \sum (-1)^{m+m_1} (1m_1 \ 1m_2 | 2m) (1m_1' \ 1-m_1 | 2-m) \end{aligned}$$

$$\times \left\langle \sum_{nn'} \int f(\mathbf{k}) Y_{1m_2}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_{nn'}} Y_{1m_1'}(\mathbf{r}_n - \mathbf{r}_{n'}) d\mathbf{k} \right\rangle_0, \quad (56)$$

Now the expression

$$\left\langle \sum_{nn'} \int f(\mathbf{k}) Y_{1m_2}(\mathbf{k}) Y_{1m_2'}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_{nn'}} Y_{1m_1}(\mathbf{r}_n) Y_{1m_1'}(\mathbf{r}_{n'}) d\mathbf{k} \right\rangle_0, \quad (57)$$

which is the last factor of the first term of (56) can be written by using the formula (2.3) of Blatt et al.⁽¹¹⁾ as

$$(9/4\pi) \sum_{LL'MM'} (1/\sqrt{(2L+1)(2L'+1)}) (1010|L0) (1010|L'0) (1m_2 1m_2'|LM) \\ \cdot (1m_1 1m_1'|L'M') \left\langle \sum_{nn'} \int f(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_{nn'}} k^2 Y_{LM}(\mathbf{k}) r_n^2 Y_{L'M'}(\mathbf{r}_n) d\mathbf{k} \right\rangle_0, \quad (58)$$

the last factor of which can of course be rewritten as

$$(N(N-1)/2) \left\langle \int f(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_{12}} k^2 r_1^2 Y_{LM}(\mathbf{k}) Y_{L'M'}(\mathbf{r}_1) d\mathbf{k} \right\rangle_0,$$

and this can further be reexpressed in the following form, if we take the individual particle model and assume that the ground state wave function is spherically symmetric and has a constant density within the sphere of radius R_0 and zero outside;

$$(N(N-1)/2) (4\pi R_0^3/3)^{-2} \int f(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}_{12}} k^2 r_1^2 Y_{LM}(\mathbf{k}) Y_{L'M'}(\mathbf{r}_1) d\mathbf{k} d\mathbf{r}_1 d\mathbf{r}_2.$$

The performance of the integration over angular variables is easy and the result is

$$(N(N-1)/2) 9R_0^{-6} \partial_{LL'} (-1)^{M_L} \int_0^\infty f(k) k^4 dk \int_0^{R_0} r^2 j_0(kr) dr \int_0^{R_0} r^4 j_L(kr) dr. \quad (59)$$

Combining (57), (58) and (59), and performing summation over magnetic quantum numbers, the first term of (56) reduces to a rather compact form and we obtain, together with other results similarly calculated from other terms of (56), the following formula

$$O_{P.E.} = (3N(N-1)/4R_0^6) \\ \cdot \left\{ -5 \int_0^\infty f(k) k^4 dk \int_0^{R_0} r^2 j_0(kr) dr \int_0^{R_0} r^4 j_0(kr) dr \right. \\ + \int_0^\infty f(k) k^4 dk \int_0^{R_0} r^2 j_0(kr) dr \int_0^{R_0} r^4 j_2(kr) dr \\ - 15 \int_0^\infty f(k) k^3 dk \int_0^{R_0} r^2 j_0(kr) dr \int_0^{R_0} r^3 j_1(kr) dr \\ \left. + 6 \int_0^\infty f(k) k^4 dk \left(\int_0^{R_0} r^3 j_1(kr) dr \right)^2 \right\}. \quad (60)$$

Now we assume for $V(r_{nn'})$ the Yukawa type interaction

$$V(r_{nn'}) = (V_0/\mu r_{nn'}) \exp[-\mu r_{nn'}],$$

then the function $f(k)$, defined in (55) takes the following form

$$f(k) = \frac{V_0}{2\pi^2\mu} \frac{1}{k^2 + \mu^2}.$$

Then the remaining integration, though somewhat lengthy, can be calculated throughout analytically and we obtain the following very simple final result:

$$O_{P.E.} = \frac{3N(N-1)}{16\pi\mu R_0} \left[-\frac{6}{(\mu R_0)^3} + \frac{27}{(\mu R_0)^5} \right] V_0, \quad (61)$$

where terms containing a very small factor $\exp(-2\mu R_0)$ are already neglected.

Now rewriting N by A as usual, and R_0 by $r_0 A^{1/3}$ we can change (61) into

$$O_{P.E.} = \frac{9|V_0|}{8\pi(\mu r_0)^4} A^{1/3} \left[1 - \frac{9}{2(\mu r_0)^2} A^{-2/3} \right], \quad (62)$$

and we see that the main term of our result is proportional to $A^{2/3}$ as is expected.

It would not be unreasonable to take $\mu^{-1} = 1.18 \times 10^{-13}$ cm, $r_0 = 1.4 \times 10^{-13}$ cm and $V_0 = 50$ Mev, then we obtain

$$O_{P.E.} = 8.95 A^{2/3} \text{ Mev.} \quad (63)$$

The surface energy with which we should compare our result is $2R^2S$ in the Bohr's paper¹⁾ (see (BI-2) and (BI-5)). As $4\pi R^2S = 15.4 A^{2/3}$ Mev, $2R^2S$ is equal to $2.48 A^{2/3}$ Mev, and so our result (63) is too large. The contribution from tensor forces would be small.

We must notice, however, that the result (63) is valid only when we assume that the central force is pure Wigner type one, and further the exchange integral is not taken into account.

If we assume, instead, that the central force is given by

$$V_c(r_{nn'}) = V(r_{nn'}) \cdot (m P_{nn'}^{(\tau)} + b P_{nn'}^{(\sigma)}), \quad (64)$$

i.e. as a sum of Majorana and Bartlett forces, m and b being taken equal to 0.8 and 0.2 in the case of saturation mixture¹²⁾, and calculate the matrix element following the procedure given by Bethe¹³⁾, we obtain for $O_{P.E.}$, instead of (56), the following formula:

$$O_{P.E.} = \left(\frac{b}{2} - m \right) \left\{ -\frac{1}{2} \sum_{i \neq j} (\psi_i(1) \psi_j(2) | [z_m^*, [z_m, V(r_{12})]] | \psi_i(1) \psi_j(2) \right\} \\ + \left(m - \frac{5b}{4} \right) \left\{ -\frac{1}{2} \sum_{i \neq j} (\psi_i(1) \psi_j(2) | [z_m^*, [z_m, V(r_{12})]] | \psi_j(1) \psi_i(2) \right\}. \quad (65)$$

The expression which appears as the coefficient of $(b/2 - m)$, in the first term of (65), is just equal to the result (63), and so if we take $m = 0.8$ and $b = 0.2$, as stated above, the contribution to surface tension from ordinary integral is estimated to be $-6.27 A^{2/3}$ Mev.

The contribution of the exchange integral is rather tedious to estimate and we have

not calculated. A simple consideration shows, however, that its contribution is not proportional to $A^{2/3}$ but to A ,

We have also to take into consideration the contribution from the kinetic energy in H_{int} to the surface tension. But it is rather easy to see that the result also is proportional to A , because the first term of (49-a), which is the main term, is proportional to the usual kinetic energy operator, when $m=m'$.

In order to obtain results which are proportional to $A^{2/3}$ from these contributions, we would be necessitated, in evaluating expectation values, to use not the ordinary spherically symmetric (individual particle) wave function but the one which correctly takes the subsidiary condition into account. In other words, as was pointed out by Hill and Wheeler⁽¹⁴⁾, the correct value of the surface energy is to be obtained from the eigenvalue of the total system for a fixed deformation of the nucleus. This is clearly identical to solve the eigenvalue problem adiabatically by regarding α_m in the Hamiltonian (47) as constant and, consequently, putting the momenta β_m zero. Such a calculation may be a higher approximation to that given in this section, and will be presented in another separate paper.

If we assume, however, that the kinetic energy part of the surface tension calculated by Hill and Wheeler⁽¹⁴⁾ is correct, then the contribution from kinetic energy to our result is $4.51 A^{2/3}$ Mev. This, combined with the above obtained contribution from the ordinary integral, gives $-1.76 A^{2/3}$ Mev. So if the contribution from the exchange integral were $4.3 A^{2/3}$ Mev, which is not an unexpectable value, correct experimental value would be obtained. Whether such a value could be obtained or not can be answered only after more detailed calculation is performed, but we may say that our result obtained in (63) is not an unreasonable one.

§ 5. Discussions

We have shown in the foot-note in § 3 that our kinetic energy operator, in the zeroth approximation, just coincides with that used by Bohr⁽¹⁻³⁾, and also shown in § 4 that the potential energy will also coincide with Bohr's one. Therefore the result of the application of our result to actual problems will be the same as that of Bohr, as far as the collective motion is concerned. So our Hamiltonian is also impossible in improving the situation concerning the inconsistency of the magnitudes of deformation obtained from quadrupole moments and level spacings of even-even nuclei. The last term of (47) will be the similar one as that obtained by Coester⁽⁶⁾ as a correction term to Bohr's Hamiltonian, but it is shown by Coester himself that this term is also insufficient in improving the above mentioned situation.

Our Hamiltonian (47), however, can be used in describing the internal or individual particle motion, at the same time with the collective surface motion, which was impossible in the original Bohr model. This fact, if it is applied correctly to the investigation of the level structure, magnetic dipole- and electric quadrupole moments and other properties of heavy nuclei, it would give us many valuable knowledges. Such investigations will be the content of forthcoming papers.

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Coulomb Interactions and the Diamagnetism of Free Electrons

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The effect of the long-range part of the Coulomb interactions on the Landau diamagnetism is investigated, applying the collective description of the Coulomb interactions developed by Bohm and Pines. It is shown that the magnitude of the diamagnetic susceptibility slightly increases due to the Coulomb interactions between electrons.

§ 1. Introduction

The extension of Landau's¹⁾ original work on the diamagnetism of a free electron gas has been undertaken by several different methods. The magnetic moment of a free electron gas consists of non-oscillatory (Landau diamagnetism) and oscillatory (de Haas-van Alphen effect) parts. When the use of a finite container to hold the electrons is considered, the surface states give rise to size dependent terms in both the oscillatory and non-oscillatory parts of the magnetic moment. The effect brought about by using a finite container has been paid attention to and has been discussed by several workers. However the effect of the Coulomb interactions between electrons on the diamagnetism has hardly been investigated.

A collective description of the Coulomb interactions in an electron gas has been developed by Bohm and Pines.²⁾ The long-range part of the Coulomb interactions between electrons is described in terms of collective fields, representing organized plasma oscillations of the system as a whole. The Hamiltonian describes these collective fields plus the system of electrons interacting via screened Coulomb forces with a range of the order of the inter electronic distance.

The aim of the present paper is to discuss the effect of the long-range part of the Coulomb interactions on the Landau diamagnetism of a free electron gas, applying the collective description of the Coulomb interactions. In § 2, the collective description of the behavior of the electrons in a magnetic field is introduced. The Hamiltonian, subsidiary conditions and the dispersion formula are derived. In § 3, the current density is calculated to the first order in the magnetic field, applying the perturbation method developed by Schafroth.³⁾ It is shown that the effect of the long-range part of the Coulomb interactions on the diamagnetism is rather small and the increase in the magnitude of the diamagnetic susceptibility for Na amounts to 6 percent.

§ 2. Collective description of electron interaction in a magnetic field

We consider a system of electrons in a magnetic field embedded in a background of

uniform positive charge whose density is equal to that of the electrons. The Hamiltonian of our system may be written as

$$H = \sum_i \pi_i^2 / 2m + 2\pi e^2 \hbar^2 \sum_{i \neq j} e^{i/\hbar \cdot k(x_i - x_j)} / k^2, \quad (2.1)$$

where $\pi_i = p_i + \frac{e}{c} A(x_i)$, $A(x_i)$ being the vector potential of the magnetic field. The second term corresponds to the Coulomb interactions expanded as a Fourier series in a box of unit volume (since we are working in a box of unit volume). The prime in the summation over k denotes a sum in which $k=0$ is excluded.*

Following Bohm and Pines, we shall introduce an equivalent Hamiltonian instead of (2.1). Our equivalent Hamiltonian is given by

$$\begin{aligned} H = & \sum_i \pi_i^2 / 2m + (4\pi)^{1/2} e / m \cdot \sum_{k < k_0} \mathcal{E}_k \cdot (\pi_i - k/2) q_k e^{i/\hbar \cdot k x_i} \\ & + 2\pi e^2 / m \cdot \sum_{k, l < k_0} (\mathcal{E}_k \cdot \mathcal{E}_l) q_k q_l e^{i/\hbar \cdot (k+l) x_i} \\ & - 1/2 \cdot \sum_{k < k_0} p_k p_{-k} + 2\pi e^2 \hbar^2 \sum_{\substack{i \neq j \\ k > k_0}} e^{i/\hbar \cdot k(x_i - x_j)} / k^2 \\ & - 2\pi n e^2 \hbar^2 \sum_{k < k_0} 1/k^2, \end{aligned} \quad (2.2)$$

with the associated set of subsidiary conditions:

$$\xi_k \Phi = \{ p_{-k} - i(4\pi e^2 \hbar^2 / k^2)^{1/2} \sum_i e^{-i/\hbar \cdot k x_i} \} \Phi = 0 \quad (k < k_0), \quad (2.3)$$

where $\mathcal{E}_k = k/|k| \cdot k_0$ is the maximum momentum, beyond which the organized oscillation is not possible. According to the estimation by Bohm and Pines, k_0 for Na is $\sim 0.68 p_0$, where p_0 is the Fermi momentum.

The equivalence of (2.2) with (2.1) may be seen by applying the unitary transformation $\Phi = S\psi$, where

$$S = \exp \left[-1/\hbar \cdot \sum_{k < k_0} (4\pi e^2 \hbar^2 / k^2)^{1/2} q_k e^{i/\hbar \cdot k x_i} \right]. \quad (2.4)$$

We shall split up the third term in (2.2) into two parts. That part for which $k+l=0$ is given by

$$2\pi n e^2 / m \cdot \sum_{k < k_0} q_k q_{-k} = \omega_p^2 / 2 \cdot \sum_{k < k_0} q_k q_{-k}, \quad (2.5)$$

where n is the total number of electrons. We shall neglect the remaining part for which $k+l \neq 0$ (random phase approximation).

Let us introduce the creation and destruction operators for the collective field a_k and a_k^* , which is defined by

* We shall drop this prime in the remainder of this paper.

$$q_i = (\hbar/2\omega)^{1/2} (a_k - a_{-k}^*),$$

$$p_k = i(\hbar\omega/2)^{1/2} (a_k^* + a_{-k}). \quad (2.6)$$

The commutation relations are given by

$$[a_k, a_{k'}^*] = \delta_{kk'},$$

$$[a_k, a_{k'}] = [a_k^*, a_{k'}^*] = 0. \quad (2.7)$$

In terms of these variables, we write our Hamiltonian and subsidiary conditions as

$$H = H_0 + H_1 + H_2 + H_{s,r}, \quad (2.8)$$

$$H_0 = \sum_i \pi_i^2/2m + \sum_{i < k_c} \hbar\omega (a_k^* a_k + 1/2), \quad (2.9a)$$

$$H_1 = e/m \cdot \sum_{i < k_c} (2\pi\hbar/\omega)^{1/2} [\mathbf{e}_i \cdot (\boldsymbol{\pi}_i - \mathbf{k}/2) a_k e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}_i} + e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}_i} a_k^* \mathbf{e}_k \cdot (\boldsymbol{\pi}_i - \mathbf{k}/2)], \quad (2.9b)$$

$$H_2 = \sum_{k < k_c} (\hbar/4\omega) (\omega_p^2 - \omega^2) (a_k^* a_k + a_k a_k^* - a_k a_{-k} - a_k^* a_{-k}^*), \quad (2.9c)$$

$$H_{s,r} = 2\pi e^2 \hbar^2 \sum_{\substack{i \neq j \\ k > k_c}} e^{i/\hbar \cdot \mathbf{k} \cdot (\mathbf{x}_i - \mathbf{x}_j)} / k^2, \quad (2.9d)$$

$$\xi_k = a_k^* + a_{-k} - (8\pi e^2 \hbar/\omega k^2)^{1/2} \sum_i e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}_i}. \quad (2.10)$$

In order to eliminate the field-particle interaction H_1 , we consider a canonical transformation from our operators $(\mathbf{x}_i, \mathbf{p}_i, a_k, a_k^*)$ to a new set of operators $(\mathbf{X}_i, \mathbf{P}_i, A_k, A_k^*)$. The relation between these two sets may be written as

$$\mathbf{x}_i = e^{-iS} \mathbf{X}_i e^{iS} \text{ etc.} \quad (2.11)$$

The generating function of our canonical transformation is given by

$$S = - (ei/m) \sum_{i < k_c} (2\pi\hbar/\omega)^{1/2} \left[\frac{\mathbf{e}_k \cdot (\boldsymbol{\Pi}_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \boldsymbol{\Pi}_i/m + k^2/2m} A_i e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}_i} \right. \\ \left. - e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}_i} A_k^* \frac{\mathbf{e}_k \cdot (\boldsymbol{\Pi}_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \boldsymbol{\Pi}_i/m + k^2/2m} \right]. \quad (2.12)$$

Our Hamiltonian is expressed in terms of new variables as follows

$$H = e^{-iS} \mathcal{H} e^{iS}$$

$$= \mathcal{H}_0 + \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{s,r} + i[\mathcal{H}_0, S] + i[\mathcal{H}_1, S]$$

$$- 1/2 \cdot [[\mathcal{H}_0, S], S] + \dots$$

$$= \mathcal{H}_0 + H_{e.p.} + \mathcal{H}_{s,r.} + \mathcal{H}_2 + i/2 \cdot [\mathcal{H}_1, S] + \dots,^* \quad (2.13)$$

where $H_{e.p.} = \mathcal{H}_1 + i[\mathcal{H}_0, S]$. The elimination of H_1 is incomplete owing to the non-

* We have neglected $i/2 [H_{e.p.}, S]$.

commutability of Π_i^2 and $\mathbf{\epsilon}_i \cdot \Pi_i$, and a weak interaction $H_{v,p.}$ between electrons and plasma oscillations remains. Making the sum of \mathcal{H}_v and terms in $i/2 \cdot [\mathcal{H}_v, S]$, which are multiplied by $(A_i^* A_i + A_i A_i^* - A_k A_{-k} - A_k^* A_{-k}^*)$, zero, we get the following dispersion relation

$$\omega^2 = \omega_p^2 + 4\pi e^2/m \cdot \sum_i \frac{2\hbar\omega \mathbf{k} \cdot \Pi_i/m + k^4/4m^2 - (\mathbf{k} \cdot \Pi_i/m)^2}{[\hbar\omega - \mathbf{k} \cdot \Pi_i/m]^2 - k^4/4m^2}. \quad (2.14)$$

For sufficiently small k , we may expand (2.14) in powers of $\mathbf{k} \cdot \Pi_i/m\hbar\omega$, and $k^2/m\hbar\omega$, and obtain

$$\omega^2 = \omega_p^2 + \sum_i (3/\hbar^2 n m^2) (\mathbf{k} \cdot \Pi_i)^2 + k^4/4\hbar^2 m^2. \quad (2.15)$$

Our Hamiltonian may thus be expressed to the lowest order of our canonical transformation as follows

$$H = H_{\text{electron}} + H_{\text{coll}} + H_{e,p.} + H_{r,p.}, \quad (2.16)$$

$$H_{\text{electron}} = \sum_i \Pi_i^2/2m - (\pi e^2 \hbar/m^2) \sum_{i: k < k_c} (1/\omega) \left[\frac{[\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)]^2}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m + k^2/2m} + \frac{[\mathbf{\epsilon}_i \cdot (\Pi_i + \mathbf{k}/2)]^2}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m - k^2/2m} \right] + 2\pi e^2 \hbar^2 \sum_{\substack{i \neq j \\ k > k_c}} e^{i/\hbar \cdot \mathbf{k} \cdot (\mathbf{X}_i - \mathbf{X}_j)} / k^2, \quad (2.17a)$$

$$H_{\text{coll}} = \sum_{k < k_c} \hbar\omega (A_k^* A_k + 1/2), \quad (2.17b)$$

$$H_{v,p.} = (e/2m^2) \sum_{i: k < k_c} (2\pi \hbar/\omega)^{1/2} \left[\left\{ \Pi_i^2 \frac{\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m + k^2/2m} - \frac{\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m + k^2/2m} \Pi_i^2 \right\} A_k e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} - A_k^* e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} \right. \\ \left. \times \left\{ \Pi_i^2 \frac{\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m + k^2/2m} - \frac{\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \Pi_i/m + k^2/2m} \Pi_i^2 \right\} \right], \quad (2.17c)$$

$$H_{r,p.} = -(\pi e^2 \hbar^2/m^2) \sum_{\substack{i \neq j \\ k < k_c}} \left[\frac{[\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)][\mathbf{\epsilon}_j \cdot (\Pi_j + \mathbf{k}/2)]}{\hbar\omega[\hbar\omega - \mathbf{k} \cdot \Pi_i/m - k^2/2m]} e^{i/\hbar \cdot \mathbf{k} \cdot (\mathbf{X}_i - \mathbf{X}_j)} \right. \\ \left. + e^{-i/\hbar \cdot \mathbf{k} \cdot (\mathbf{X}_i - \mathbf{X}_j)} \frac{[\mathbf{\epsilon}_i \cdot (\Pi_i - \mathbf{k}/2)][\mathbf{\epsilon}_j \cdot (\Pi_j + \mathbf{k}/2)]}{\hbar\omega[\hbar\omega - \mathbf{k} \cdot \Pi_i/m - k^2/2m]} \right]. \quad (2.17d)$$

In obtaining (2.16), we have neglected a number of terms which are quadratic in the field variables and are multiplied by a phase factor with a nonvanishing argument $\exp[i(\mathbf{k} + \mathbf{l}) \cdot \mathbf{X}_i]$.

The second term in H_{electron} , which may be interpreted as a sum of the self energies of electrons, becomes approximately²⁾

$$E_{\text{self}} = -(n'/3n) \sum_i \Pi_i^2/2m - (3n'/40) (k_c^2/m), \quad (2.18)$$

where $n' = 4\pi k_c^3/3\hbar^3$. Therefore we find

$$H_{\text{electron}} = \sum_i p_i^2 / 2m^* + 2\pi e^2 \hbar^2 \sum_{\substack{i \neq j \\ l > l_c}} e^{i/\hbar \cdot k(X_i - X_j)} / k^2 + \text{const.}, \quad (2 \cdot 19)$$

where $m^* = m \times 3n / (3n - n')$. Thus the "new" electrons behave as if they had an effective mass m^* , which is slightly larger than the "bare" electron mass. $H_{\text{r.p.}}$ describes very weak electron-electron interaction. We may safely disregard $H_{\text{r.p.}}$ in comparison with the screened Coulomb interactions in considering the effects of electron-electron interactions.

Our new subsidiary conditions in lowest order of our canonical transformation are given by

$$(\xi_k)_{\text{new}} \psi = \sum_i \frac{1}{1 - (1/\hbar\omega)^2 (\mathbf{k} \cdot \mathbf{II}_i / m - k^2 / 2m)^2} e^{i/\hbar \cdot \mathbf{k} X_i} \psi = 0 \quad (k < k_c). \quad (2 \cdot 20)$$

It should be noted that in our new representation, the subsidiary condition (2·20) continue to commute with the Hamiltonian (2·16) within the approximations we have made. This follows since the commutation relations are unchanged by a canonical transformation.

Now let us consider the current density operator, which is given by

$$\mathbf{j}(\mathbf{x}) = -e/2m \cdot \sum_i [\pi_i \delta(\mathbf{x} - \mathbf{x}_i) + \delta(\mathbf{x} - \mathbf{x}_i) \pi_i]. \quad (2 \cdot 21)$$

The effect of our transformation on the current density operator may be obtained in similar fashion. Our new current density operator is given by

$$\mathbf{j}(\mathbf{x}) = e^{-iS} \mathbf{J}(\mathbf{x}) e^{iS} = \mathbf{J}(\mathbf{x}) + \mathbf{J}^{(1)}(\mathbf{x}) + \mathbf{J}^{(2)}(\mathbf{x}) + \dots, \quad (2 \cdot 22)$$

$$\mathbf{J}(\mathbf{x}) = -e/2m \cdot \sum_i [\mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) + \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i], \quad (2 \cdot 23a)$$

$$\begin{aligned} \mathbf{J}^{(1)}(\mathbf{x}) &= i[\mathbf{J}(\mathbf{x}), S] \\ &= -e^2/2m^2 \cdot \sum_{ik < k_c} (2\pi\hbar/\omega)^{1/2} [\{ \mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) A_i(\mathbf{k}) - A_i(\mathbf{k}) (\mathbf{II}_i - \mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \\ &\quad + \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i A_i(\mathbf{k}) - A_i(\mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) (\mathbf{II}_i - \mathbf{k}) \} A_k e^{i/\hbar \cdot \mathbf{k} X_i} \\ &\quad + A_i^* e^{-i/\hbar \cdot \mathbf{k} X_i} \{ A_i(\mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i - \delta(\mathbf{x} - \mathbf{X}_i) (\mathbf{II}_i - \mathbf{k}) A_i(\mathbf{k}) \\ &\quad + A_i(\mathbf{k}) \mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) - (\mathbf{II}_i - \mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) A_i(\mathbf{k}) \}], \end{aligned} \quad (2 \cdot 23b)$$

$$\begin{aligned} \mathbf{J}^{(2)}(\mathbf{x}) &= -1/2 \cdot [[\mathbf{J}(\mathbf{x}), S], S] \\ &= -e^3/4m^3 \cdot \sum_{k < k_c} (2\pi\hbar/\omega) [- \sum_{ij} \{ \mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) A_i(\mathbf{k}) \\ &\quad - A_i(\mathbf{k}) (\mathbf{II}_i - \mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) + \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i A_i(\mathbf{k}) - A_i(\mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \\ &\quad \times (\mathbf{II}_i - \mathbf{k}) \} e^{i/\hbar \cdot \mathbf{k} (X_i - X_j)} A_j(\mathbf{k}) - \sum_{ij} e^{-i/\hbar \cdot \mathbf{k} X_i} \{ A_i(\mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i \\ &\quad - \delta(\mathbf{x} - \mathbf{X}_i) (\mathbf{II}_i - \mathbf{k}) A_i(\mathbf{k}) + A_i(\mathbf{k}) \mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) - (\mathbf{II}_i - \mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \\ &\quad \times A_i(\mathbf{k}) \} A_j(\mathbf{k}) e^{i/\hbar \cdot \mathbf{k} X_j} + \sum_i [e^{-i/\hbar \cdot \mathbf{k} X_i} \{ A_i(\mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \mathbf{II}_i - \delta(\mathbf{x} - \mathbf{X}_i) \\ &\quad \times (\mathbf{II}_i - \mathbf{k}) A_i(\mathbf{k}) + A_i(\mathbf{k}) \mathbf{II}_i \delta(\mathbf{x} - \mathbf{X}_i) - (\mathbf{II}_i - \mathbf{k}) \delta(\mathbf{x} - \mathbf{X}_i) \} \end{aligned}$$

$$\times A_i(\mathbf{k})\}, A_i(\mathbf{k})e^{i[\hat{p}\cdot\mathbf{k}X_i]}A_kA_k^*+\dots], \quad (2\cdot23c)$$

where

$$A_i(\mathbf{k}) = \frac{\epsilon_k \cdot (\mathbf{II}_i - \mathbf{k}/2)}{\hbar\omega - \mathbf{k} \cdot \mathbf{II}_i/m + k^2/2m}. \quad (2\cdot24)$$

In obtaining (2·23c), we have neglected a number of terms, which are multiplied by $A_k^*A_l$, $A_lA_l^*$ ($\mathbf{k} \neq \mathbf{l}$), A_kA_l or $A_k^*A_l^*$. These terms make no contribution to the magnetic moment as will be seen in the next section.

We have treated ω as a pure number thus far, although we see from (2·14) that ω is, in fact, an operator since it contains \mathbf{II}_i . We have ignored this fact in carrying out our canonical transformation. This approximation is justified, because the dependence of ω on \mathbf{II}_i is of the order of $\langle (\mathbf{k} \cdot \mathbf{II}_i/m\hbar\omega)^2 \rangle_{Av}$, which is small.^{2)*}

§ 3. Landau diamagnetism

We shall define Hermitian operators $\hat{\xi}_k^+$ and $\hat{\xi}_k^-$ by

$$\begin{aligned} \hat{\xi}_k^+ &= (1/2) (\hat{\xi}_k + \text{h. c.}), \\ \hat{\xi}_k^- &= (1/2i) (\hat{\xi}_k - \text{h. c.}), \end{aligned} \quad \begin{pmatrix} k < k_c \\ k_z > 0 \end{pmatrix} \quad (3\cdot1)$$

where h. c. means Hermitian conjugate. Since H commutes with $\hat{\xi}_k^\pm$ within our approximation, a set of normalized orthogonal eigenfunctions of H forms an orthonormal set of simultaneous eigenfunctions of H and $\hat{\xi}_k^{\pm(4)}$:

$$\begin{aligned} H\psi_{n\zeta'} &= E_{n\zeta'}\psi_{n\zeta'}, \\ \hat{\xi}_k^\pm \psi_{n\zeta'} &= \zeta'^\pm \psi_{n\zeta'}, \quad (\zeta' \equiv \dots, \zeta_k'^+, \dots, \zeta_k'^-, \dots) \end{aligned} \quad (3\cdot2)$$

The true eigenfunctions of our system correspond to those with $\zeta_k'^\pm = 0$:

$$\begin{aligned} H\psi_{n\zeta} &= E_{n\zeta}\psi_{n\zeta}, \\ \hat{\xi}_k^\pm \psi_{n\zeta} &= 0. \end{aligned} \quad (3\cdot3)$$

The mean current density at temperature T is given by

$$\mathbf{i}(\mathbf{x}) = \sum_n \langle \psi_{n0}, \mathbf{j}(\mathbf{x}) \exp \beta(\zeta n + \mathcal{Q} - H) \cdot \psi_{n0} \rangle, \quad (3\cdot4)$$

where

\mathcal{Q} = thermodynamic potential

$\beta = 1/kT$

ζ = chemical potential.

The mean current density (3·4) may be rewritten as

* For weak magnetic field, the value of $\langle (\mathbf{k} \cdot \mathbf{II}_i/m\hbar\omega)^2 \rangle_{Av}$ for Na is $\sim 1/16$.

$$\begin{aligned}
\mathbf{i}(\mathbf{x}) &= \sum_n \sum_{\mathbf{k}'} (\psi_{n\mathbf{k}'}, \mathbf{j}(\mathbf{x}) \exp \beta(\zeta n + \Omega - H) \cdot \prod_{\substack{k < k_0 \\ k_z > 0}} \delta(\xi_k^+) \delta(\xi_k') \psi_{n\mathbf{k}'}) \\
&= \text{Tr} [\mathbf{j}(\mathbf{x}) \exp \beta(\zeta n + \Omega - H) \cdot \prod_{\substack{k < k_0 \\ k_z < 0}} \delta(\xi_k^+) \delta(\xi_k^-)].
\end{aligned} \quad (3.5)$$

A perturbation treatment of the magnetic field leads to a linear relationship between the current density $\mathbf{i}(\mathbf{x})$ and the magnetic field:

$$i_\mu(\mathbf{x}) = \sum_\nu \int K_{\mu\nu}(\mathbf{x} - \mathbf{x}') A_\nu(\mathbf{x}') d\mathbf{x}'. \quad (3.6)$$

Written in momentum space, this reads

$$i_\mu(\mathbf{q}) = \sum_\nu K_{\mu\nu}(\mathbf{q}) A_\nu(\mathbf{q}). \quad (3.7)$$

Gauge invariance and the equation of continuity require

$$K_{\mu\nu}(\mathbf{q}) = -(\mathbf{q}_\mu \mathbf{q}_\nu - \delta_{\mu\nu} q^2) K(q^2). \quad (3.8)$$

If $K(q^2)$ is expanded as

$$K(q^2) = \alpha_0 + \alpha_1 q^2 + \dots, \quad (3.9)$$

then (3.6) can be written as

$$\mathbf{i}(\mathbf{x}) = \alpha_0 \hbar^2 \text{rot } \mathbf{H} + \dots, \quad (3.10)$$

where $\mathbf{H} = \text{rot } \mathbf{A}(\mathbf{x})$. Hence the susceptibility χ may be obtained by the relation $\chi = \alpha_0 \cdot \hbar^2/c$.

As we are concerned with the effect of the long-range part of the Coulomb interactions on the Landau diamagnetism, we shall omit the screened Coulomb interactions. Furthermore we may disregard the states in which plasma oscillations are excited, because the values of the energies of plasma quanta are for alkali metals of the order of several electron volts. Therefore our Hamiltonian may be given by

$$H = \sum_i \mathbf{p}_i^2 / 2m^* + H_{\text{e.p.}} + \sum_{k < k_0} \hbar \omega / 2 \quad (3.11)$$

$$\begin{aligned}
&\approx \sum_i \mathbf{p}_i^2 / 2m^* + (e/2m^*c) \sum_i \{ \mathbf{p}_i \cdot \mathbf{A}(\mathbf{X}_i) + \mathbf{A}(\mathbf{X}_i) \cdot \mathbf{p}_i \} \\
&+ (e^2/2m^2c) \sum_{i, k < k_0} (1/\hbar \omega) (2\pi\hbar/\omega)^{1/2} (\hbar/i) [\{ \mathbf{p}_i \cdot [\mathbf{\epsilon}_k \times \text{rot } \mathbf{A}(\mathbf{X}_i)] \\
&+ [\mathbf{\epsilon}_k \times \text{rot } \mathbf{A}(\mathbf{X}_i)] \cdot \mathbf{p}_i \} A_k e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} - A_k^* e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} \{ \mathbf{p}_i \cdot [\mathbf{\epsilon}_k \times \text{rot } \mathbf{A}(\mathbf{X}_i)] \\
&+ [\mathbf{\epsilon}_k \times \text{rot } \mathbf{A}(\mathbf{X}_i)] \cdot \mathbf{p}_i \}] + \sum_{k < k_0} \hbar \omega / 2.
\end{aligned} \quad (3.12)$$

In obtaining (3.12) we have neglected the terms which are quadratic in the magnetic field, and have approximated $1/(\hbar \omega - \mathbf{k} \cdot \mathbf{p}_i/m + k^2/2m)$ by $1/\hbar \omega$ in $H_{\text{e.p.}}$. We shall treat ω as a pure number, then we may drop the zero point energy.

Using the formalism of second quantization, we get

$$H = H_0 + H' + H_{e.p.}, \quad (3.13)$$

$$H_0 = \sum_{\nu\sigma} (\mathbf{p}^2/2m^*) b_{\nu\sigma}^* b_{\nu\sigma} \\ = \sum_{\nu\sigma} \epsilon_{\nu}(\mathbf{p}) b_{\nu\sigma}^* b_{\nu\sigma}, \quad (3.14a)$$

$$H' = (e/2m^*c) \sum_{\nu\nu'\sigma} (\mathbf{p} + \mathbf{p}') \cdot \mathbf{A}(\mathbf{p} - \mathbf{p}') b_{\nu'\sigma}^* b_{\nu\sigma}, \quad (3.14b)$$

$$H_{e.p.} = (e^2/2m^2c) \sum_{\nu\nu'\sigma < k_c} (1/\hbar\omega) (2\pi\hbar/\omega)^{1/2} [-(\mathbf{p} + \mathbf{p}' + \mathbf{k}) \cdot [\boldsymbol{\epsilon}_k \times [\mathbf{p} - \mathbf{p}' + \mathbf{k} \\ \times \mathbf{A}(\mathbf{p} - \mathbf{p}' + \mathbf{k})]] b_{\nu'\sigma}^* b_{\nu\sigma} A_k + (\mathbf{p} + \mathbf{p}' + \mathbf{k}) \cdot [\boldsymbol{\epsilon}_k \times [\mathbf{p} - \mathbf{p}' - \mathbf{k} \\ \times \mathbf{A}(\mathbf{p} - \mathbf{p}' - \mathbf{k})]] b_{\nu'\sigma}^* b_{\nu\sigma} A_k^*], \quad (3.14c)$$

where

$$[b_{\nu\sigma}, b_{\nu'\sigma'}^*]_+ = \delta_{\nu\nu'} \delta_{\sigma\sigma'}, \quad (3.15) \\ [b_{\nu\sigma}, b_{\nu'\sigma'}]_- = [b_{\nu'\sigma'}^*, b_{\nu\sigma}^*]_- = 0.$$

σ refers to the electron spin and takes on two values corresponding to the two orientations of the electron spin. We have put

$$\mathbf{A}(\mathbf{p}) = \int \mathbf{A}(\mathbf{x}) e^{i/\hbar \cdot \mathbf{p} \cdot \mathbf{x}} d\mathbf{x}, \quad (3.16)$$

so that

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{p}} \mathbf{A}(\mathbf{p}) e^{-i/\hbar \cdot \mathbf{p} \cdot \mathbf{x}}, \quad (3.17)$$

The subsidiary conditions are given by

$$\hat{\xi}_k^+ = (1/2) \sum_{\nu\sigma} g(\mathbf{k}, \mathbf{p}) (b_{\nu+k\sigma}^* b_{\nu\sigma} + b_{\nu\sigma}^* b_{\nu+k\sigma}), \\ \hat{\xi}_k^- = (1/2i) \sum_{\nu\sigma} g(\mathbf{k}, \mathbf{p}) (b_{\nu+k\sigma}^* b_{\nu\sigma} - b_{\nu\sigma}^* b_{\nu+k\sigma}), \quad (3.18)$$

where

$$g(\mathbf{k}, \mathbf{p}) = \frac{1}{1 - (1/\hbar\omega)^2 (\mathbf{k} \cdot \mathbf{p}/m + k^2/2m)} \quad (3.19)$$

Here we have neglected the dependence of $\hat{\xi}_k^\pm$ on the magnetic field. The justification for this approximation is shown in Appendix II.

We next consider the current density operator which is given by (2.22). Since the states in which plasma oscillations are excited are disregarded among the terms which appear in $\mathbf{J}^{(2)}(\mathbf{x})$ as quadratic in the collective field variables only the terms which are multiplied by A, A^* need to be retained. $\mathbf{J}^{(1)}(\mathbf{x})$ is linear in the collective field variables, therefore it contributes to the magnetic moment, when combined with $H_{e.p.}$. As we are going to calculate the mean current density to the first order in the magnetic field, we need only the zeroth order terms in the magnetic field in $\mathbf{J}^{(1)}(\mathbf{x})$. Thus we have

$$\mathbf{j}(\mathbf{x}) = \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_0^{(1)}(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x}), \quad (3.20)$$

$$\mathbf{J}_0(\mathbf{x}) = - (e/2m) \sum_{pp'\sigma} (\mathbf{p} + \mathbf{p}') e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma}, \quad (3 \cdot 21a)$$

$$\mathbf{J}_1(\mathbf{x}) = - (e^2/mc) \sum_{pp'\sigma} \mathbf{A}(\mathbf{x}) e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma}, \quad (3 \cdot 21b)$$

$$\begin{aligned} \mathbf{J}_0^{(1)}(\mathbf{x}) = & - (e^2/2m^2) \sum_{\substack{pp'\sigma \\ k < k_c}} (1/\hbar\omega) (2\pi\hbar/\omega)^{1/2} [\{\boldsymbol{\varepsilon}_k \cdot (\mathbf{p}' + \mathbf{k}/2) (\mathbf{p} + \mathbf{p}' + \mathbf{k}) \\ & - \boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) (\mathbf{p} + \mathbf{p}' - \mathbf{k})\} e^{i/\hbar \cdot (\mathbf{p}' - \mathbf{p} + \mathbf{k}) \cdot \mathbf{x}} b_{p\sigma}^* b_{p'\sigma} \mathbf{A}_k \\ & - \{\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) (\mathbf{p} + \mathbf{p}' - \mathbf{k}) - \boldsymbol{\varepsilon}_k \cdot (\mathbf{p}' - \mathbf{k}/2) (\mathbf{p} + \mathbf{p}' + \mathbf{k})\} \\ & \times e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}' - \mathbf{k}) \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma} \mathbf{A}_k^*], \end{aligned} \quad (3 \cdot 21c)$$

$$\mathbf{J}_0^{(2)}(\mathbf{x}) = - (e/2m) \sum_{pp'\sigma} (\mathbf{p} + \mathbf{p}') [\alpha - \gamma (\mathbf{p}' - \mathbf{p})^2] e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma}, \quad (3 \cdot 21d)$$

$$\begin{aligned} \mathbf{J}_1^{(2)}(\mathbf{x}) = & - (e^2/mc) \sum_{pp'\sigma} \mathbf{A}(\mathbf{x}) [\alpha - \gamma (\mathbf{p}' - \mathbf{p})^2] e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma}, \\ & + \gamma (e^2/mc) \sum_{pp'\sigma} [(\hbar/i) \nabla (\mathbf{A}(\mathbf{x}) \cdot (\mathbf{p} - \mathbf{p}')) + (\hbar/i)^2 \text{grad div } \mathbf{A}(\mathbf{x})] \\ & \times e^{i/\hbar \cdot (\mathbf{p} - \mathbf{p}') \cdot \mathbf{x}} b_{p'\sigma}^* b_{p\sigma}, \end{aligned} \quad (3 \cdot 21e)$$

where

$$\alpha = (e^2/3m^2) \sum_{k < k_c} (2\pi\hbar/\omega) (1/\hbar\omega)^2 k^2, \quad (3 \cdot 22)$$

$$\gamma = (e^2/6m^2) \sum_{k < k_c} (2\pi\hbar/\omega) (1/\hbar\omega)^2. \quad (3 \cdot 23)$$

In obtaining (3·21c), (3·21d) and (3·21e), we have approximated $1/(\hbar\omega - \mathbf{k} \cdot \mathbf{H}_i/m + k^2/2m)$ by $1/\hbar\omega$ and we have replaced $A_k A_k^*$ by I because of $\langle A_k A_k^* \rangle_{\text{vac}} = \langle I + A_k^* A_k \rangle_{\text{vac}} = I$. The terms which depend on two electron coordinates in $\mathbf{J}^{(2)}(\mathbf{x})$ cancel with each other to the first order in the magnetic field.

If we label the eigenstates of H_0 by ν, ν', ν'', \dots and call the corresponding eigenvalues of H_0 $E_\nu, E_{\nu'}, E_{\nu''}, \dots$ and if we apply the perturbation method⁽³⁾, we get the following expression for the mean current density to the first order in the magnetic field:

$$\begin{aligned} \mathbf{i}(\mathbf{x}) = & \sum_{\substack{\nu\nu' \\ k < k_c \\ k_z > 0}} \langle \prod_{k < k_c} \delta(\hat{\varepsilon}_k^+) \delta(\hat{\varepsilon}_k^-) \rangle_{\nu\nu'} \exp \beta(\zeta n + \Omega_0 - E_\nu) \\ & \times [\langle \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x}) \rangle_{\nu'\nu} + \sum_{\nu''} \frac{1 - e^{\beta(E_\nu - E_{\nu''})}}{E_\nu - E_{\nu''}} \\ & \times \{ \langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle_{\nu'\nu''} \langle H' - \Omega' \rangle_{\nu''\nu} + \langle \mathbf{J}_0^{(1)}(\mathbf{x}) \rangle_{\nu'\nu''} \langle H_{\text{e.p.}} - \Omega_{\text{e.p.}} \rangle_{\nu''\nu} \}], \end{aligned} \quad (3 \cdot 24)$$

where Ω_0 is the thermodynamic potential of the unperturbed system, and Ω' and $\Omega_{\text{e.p.}}$ are the corrections to the thermodynamic potential due to H' and $H_{\text{e.p.}}$ in first order respectively.

Ω_0 , Ω' and $\Omega_{\text{e.p.}}$ are determined by

$$1 = \text{Tr} [\exp (\zeta n + \Omega - H) \cdot \prod_{\substack{k < k_c \\ k_z > 0}} \delta(\hat{\varepsilon}_k^+) \delta(\hat{\varepsilon}_k^-)]$$

$$\begin{aligned}
&= \sum_{\nu\nu'} \langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu'} \exp(\zeta n + Q_0 - E_\nu) \cdot [\delta_{\nu\nu'} \\
&\quad + \frac{1 - e^{i(E_\nu - E_{\nu'})}}{E_{\nu'} - E_\nu} \langle Q' + Q_{e.p.} - H' - H_{e.p.} \rangle_{\nu'\nu} + \dots]. \quad (3 \cdot 25)
\end{aligned}$$

We get

$$e^{-\beta Q_0} = \sum_\nu \langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu} e^{\beta(\zeta n - E_\nu)}, \quad (3 \cdot 26)$$

$$Q' = \beta^{-1} e^{\beta Q_0} \sum_{\nu\nu'} \langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu'} e^{\beta(\zeta n - E_\nu)} \frac{1 - e^{i(E_\nu - E_{\nu'})}}{E_{\nu'} - E_\nu} \langle H' \rangle_{\nu\nu'}, \quad (3 \cdot 27)$$

$$Q_{e.p.} = \beta^{-1} e^{\beta Q_0} \sum_{\nu\nu'} \langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu'} e^{\beta(\zeta n - E_\nu)} \frac{1 - e^{i(E_\nu - E_{\nu'})}}{E_{\nu'} - E_\nu} \langle H_{e.p.} \rangle_{\nu'\nu}. \quad (3 \cdot 28)$$

We can write

$$\delta(\xi_k^\pm) = (1/2\pi) \int_{-\infty}^{\infty} d\lambda_k^\pm \exp(i\lambda_k^\pm \xi_k^\pm). \quad (3 \cdot 29)$$

Therefore

$$\begin{aligned}
&\langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu'} \\
&= (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \langle \exp i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-) \rangle_{\nu\nu'}, \\
&= (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} \langle \{i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-)\}^n \rangle_{\nu\nu'} / n!. \quad (3 \cdot 30)
\end{aligned}$$

If $\nu = \nu'$,*

$$\begin{aligned}
&\langle H \delta(\xi_k^+) \delta(\xi_k^-) \rangle_{\nu\nu} \\
&= (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} \langle \{i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-)\}^n \rangle_{\nu\nu} / n! \\
&\approx (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} [1/2 \cdot \langle \{i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-)\}^2 \rangle_{\nu\nu}]^n / n! \\
&= (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} [-1/8 \cdot \sum_{\substack{pk < k_0 \\ k_g > 0}} [g^2(k, p) \{n_{p\sigma}(1 - n_{p+k\sigma}) \\
&\quad (1 - n_{p+k\sigma})\}^2 - O(1/n)]. \quad \text{In general, the leading terms in } 2n\text{-th order terms in } \lambda_k^\pm \text{ correspond to the processes}
\end{aligned}$$

* The process shown in Fig. 1, which describes a two-step process leading back to the initial state: $p \rightarrow p+k \rightarrow p$, contribute to the terms which are quadratic in λ_k^\pm . The contributions to the fourth order terms in λ_k^\pm come from the processes shown in Fig. 2. Fig. 2a describes a two-electron process in which each electron suffers scatterings like those shown in Fig. 1. Fig. 2b corresponds to four-step processes of one electron: $p \rightarrow p+k \rightarrow p \rightarrow p+k' \rightarrow p$, etc. The ratio of the magnitude of the contributions from the processes shown in Fig. 2b and Fig. 2a is of the order of $\sum_{\sigma, p, k, k' < k_0} g^2(k, p) g^2(k', p) n_{p\sigma} (1 - n_{p+k\sigma}) (1 - n_{p+k'\sigma}) / [\sum_{\sigma, p, k < k_0} g^2(k, p) n_{p\sigma} (1 - n_{p+k\sigma})]^2 \sim O(1/n)$. In general, the leading terms in $2n$ -th order terms in λ_k^\pm correspond to the processes in which each of n electrons suffers scatterings like those shown in Fig. 1. Terms of odd powers in λ_k^\pm do not contribute, because they are odd functions and so the integrals vanish.

$$+n_{p+k\sigma}(1-n_{p\sigma})\}(\lambda_k^{2+}+\lambda_k^{2-})^n/n!, \quad (3.31)$$

where we have neglected terms of the order of $1/n$ as compared with the terms of the order of 1.

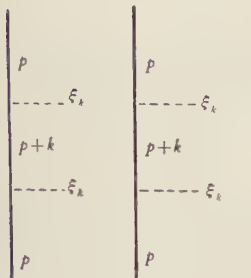


Fig. 1

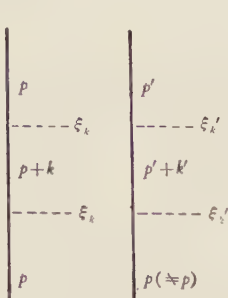


Fig. 2a

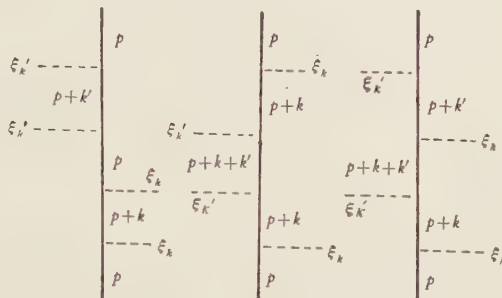


Fig. 2b

Substituting (3.31) into (3.26), we obtain

$$\begin{aligned} e^{-\beta\Omega_0} &\approx (1/2\pi)^{n'} \int \prod_{\substack{n < k_\sigma \\ k_z > 0}} d\lambda_k^+ d\lambda_k^- \exp \left[-1/4 \cdot \sum_{\substack{qk < k_\sigma \\ k_z > 0}} g^2(\mathbf{k}, \mathbf{q}) \{f_q(1-f_{q+k}) \right. \\ &\quad \left. + f_{q+k}(1-f_q)\} (\lambda_k^{2+} + \lambda_k^{2-}) \right] \cdot \prod_p [1 + e^{-\beta(\varepsilon(p) - \zeta)}]^2 \\ &= (1/2\pi)^{n'} \prod_{\substack{k < k_\sigma \\ k_z > 0}} \frac{4\pi}{\sum_q g^2(\mathbf{k}, \mathbf{q}) \{f_q(1-f_{q+k}) + f_{q+k}(1-f_q)\}} \\ &\quad \times \prod_p [1 + e^{-\beta(\varepsilon(p) - \zeta)}]^2, \end{aligned} \quad (3.32)$$

where $f_p = 1/\{1 + \exp \beta(\varepsilon(p) - \zeta)\}$ is the Fermi function.

In (3.27) the part for which $\nu = \nu'$ is proportional to $\int \mathbf{A}(\mathbf{x}) d\mathbf{x}$, which can be made to vanish by a gauge transformation. Accordingly we find

$$\begin{aligned} \Omega' &\approx \beta^{-1} e^{\beta\Omega_0} \sum_{\nu \neq \nu'} (1/2\pi)^{n'} \prod_{\substack{k < k_\sigma \\ k_z > 0}} d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} \left[1/2 \cdot \sum_p \left\langle \left\{ i \sum_{\substack{k < k_\sigma \\ k_z > 0}} (\lambda_k^+ \tilde{\xi}_k^+ + \lambda_k^- \tilde{\xi}_k^-) \right\}^2 \right\rangle_{\nu\nu'} \right]^n / n! \\ &\quad \times \left[\left\langle i \sum_{\substack{k < k_\sigma \\ k_z > 0}} (\lambda_k^+ \tilde{\xi}_k^+ + \lambda_k^- \tilde{\xi}_k^-) \right\rangle_{\nu\nu'} + 1/2 \left\langle \left\{ i \sum_{\substack{k < k_\sigma \\ k_z > 0}} (\lambda_k^+ \tilde{\xi}_k^+ + \lambda_k^- \tilde{\xi}_k^-) \right\}^2 \right\rangle_{\nu\nu'} \right] \\ &\quad \times e^{\beta(\zeta n - H_{\nu'})} \frac{1 - e^{\beta(E_{\nu'} - E_\nu)}}{E_{\nu'} - E_\nu} \langle H' \rangle_{\nu'\nu}, \end{aligned} \quad (3.33)$$

$\langle H' \rangle_{\nu'\nu}$ corresponds to an inelastic scattering of one electron ($\mathbf{p} \rightarrow \mathbf{p} + \mathbf{k}$). This momentum change must be compensated by ξ_k^\pm . Then the integral is seen to be of the form

$$\int \prod_{\substack{k < k_\sigma \\ k_z > 0}} d\lambda_k^+ d\lambda_k^- \exp \left[-1/4 \cdot \sum_{\substack{qk < k_\sigma \\ k_z > 0}} g^2(\mathbf{k}, \mathbf{q}) \{f_q(1-f_{q+k}) + f_{q+k}(1-f_q)\} (\lambda_k^{2+} + \lambda_k^{2-}) \right]$$

$$\begin{aligned} & \times [A(\mathbf{k}, \mathbf{p}) \lambda_k^+ + B(\mathbf{k}, \mathbf{p}) \lambda_k^- + C(\mathbf{k}, \mathbf{k}', \mathbf{p}) (\lambda_{k-k'}^+ \lambda_{k'}^+ - \lambda_{k-k'}^- \lambda_{k'}^- \\ & + i \lambda_{k-k'}^+ \lambda_{k'}^- + i \lambda_{k-k'}^- \lambda_{k'}^+)] = 0, \end{aligned} \quad (3 \cdot 34)$$

because

$$\begin{aligned} & \int_{-\infty}^{\infty} d\lambda e^{-\alpha^2 \lambda^2} \lambda = 0, \\ & \iint_{-\infty}^{\infty} d\lambda^+ d\lambda^- e^{-\alpha^2 (\lambda^{2+} + \lambda^{2-})} (\lambda^{2+} - \lambda^{2-}) = 0. \end{aligned} \quad (3 \cdot 35)$$

Therefore

$$\Omega' \approx 0. \quad (3 \cdot 36)$$

$H_{e.p.}$ has no diagonal matrix elements and $\hat{\varepsilon}_k^\pm$ do not depend on the collective field variables, hence $\langle H \partial(\hat{\varepsilon}_k^\pm) \partial(\hat{\varepsilon}_k^\pm) \rangle_{\nu\nu'} \langle H_{e.p.} \rangle_{\nu\nu'} = 0$. Therefore

$$\Omega_{e.p.} = 0. \quad (3 \cdot 37)$$

We now proceed to calculate the mean current density. We shall split up the mean current density (3.24) into two parts: the $\mathbf{i}_1(\mathbf{x})$ for which $\nu = \nu'$ and the remaining part $\mathbf{i}_2(\mathbf{x})$ for which $\nu \neq \nu'$.

(a) $\mathbf{i}_1(\mathbf{x})$

$$\begin{aligned} \mathbf{i}_1(\mathbf{x}) & \approx \sum_{\nu} (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} [1/2 \cdot \langle \{i\} \sum (\lambda_k^+ \hat{\varepsilon}_k^- + \lambda_k^- \hat{\varepsilon}_k^+) \rangle_{\nu\nu}^2]^{n/n!} \\ & \times e^{\beta(\sum n + \Omega_0 - E_{\nu})} [\langle \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x}) \rangle_{\nu\nu} + \sum_{\nu'} \frac{1 - e^{\beta(E_{\nu} - E_{\nu'})}}{E_{\nu} - E_{\nu'}} \\ & \times \{ \langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle_{\nu\nu'} \langle H' \rangle_{\nu'\nu} + \langle \mathbf{J}_0^{(1)} \rangle_{\nu\nu'} \langle H_{e.p.} \rangle_{\nu'\nu} \}] \\ & = \sum_{\dots n_{p\sigma} \dots} (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} [-1/8 \cdot \sum_{\substack{r\sigma k < k_0 \\ k_z > 0}} g^2(\mathbf{k}, \mathbf{r}) \{n_{r\sigma}(1 - n_{r+k\sigma}) \\ & + n_{r+k\sigma}(1 - n_{r\sigma})\} (\lambda_k^{2+} + \lambda_k^{2-})]^{n/n!} \cdot \exp \beta [\zeta \sum_{r\sigma} n_{r\sigma} + \Omega_0 - \sum_{r\sigma} \varepsilon(\mathbf{r}) n_{r\sigma}] \\ & \times [-(e^2/mc)(1 + \alpha) \mathbf{A}(\mathbf{x}) \sum_{p\sigma} n_{p\sigma} \\ & + \gamma(e^2/mc)(\hbar/i)^2 \text{grad div } \mathbf{A}(\mathbf{x}) \sum_{p\sigma} n_{p\sigma} \\ & - (e^2/4mm^*c) \sum_{pq\sigma} (1 + \alpha - \gamma q^2) (2\mathbf{p} - \mathbf{q}) e^{-i|\vec{r}_0 \cdot \mathbf{q}|x} \\ & \times \mathbf{A}(\mathbf{q}) \cdot (2\mathbf{p} - \mathbf{q}) \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} - \mathbf{q}) \}}{\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} - \mathbf{q})} - n_{p\sigma}(1 - n_{p-q\sigma}) \\ & - \sum_{\substack{p\sigma \\ k < k_0}} (e^2/2m^*c) (1/\hbar\omega)^2 (2\pi\hbar/\omega) [2\mathbf{k} \cdot \varepsilon_k \cdot (\mathbf{p} - \mathbf{k}/2) \{(\mathbf{p} \cdot \mathbf{q})(\varepsilon_k \cdot \mathbf{A}(\mathbf{q})) \} \end{aligned}$$

$$\begin{aligned}
& -(\mathbf{p} \cdot \mathbf{A}(\mathbf{q}))(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})\} + k\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) \{(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})(\mathbf{q} \cdot \mathbf{A}(\mathbf{q})) - q^2(\boldsymbol{\varepsilon}_k \cdot \mathbf{A}(\mathbf{q}))\} \\
& + \{q\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) - 2\mathbf{p}(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})\} \{(\mathbf{p} \cdot \mathbf{q})(\boldsymbol{\varepsilon}_k \cdot \mathbf{A}(\mathbf{q})) - (\mathbf{p} \cdot \mathbf{A}(\mathbf{q}))(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})\} \\
& \times e^{-i/\hbar \cdot \mathbf{q} \cdot \boldsymbol{\alpha}} \frac{1 - \exp \beta \{\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} - \mathbf{k} - \mathbf{q}) - \hbar\omega\}}{\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} - \mathbf{k} - \mathbf{q}) - \hbar\omega} n_{p\sigma}(1 - n_{p-k-q\sigma})].
\end{aligned} \quad (3.38)$$

Carrying out the summations over $n_{p\sigma}$, using (3.32) and transforming to the momentum space, we get for the kernel, which is defined by (3.7),

$$K_{\mu\nu}(\mathbf{q}) = K_{\mu\nu}'(\mathbf{q}) + K_{\mu\nu}''(\mathbf{q}), \quad (3.39)$$

where $K_{\mu\nu}'(\mathbf{q})$ is the contribution to the kernel from $\mathbf{J}(\mathbf{x})$ and $\mathbf{J}^{(2)}(\mathbf{x})$, and $K_{\mu\nu}''(\mathbf{q})$ is that from $\mathbf{J}^{(1)}(\mathbf{x})$. They are given by

$$\begin{aligned}
K_{\mu\nu}'(\mathbf{q}) = & (2e^2/mc)\gamma \sum_p \{q_\mu q_\nu f_p - (\beta q^2/m^*)p_\mu p_\nu f_p(1-f_p)\} \\
& - (2e^2/mmc^*) (1+\alpha) \sum_p [(\beta^2/4m^*) \{q^2 p_\mu p_\nu \\
& + \sum_\lambda (q_\mu q_\lambda p_\nu p_\lambda + q_\nu q_\lambda p_\mu p_\lambda)\} \{f_p(1-f_p) - 2f_p^2(1-f_p)\} \\
& + (\beta^3/6m^{2*}) \sum_{\lambda\rho} q_\lambda q_\rho p_\mu p_\nu p_\lambda p_\rho \{6f_p^2(1-f_p)^2 - f_p(1-f_p)\} \\
& - (q_\mu q_\nu/4)\beta f_p(1-f_p)] + O(q^4),
\end{aligned} \quad (3.40)$$

$$\begin{aligned}
K_{\mu\nu}''(\mathbf{q}) = & - (e^4/m^4c) \sum_{p,k < k_c} (1/\hbar\omega)^2 (2\pi\hbar/\omega) [k_\mu \boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) \\
& \times \{(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})q_\nu - q^2 k_\nu/k\} (1 - e^{\beta\eta})/\eta \cdot f_p(1-f_{p-k}) \\
& + \{q_\mu \boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) - 2p_\mu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})\} \{(\mathbf{p} \cdot \mathbf{q})k_\nu/k - (\boldsymbol{\varepsilon}_k \cdot \mathbf{q})p_\nu\} (1 - e^{\beta\eta})/\eta \\
& \times f_p(1-f_{p-k}) + 2k_\mu \boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2) \{(\mathbf{p} \cdot \mathbf{q})k_\nu/k - (\boldsymbol{\varepsilon}_k \cdot \mathbf{q})p_\nu\} \\
& \times \{-\mathbf{q} \cdot (\mathbf{p} - \mathbf{k})/m \cdot (\beta\eta e^{\beta\eta} - e^{\beta\eta} + 1)/\eta^2 \cdot f_p(1-f_{p-k}) \\
& + \mathbf{q} \cdot (\mathbf{p} - \mathbf{k})/m \cdot (1 - e^{\beta\eta})/\eta \cdot \beta f_p f_{p-k}(1-f_{p-k})] + O(q^4),
\end{aligned} \quad (3.41)$$

where

$$\eta \equiv -\hbar\omega + \mathbf{k} \cdot \mathbf{p}/m - k^2/2m.$$

Performing the summations, we get the following gauge invariant expressions for the kernels (see Appendix I)

$$K_{\mu\nu}'(\mathbf{q}) = - (q_\mu q_\nu - \delta_{\mu\nu} q^2) [- (2\pi e^2/3mc) (2m^* \zeta)^{1/2} (1 + \alpha + 8m^* \zeta \gamma)/\hbar^3] + O(q^4), \quad (3.42)$$

$$K_{\mu\nu}''(\mathbf{q}) = - (q_\mu q_\nu - \delta_{\mu\nu} q^2) (2\pi e^2/3mc) (2m^* \zeta)^{1/2} \varepsilon/\hbar^3 + O(q^4), \quad (3.43)$$

where

$$-\varepsilon = (45/4) (\zeta/\hbar\omega) \alpha [-16/45 + 2\kappa/27 - \kappa^3/360$$

$$+ (\zeta/\hbar\omega) (1/\kappa^2) \{-2/9 + 5\kappa/12 - 38\kappa^2/225 + 2\kappa^4/9 - 7\kappa^5/36 - 2\kappa^6/27 + \kappa^7/45\}], \quad (3.44)$$

and $\kappa = k_c/p_0$, p_0 being the Fermi momentum. In (3.43) and (3.44) we have neglected terms which vanish as $T \rightarrow 0$.

The susceptibility is given by

$$\begin{aligned} \chi &= - (4\pi\mu^2/3\hbar^3) (3\pi^2 n) (1 + \alpha + 8m^*\zeta\gamma - \varepsilon) \\ &= \chi_0 + \chi_0 (\alpha + 8m^*\zeta\gamma - \varepsilon) \\ &= \chi_0 + J\chi_0, \end{aligned} \quad (3.45)$$

where $\mu = e\hbar/2mc$ is the Bohr magneton and χ_0 is the diamagnetic susceptibility for the perfect electron gas.

From (3.22) and (3.23) we get

$$\alpha \approx (n'/n) (k_c^2/10m\hbar\omega), \quad (3.46)$$

$$\gamma \approx (n'/n) (1/12m\hbar\omega). \quad (3.47)$$

According to the estimation by Bohm and Pines, k_c for Na is $\sim 0.68 p_0$, so that $n' \sim 0.16 n$ and $k_c^2/2m \sim 0.46 \zeta$. Inserting these values and $\zeta/\hbar\omega \sim 1/1.56$, we find that the correction to the susceptibility for Na amounts to about 6 percent:

$$\begin{aligned} \chi_0 (\alpha + 8m^*\zeta\gamma) &\sim 0.077 \chi_0 \quad (\text{contribution from } \mathbf{J}^{(2)}(\mathbf{x})), \\ -\chi_0 \varepsilon &\sim -0.021 \chi_0 \quad (\text{contribution from } \mathbf{J}^{(1)}(\mathbf{x})), \\ J\chi_0/\chi_0 &\sim 0.056. \end{aligned}$$

In carrying out the summations over $n_{p\sigma}$ in (3.38), we have neglected the cases where \mathbf{r}_σ or $\mathbf{r} + \mathbf{k}_\sigma$ in $\langle H\delta(\xi_k^\pm)\delta(\xi_{-}^\pm) \rangle_{vv}$ coincides with \mathbf{p}_0 , $\mathbf{p} - \mathbf{q}_\sigma$ or $\mathbf{p} - \mathbf{k} - \mathbf{q}_\sigma$. These cases represent the effect of the subsidiary conditions. However this effect is very small. It can be shown that the contribution to the susceptibility is of the order of $J\chi_0/n$.

(b)

$$\begin{aligned} i_2(\mathbf{x}) &\approx \sum_{\nu, \nu'} \exp i\beta(\zeta n + \mathcal{Q}_0 - E_\nu) \cdot (1/2\pi)^{n'} \int H d\lambda_k + d\bar{\lambda}_k \sum_{n=0}^{\infty} [1/2 \cdot \sum_{\sigma} \\ &\quad \times \langle \{i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-)\}^2 \rangle_{\sigma\sigma}]^n / n! \cdot [\langle i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-) \rangle_{\nu\nu'} \\ &\quad + 1/2 \langle \{i \sum (\lambda_k^+ \xi_k^+ + \lambda_k^- \xi_k^-)\}^2 \rangle_{\nu\nu'}] [\langle \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x}) \rangle_{\nu'\nu} \\ &\quad + \sum_{\nu''} \frac{1 - e^{\beta(E_\nu - E_{\nu''})}}{E_\nu - E_{\nu''}} \{ \langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle_{\nu'\nu''} \langle H' \rangle_{\nu''\nu} \\ &\quad + \langle \mathbf{J}_0^{(1)}(\mathbf{x}) \rangle_{\nu'\nu''} \langle H_{e,p} \rangle_{\nu''\nu} \}]. \end{aligned} \quad (3.48)$$

$\langle \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x}) \rangle_{\nu'\nu}$ in (3.48) corresponds to an inelastic scattering of one electron. The momentum defect arising in $\mathbf{J}_1(\mathbf{x}) + \mathbf{J}_1^{(2)}(\mathbf{x})$ must be compensated by the scattering by ξ_k^\pm . The same argument as in the evaluation of \mathcal{Q} leads to that the contribution from

this one electron process to the mean current density vanishes. Since ξ_k^\pm do not depend on the collective field variables, $\mathbf{J}_0^{(1)}(\mathbf{x})$ makes no contribution. Hence the sole contribution comes from the terms which contain $\langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle \langle H' \rangle$. As the momentum change of the electron by ξ_k^\pm is $\pm \mathbf{k}(\mathbf{k} \cdot \mathbf{k}_c)$, the dependence of the mean current density on the vector potential of the magnetic field is of the form

$$\mathbf{i}_2(\mathbf{x}) = \sum_{\mathbf{k} < \mathbf{k}_c} [\dots] e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}} \mathbf{A}(\mathbf{k}), \quad (3.49)$$

i. e., there are no Fourier components with $k > k_c$. Therefore this current density has nothing to do with the Landau diamagnetism. Thus we find that our final result is given by (3.45). (Actually $\mathbf{i}_2(\mathbf{x})$ vanishes. See Appendix III).

§ 4. Discussion

We have developed the collective description of the behavior of electrons in the magnetic field and have investigated the effect of the long-range part of the Coulomb interactions on the Landau diamagnetism. It has been shown that the long-range part of the Coulomb interactions slightly increases the magnitude of the diamagnetic susceptibility. Our model may be applied to alkali metals because the effect of the periodicity of the crystal lattice may be taken into account by the replacement of the electron mass m by $m_{\text{eff}}^{(b)}$. Our result is qualitatively in agreement with that obtained by Pines.* Pines⁽⁶⁾ has calculated the energy of one electron in the absence of the magnetic field and inserted it into the formula for the diamagnetic susceptibility. He has estimated the increase in the magnitude of the diamagnetic susceptibility for alkali metal to be about 10 percent.

The experimental value of the diamagnetic susceptibility for Na seems to be considerably smaller than the theoretical one, therefore other effects would have to be considered. The screened Coulomb interactions, of which effect we have not considered in the present paper, would not have a much effect on the diamagnetism. However this effect remains to be examined.

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Appendix I

$K'_{\mu\nu}(q)$ is given by

$$\begin{aligned} K'_{\mu\nu}(q) = & -(2e^2/mc)(1+\alpha)\delta_{\mu\nu}\sum_p f_p + (2e^2/mc)\gamma q_\mu q_\nu \sum_p f_p \\ & - (2e^2/mm^*c)\sum_p (1+\alpha-\gamma q^2)(p_\mu - q_\mu/2)(p_\nu - q_\nu/2) \\ & \times \frac{1 - \exp\{\beta\{\varepsilon(p) - \varepsilon(p-q)\}\}}{\varepsilon(p) - \varepsilon(p-q)} f_p(1-f_{p-q}). \end{aligned} \quad (A1)$$

* While preparing our manuscript, we were informed of Prof. Pines recent work through his preprint sent to Mr. H. Watanabe. We thank Mr. Watanabe for his kindness.

Expanding (A1) in powers of q , we get

$$\begin{aligned}
 K'_{\mu\nu}(\mathbf{q}) = & -(2e^2/mm^*c)(1+\alpha)\sum_p[(\beta^2/4m^*)\{q^2 p_\mu p_\nu + \sum_\lambda (q_\mu q_\lambda p_\nu p_\lambda \\
 & + q_\nu q_\lambda p_\mu p_\lambda)\}\{f_p(1-f_p) - 2f_p^2(1-f_p)\} \\
 & + (\beta^3/6m^{2*})\sum_{\lambda\rho} q_\lambda q_\rho p_\mu p_\nu p_\lambda p_\rho \{6f_p^2(1-f_p)^2 - f_p(1-f_p)\} \\
 & - (q_\mu q_\nu/4)\beta f_p(1-f_p)] + (2e^2/mc)\gamma\sum_p [q_\mu q_\nu f_p \\
 & - (\beta q^2/m^*)p_\mu p_\nu f_p(1-f_p)] + O(q^4). \quad (\text{A2})
 \end{aligned}$$

Replacing the summations over p by the integration, we obtain

$$\begin{aligned}
 \sum_p f_p &= (8\pi/3h^3)(2m^{3*})^{1/2}[\zeta^{1/2} + (\pi^2/8\beta^2)\zeta^{-1/2} + \dots], \\
 \sum_p f_p(1-f_p) &= (4\pi/h^3)(2m^{3*})^{1/2}\beta^{-1}[\zeta^{1/2} - (\pi^2/24\beta^2)\zeta^{-3/2} + \dots], \\
 \sum_p p_\mu p_\nu f_p(1-f_p) &= \delta_{\mu\nu}(8\pi/3h^3)m^*(2m^{3*})^{1/2}\beta^{-1}[\zeta^{3/2} + (\pi^2/8\beta^2)\zeta^{-1/2} + \dots], \\
 \sum_p p_\mu p_\nu f_p^2(1-f_p) &= \delta_{\mu\nu}(8\pi/3h^3)m^*(2m^{3*})^{1/2}\beta^{-1}[\zeta^{5/2}/2 - (3/4\beta)\zeta^{1/2} \\
 & + (\pi^2/16\beta^2)\zeta^{-1/2} + \dots], \\
 \sum_p p_\mu p_\nu p_\lambda p_\rho f_p(1-f_p) &= (\delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda})(16\pi/15h^3) \\
 & \times m^{2*}(2m^{3*})^{1/2}\beta^{-1}[\zeta^{5/2} + (5\pi^2/8\beta^2)\zeta^{1/2} + \dots], \\
 \sum_p p_\mu p_\nu p_\lambda p_\rho f_p^2(1-f_p)^2 &= (\delta_{\mu\nu}\delta_{\lambda\rho} + \delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda})(16\pi/15h^3) \\
 & \times m^{2*}(2m^{3*})^{1/2}\beta^{-1}[\zeta^{5/2}/6 + (5/8\beta^2)(\pi^2/6-1)\zeta^{1/2} + \dots]. \quad (\text{A3})
 \end{aligned}$$

Inserting (A3) into (A2), we get

$$\begin{aligned}
 K'_{\mu\nu}(\mathbf{q}) = & -(q_\mu q_\nu - \delta_{\mu\nu} q^2)[-(2\pi e^2/3mc)(2m^*\zeta)^{1/2}(1+\alpha+8m^*\zeta\gamma)/h^3] \\
 & + O(q^4). \quad (\text{A4})
 \end{aligned}$$

$K''_{\mu\nu}(\mathbf{q})$ is given by

$$\begin{aligned}
 K''_{\mu\nu}(\mathbf{q}) = & -(e^4/m^4c)\sum_{p, k < k_c} (1/\hbar\omega)^2 (2\pi\hbar/\omega)[2k_\mu(\mathbf{\epsilon}_k \cdot (\mathbf{p}-\mathbf{k}/2)) \\
 & \times \{(\mathbf{p} \cdot \mathbf{q})k_\nu/k - (\mathbf{\epsilon}_k \cdot \mathbf{q})p_\nu\} + k_\mu(\mathbf{\epsilon}_k \cdot (\mathbf{p}-\mathbf{k}/2))\{(\mathbf{\epsilon}_k \cdot \mathbf{q})q_\nu \\
 & - q^2 k_\nu/k\} + \{q_\mu(\mathbf{\epsilon}_k \cdot (\mathbf{p}-\mathbf{k}/2)) - 2p_\mu(\mathbf{\epsilon}_k \cdot \mathbf{q})\}\{(\mathbf{p} \cdot \mathbf{q})k_\nu/k \\
 & - (\mathbf{\epsilon}_k \cdot \mathbf{q})p_\nu\}] \frac{1 - \exp\{\beta(\eta + \rho)\}}{\eta + \rho} f_p(1-f_{p-k-q}), \quad (\text{A5})
 \end{aligned}$$

where

$$\eta \equiv -\hbar\omega + \mathbf{k} \cdot \mathbf{p}/m - k^2/2m,$$

$$\rho \equiv \mathbf{q} \cdot (\mathbf{p}-\mathbf{k})/m - q^2/2m.$$

Expanding $\{1 - e^{\beta(\eta + \rho)}\} / (\eta + \rho)$ in powers of ρ , we get

$$\frac{1 - e^{\beta(\eta + \rho)}}{\eta + \rho} = \frac{1 - e^{\beta\eta}}{\eta} - \frac{\rho}{\eta^2} (\beta\eta e^{\beta\eta} - e^{\beta\eta} + 1) + O(\rho^2). \quad (\text{A6})$$

Also we find

$$f_p(1 - f_{p-k-q}) = f_p(1 - f_{p-k}) + \mathbf{q} \cdot (\mathbf{p} - \mathbf{k}) / m \cdot \beta f_p f_{p-k}(1 - f_{p-k}) + O(q^2). \quad (\text{A7})$$

Inserting (A6) and (A7) into (A5), we obtain (3.41), which yields in the limit of $T=0$

$$\begin{aligned} K''_{\mu\nu}(\mathbf{q}) = & (e^4/m^4c) \sum_{p, k < k_c} (1/\hbar\omega)^3 (2\pi\hbar/\omega) [k_\mu(\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2)) \{q_\nu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q}) \\ & - q^2 k_\nu/k\} [f_p(1 - f_{p-k})]_{T=0} + \{q_\mu(\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2)) - 2p_\mu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q})\} \\ & \times \{(\mathbf{p} \cdot \mathbf{q})k_\nu/k - (\boldsymbol{\varepsilon}_k \cdot \mathbf{q})p_\nu\} [f_p(1 - f_{p-k})]_{T=0} \\ & + 2k_\mu(\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2)) \{(\mathbf{p} \cdot \mathbf{q})k_\nu/k - (\boldsymbol{\varepsilon}_k \cdot \mathbf{q})p_\nu\} \{ \mathbf{q} \cdot (\mathbf{p} - \mathbf{k}) / m\hbar\omega \\ & \times [f_p(1 - f_{p-k})]_{T=0} + \mathbf{q} \cdot (\mathbf{p} - \mathbf{k}) / m \cdot [\beta f_p f_{p-k}(1 - f_{p-k})]_{T=0} \} \\ & + O(q^4), \end{aligned} \quad (\text{A8})$$

where we have put $\gamma_i \approx -\hbar\omega$. The integrations over \mathbf{p} and \mathbf{k} are elementary, but tedious. For example the first term becomes

$$\begin{aligned} & \sum_{p, k < k_c} k_\mu(\boldsymbol{\varepsilon}_k \cdot (\mathbf{p} - \mathbf{k}/2)) \{q_\nu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q}) - q^2 k_\nu/k\} [f_p(1 - f_{p-k})]_{T=0} \\ & = (1/\hbar^6) \int d\mathbf{k} k_\mu \{q_\nu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q}) - q^2 k_\nu/k\} \int_{p < p_0, |\mathbf{p} - \mathbf{k}| > p_0} p^2 dp \int (p \cos \theta - k/2) \sin \theta d\theta d\varphi \\ & = (2\pi/\hbar^6) \int_{k < k_c} d\mathbf{k} k_\mu \{q_\nu(\boldsymbol{\varepsilon}_k \cdot \mathbf{q}) - q^2 k_\nu/k\} / k \int_{p_0 - k}^{p_0} p dp \int_{p_0}^{p+k} \{(p^2 + k^2 - s^2)/2k - k/2\} s ds \\ & = -(8\pi^2/45\hbar^6) p_0^3 k_c^5 (q_\mu q_\nu - \delta_{\mu\nu} q^2), \end{aligned}$$

where $s^2 = p^2 + k^2 - 2pk \cos \theta$. Other terms may be evaluated similarly. The result is given by (3.43).

The mean total number of electrons may be given by

$$\bar{n} = \frac{\text{Tr}[n \exp \beta(\zeta n - H) \cdot H \partial(\xi_k^+) \partial(\xi_k^-)]}{\text{Tr}[\exp \beta(\zeta n - H) \cdot H \partial(\xi_k^+) \partial(\xi_k^-)]}, \quad (\text{A9})$$

which, with the help of (3.26), can be written as

$$\bar{n} = \beta^{-1} \frac{\partial}{\partial \zeta} \log e^{-\beta \Omega_0}. \quad (\text{A10})$$

The insertion of (3.32) into (A10) gives

$$\bar{n} = 2 \sum_p f_p - \frac{\sum_p g^2(\mathbf{k}, \mathbf{p}) \{f_p(1-f_{p+k})(1-f_p-f_{p+k}) + f_{p+k}(1-f_p)(1-f_{p+k}-f_p)\}}{\sum_p g^2(\mathbf{k}, \mathbf{p}) \{f_p(1-f_{p+k}) + f_{p+k}(1-f_p)\}}. \quad (\text{A11})$$

The second term tends to zero as $T \rightarrow 0$. Therefore

$$\bar{n} = (16\pi/3\hbar^3) (2m^{3*})^{1/2} \zeta^{3/2} = (8\pi p_0^3/3\hbar^3). \quad (\text{A12})$$

Appendix II

The subsidiary conditions are given by (3.1):

$$\xi_k^+ = \frac{1}{2} \sum_i \left\{ \frac{1}{1 - (1/\hbar\omega)^2 (\mathbf{k} \cdot \mathbf{P}_i/m - k^2/2m)^2} e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} + \text{h. c.} \right\}. \quad (\text{A13})$$

Expanding (A13), we get to a good approximation

$$\begin{aligned} \xi_k^+ = \frac{1}{2} \sum_i \left[\left\{ \frac{1}{1 - (1/\hbar\omega)^2 (\mathbf{k} \cdot \mathbf{P}_i/m - k^2/2m)^2} + \frac{1}{(\hbar\omega)^2} \frac{e}{m^2 c} \right\} (\mathbf{k} \cdot \mathbf{p}_i) (\mathbf{k} \cdot \mathbf{A}(\mathbf{X}_i)) \right. \\ \left. + (\mathbf{k} \cdot \mathbf{A}(\mathbf{X}_i)) (\mathbf{k} \cdot \mathbf{P}_i) - k^2 (\mathbf{k} \cdot \mathbf{A}(\mathbf{X}_i)) \right\} e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{X}_i} + \text{h. c.} \right]. \quad (\text{A14}) \end{aligned}$$

Using the formalism of second quantization, we obtain

$$\begin{aligned} \xi_k^+ = \frac{1}{2} \sum_{p\sigma} g(\mathbf{k}, \mathbf{p}) (b_{p+k\sigma}^* b_{p\sigma} + b_{p\sigma}^* b_{p+k\sigma}) \\ + \frac{1}{2} \sum_{p p' \sigma} [\theta(\mathbf{k}, \mathbf{p}, \mathbf{p}') b_{p'\sigma}^* b_{p\sigma} + \theta^*(\mathbf{k}, \mathbf{p}, \mathbf{p}') b_{p\sigma}^* b_{p'\sigma}] \\ = \xi_{k0}^+ + \xi_{k1}^+, \quad (\text{A15}) \end{aligned}$$

where

$$\begin{aligned} \theta(\mathbf{k}, \mathbf{p}, \mathbf{p}') &= (1/\hbar\omega)^2 (e/m^2 c) \{ \mathbf{k} \cdot (\mathbf{p} + \mathbf{p}' + \mathbf{k}) - k^2 \} \mathbf{k} \cdot \mathbf{A}(\mathbf{p} - \mathbf{p}' + \mathbf{k}), \\ \theta^*(\mathbf{k}, \mathbf{p}, \mathbf{p}') &= (1/\hbar\omega)^2 (e/m^2 c) \{ \mathbf{k} \cdot (\mathbf{p} + \mathbf{p}' + \mathbf{k}) - k^2 \} \mathbf{k} \cdot \mathbf{A}(\mathbf{p}' - \mathbf{p} - \mathbf{k}). \end{aligned} \quad (\text{A16})$$

Analogously

$$\begin{aligned} \xi_k^- &= \frac{1}{2i} \sum_{p\sigma} g(\mathbf{k}, \mathbf{p}) (b_{p+k\sigma}^* b_{p\sigma} - b_{p\sigma}^* b_{p+k\sigma}) \\ &+ \frac{1}{2i} \sum_{p p' \sigma} [\theta(\mathbf{k}, \mathbf{p}, \mathbf{p}') b_{p'\sigma}^* b_{p\sigma} - \theta^*(\mathbf{k}, \mathbf{p}, \mathbf{p}') b_{p\sigma}^* b_{p'\sigma}] \\ &= \xi_{k0}^- + \xi_{k1}^-. \quad (\text{A17}) \end{aligned}$$

The contribution from ξ_k^+ to the mean current density in first order in the the magnetic field is given by

$$\sum_{\nu \nu' p} (1/2\pi)^{\nu'} \int d\lambda_k^+ d\lambda_{k1}^- \langle i \sum (\lambda_k^+ \xi_{k1}^+ + \lambda_{k1}^- \xi_{k1}^-) \rangle_{\nu \nu'}$$

$$\begin{aligned} & \times \langle \exp i \sum_{\nu} (\lambda_k^+ \hat{\xi}_{k_0}^+ + \lambda_k^- \hat{\xi}_{k_0}^-) \rangle_{\nu\nu} \cdot \langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle_{\nu\nu} \\ & \times \exp \beta(\zeta n + \mathcal{Q}_0 - E_\nu). \end{aligned} \quad (\text{A18})$$

We see that the integral is of the form

$$\begin{aligned} & \int H d\lambda_k^+ d\lambda_k^- \sum_{\substack{k < k_0 \\ k_z > 0}} \{a(\mathbf{k})\lambda_k^+ + b(\mathbf{k})\lambda_k^-\} \exp[-1/4 \cdot \sum_{\substack{pk < k_0 \\ k_z > 0}} g^2(\mathbf{k}, \mathbf{p})] \\ & \times \{f_p(1-f_{p+k}) + f_{p+k}(1-f_p)\} (\lambda_k^{2+} + \lambda_k^{2-}), \end{aligned} \quad (\text{A19})$$

which vanishes.

Appendix III

$$\begin{aligned} \mathbf{i}_2(\mathbf{x}) \approx & \sum_{\nu \neq \nu'} \exp \beta(\zeta n + \mathcal{Q}_0 - E_\nu) \cdot (1/2\pi)^{n'} \int H d\lambda_k^+ d\lambda_k^- \sum_{n=0}^{\infty} [\\ & 1/2 \cdot \sum_{\rho} \langle \{i \sum (\lambda_k^+ \hat{\xi}_k^+ + \lambda_k^- \hat{\xi}_k^-) \}^2 \rangle_{\rho\rho}]^n / n! [\langle i \sum (\lambda_k^+ \hat{\xi}_k^+ + \lambda_k^- \hat{\xi}_k^-) \rangle_{\nu\nu'} \\ & + 1/2 \langle \{i \sum (\lambda_k^+ \hat{\xi}_k^+ + \lambda_k^- \hat{\xi}_k^-) \}^2 \rangle_{\nu\nu'} \cdot \left[\sum_{\nu''} \frac{1 - e^{\beta(E_\nu - E_{\nu''})}}{E_\nu - E_{\nu''}} \right. \\ & \left. \times \langle \mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x}) \rangle_{\nu''\nu'''} \langle H' \rangle_{\nu''\nu} \right]]. \end{aligned} \quad (\text{A20})$$

We need to consider only the processes in which one electron suffers an inelastic scattering (the elastic scattering should be excluded) by H' and another electron is scattered inelastically by $\mathbf{J}_0(\mathbf{x}) + \mathbf{J}_0^{(2)}(\mathbf{x})$.

If the inelastic scattering of two electrons $(\mathbf{p}_1, \mathbf{p}_2 \rightarrow \mathbf{p}_1 \pm \mathbf{k}, \mathbf{p}_2 \pm \mathbf{k})$ is done by H' and $\mathbf{J} + \mathbf{J}^{(2)}$, $\langle H \hat{\sigma}(\hat{\xi}_k^+) \hat{\sigma}(\hat{\xi}_k^-) \rangle_{\nu\nu'}$ must correspond to the scattering $(\mathbf{p}_1 \pm \mathbf{k}, \mathbf{p}_2 \pm \mathbf{k} \rightarrow \mathbf{p}_1, \mathbf{p}_2)$. Then the expression for the mean current density contains the factor

$$\begin{aligned} & \int H d\lambda_k^+ d\lambda_k^- \sum_{\substack{p_1 \neq p_2 \\ k < k_0 \\ k_z > 0}} g(\mp \mathbf{k}, \mathbf{p}_1) g(\pm \mathbf{k}, \mathbf{p}_2) (\lambda_k^{2+} - \lambda_k^{2-} \pm 2i\lambda_k^+ \lambda_k^-) \\ & \times \exp[-1/4 \cdot \sum_{\substack{qk < k_0 \\ k_z > 0}} g(\mathbf{k}, \mathbf{q}) \{f_q(1-f_{q+k}) + f_{q+k}(1-f_q)\} (\lambda_k^{2+} + \lambda_k^{2-})] \\ & \times F(\mathbf{k}, \mathbf{p}_1, \mathbf{p}_2), \end{aligned} \quad (\text{A21})$$

which vanishes, because

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda^+ d\lambda^- (\lambda^{2+} - \lambda^{2-}) \exp[-a^2(\lambda^{2+} + \lambda^{2-})] = 0, \\ & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\lambda^+ d\lambda^- \lambda^+ \lambda^- \exp[-a^2(\lambda^{2+} + \lambda^{2-})] = 0. \end{aligned} \quad (\text{A22})$$

Thus we are left with the process in which two electrons $(\mathbf{p}_1, \mathbf{p}_2)$ are scattered into the states $(\mathbf{p}_1 \pm \mathbf{k}, \mathbf{p}_2 \mp \mathbf{k})$ by H' and $\mathbf{J} + \mathbf{J}^{(2)}$. We have

$$\begin{aligned}
 \mathbf{i}_2(\mathbf{x}) \approx & e^{i\Omega_0} \left[H(1 + e^{-\beta(\varepsilon(\mathbf{r}) - \zeta)})^2 \cdot (1/2\pi)^{n'} \int_{\substack{k < k_c \\ k_z > 0}} H d\lambda_k^+ d\lambda_k^- \right. \\
 & \times \exp \left[-1/4 \cdot \sum_{\substack{qk < k_c \\ k_z > 0}} g^2(\mathbf{k}, \mathbf{q}) \{ f_q(1 - f_{q+k}) + f_{q+k}(1 - f_q) \} (\lambda_k^{2+} + \lambda_k^{2-}) \right] \\
 & \times \left[-1/8 \cdot \sum_{\substack{p_1 \neq p_2 \\ l < k_c \\ k_z > 0}} g(-\mathbf{k}, \mathbf{p}_1) g(\mathbf{k}, \mathbf{p}_2) (\lambda_k^{2+} + \lambda_k^{2-}) \right. \\
 & \times \{ -(2e^2/mm^*c) (1 + \alpha - \gamma k^2) \left[(\mathbf{p}_2 + \mathbf{k}/2) e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}} (\mathbf{p}_1 \cdot \mathbf{A}(\mathbf{k})) \right. \\
 & \times \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}_1) - \varepsilon(\mathbf{p}_1 - \mathbf{k}) \}}{\varepsilon(\mathbf{p}_1) - \varepsilon(\mathbf{p}_1 - \mathbf{k})} + (\mathbf{p}_1 - \mathbf{k}/2) e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}} (\mathbf{p}_2 \cdot \mathbf{A}(-\mathbf{k})) \\
 & \times \left. \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}_2) - \varepsilon(\mathbf{p}_2 + \mathbf{k}) \}}{\varepsilon(\mathbf{p}_2) - \varepsilon(\mathbf{p}_2 + \mathbf{k})} \right] f_{p_1} f_{p_2} (1 - f_{p_1 - \mathbf{k}}) (1 - f_{p_2 + \mathbf{k}}) \} \\
 & - 1/8 \cdot \sum_{\substack{p_1 \neq p_2 \\ k < k_c \\ k_z > 0}} g(\mathbf{k}, \mathbf{p}_1) g(-\mathbf{k}, \mathbf{p}_2) (\lambda_k^{2+} + \lambda_k^{2-}) \{ -(2e^2/mm^*c) \\
 & \times (1 + \alpha - \gamma k^2) \left[(\mathbf{p}_2 - \mathbf{k}/2) e^{i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}} (\mathbf{p}_1 \cdot \mathbf{A}(-\mathbf{k})) \right. \\
 & \times \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}_1) - \varepsilon(\mathbf{p}_1 + \mathbf{k}) \}}{\varepsilon(\mathbf{p}_1) - \varepsilon(\mathbf{p}_1 + \mathbf{k})} + (\mathbf{p}_1 + \mathbf{k}/2) e^{-i/\hbar \cdot \mathbf{k} \cdot \mathbf{x}} (\mathbf{p}_2 \cdot \mathbf{A}(\mathbf{k})) \\
 & \times \left. \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}_2) - \varepsilon(\mathbf{p}_2 - \mathbf{k}) \}}{\varepsilon(\mathbf{p}_2) - \varepsilon(\mathbf{p}_2 - \mathbf{k})} \right] f_{p_1} f_{p_2} (1 - f_{p_1 + \mathbf{k}}) (1 - f_{p_2 - \mathbf{k}}) \} \left. \right]. \quad (\text{A23})
 \end{aligned}$$

Here we have fixed the gauge by $\text{div } \mathbf{A}(\mathbf{x}) = 0$. Each term in (A23) has the following sum as a factor

$$\sum_p (\mathbf{p} \cdot \mathbf{A}(\pm \mathbf{k})) g(\mp \mathbf{k}, \mathbf{p}) \frac{1 - \exp \beta \{ \varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} \mp \mathbf{k}) \}}{\varepsilon(\mathbf{p}) - \varepsilon(\mathbf{p} \mp \mathbf{k})} f_p (1 - f_{p \mp \mathbf{k}}). \quad (\text{A24})$$

Since the summand except $(\mathbf{p} \cdot \mathbf{A}(\mathbf{k}))$ is a function of \mathbf{p} and $(\mathbf{p} \cdot \mathbf{k})$, (A24) is easily seen to be proportional to $(\mathbf{k} \cdot \mathbf{A}(\mathbf{k}))/k$, which vanishes on account of $\text{div } \mathbf{A}(\mathbf{x}) = 0$. Therefore we find

$$\mathbf{i}_2(\mathbf{x}) \approx 0. \quad (\text{A25})$$

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Note added in proof: It is expected that the following term which should be added to the current density operator (2.21) will make some contribution to the susceptibility:

$$A_j^i(x) = -e^2/m \cdot \sum_{i,k} (2\pi\hbar/\omega)^{1/2} \varepsilon_k e^{i/\hbar \cdot kx_i} (a_k - a_k^* - k) \delta(x - x_i).$$

The effect of this term on the diamagnetism together with the effect of the screened Coulomb interactions will be considered in a subsequent paper.

Letters to the Editor

Interaction of π -Mesons with π -Mesons

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In this note I should like to point out that the π - π cross section inferred from the Cosmotron data is not inconsistent with that derived from the cosmic ray experiments by means of the Landau theory of multiple production, thus giving an indirect support to the latter theory.

The experimental data¹⁾ on the reaction: $\pi + N \rightarrow \pi + \pi + N$, (N being a nucleon), induced by the 1.4 Gev beam of pions from the Cosmotron have been analyzed by Ito and Minami²⁾ based on the Kovacs model³⁾. Having investigated the angular and the energy distribution of the produced pions and the recoil nucleon, they have indicated the probable existence of a fairly strong π - π interaction. If we denote, therefore, by P the effective number of slow pions in the surrounding field of a nucleon, we can, to a rough approximation, put

$$\begin{aligned} \sigma(\pi + N \rightarrow (n+1)\pi + N) \\ \approx P \cdot \sigma(\pi + \pi \rightarrow \pi + n\pi), \\ (n=1, 2, \dots). \end{aligned} \quad (2)$$

It will not be unreasonable to assume, say, $P \sim 0.2$ -1.0, so that we can conjecture, for the energy region concerned, as

$$\sigma_{\pi-\pi} \sim (\hbar/m_{\pi}c)^2, \quad (2)$$

since the total cross section for the inelastic π - N scattering can be estimated to be about $(25 \pm 3) \cdot 10^{-27} \text{ cm}^2$.¹⁾

Another argument for such an interaction has been presented by Dyson⁴⁾ and Takeda⁵⁾ independently. In order to explain the pronounced maximum of $\pi^- - p$ cross section at about 900 Mev, they introduced their respective models of strong pion-pion interaction at this energy. Takeda, in particular makes an estimate of $\sigma_{\pi-\pi}$, putting $P=0.4$. His model can also lead to the same order of magnitude with (2) for the total pion-pion cross section at the energy region, say, 600 Mev \sim 1.2 Gev.

The Bristol group⁶⁾ has, on the other hand, recently performed a precise examination on the nature of secondary particles of 19 cosmic ray jet showers of primary energies 10^{-3} - 10^{-4} Gev, and found the ratio

$$N_y/N_{\pi^0} = 0.65 \pm 0.25, \quad (3)$$

where N_y denotes the number of neutral particles producing secondary stars and N_{π^0} denotes the number of neutral pions which are detected by electron pairs. (3) gives an upper limit to the ratio of neutral heavy mesons (θ^0) to neutral pions; but such a value could hardly be explained by Fermi's statistical theory of multiple production in the original form⁷⁾, since the latter would give a definitely higher value for this ratio*.

* Here we have assigned, according to Nishijima's scheme⁸⁾, the isotopic spin 1/2 to heavy mesons (statistical weight 2 for θ^0), because this assignment is consistent with other experimental data

Thus Fujimoto suggests¹⁰⁾ that the modification of the Fermi theory proposed by Landau¹¹⁾ will indeed be necessary. That is to say, the secondary particles we observe should be supposed to have been separated from the "meson gas" of a comparatively low temperature.

Making use of the formulae for the number density of an ideal Bose gas, we can deduce from (3) the "critical temperature", T_c , at which the free flight of the particles from the meson gas takes place**:

$$kT_c \approx (1.1 \pm 0.5) m_\pi c^2. \quad (4)$$

Then the energy density and the number density of the meson gas corresponding to this critical temperature determine the average "thermal energy" w (including rest energy) of the pions just before their separation from the cloud:

$$w \approx (4.0 \pm 1.0) m_\pi c^2. \quad (5)$$

Now let us estimate the π - π cross section at this energy region. According to Landau's picture, the mean free path of the mesons becomes approximately equal to the size of the meson cloud $\hbar/m_\pi c$ just when the latter reaches the critical temperature. It would be natural, further, to

assume that the cross sections $\sigma_{\pi-\pi}$, $\sigma_{0-\pi}$ cannot be extraordinarily larger than $\sigma_{\pi-\pi}$. Thus we are led to the expression

$$\sigma_{\pi-\pi} \lesssim (\text{mean free path} \times \text{density of pions})^{-1} \approx (\hbar/m_\pi c)^2 \quad (6)$$

which may be consistent with (2).

The above reasoning, as I hope, would make the existence of a fairly strong π - π interaction, whose cross section is of the order of $(\hbar/m_\pi c)^2$ for the energy near 1 Gev, more plausible.

A more detailed account will appear in this journal.

In conclusion I express my sincere gratitude to Prof. Y. Fujimoto and Dr. J. Nishimura for their discussions and kind information concerning the results of their analysis before publication, and to Dr. S. Minami for his discussions.

of the Bristol group⁶⁾.

Further we leave out the possibility of heavy mesons being composite particles (which hypothesis could lower the expected ratio to a certain degree), because Nishimura's recent analysis⁹⁾ on the transverse momenta of the secondary particles emitted from the jet stars seems to contradict with such a point of view.

** The numerical values in the following are calculated assuming a zero spin for the heavy meson. A heavy meson with spin ≥ 1 will lead to a lower T_c ($\leq 0.75 m_\pi c^2/k$) and a larger $\sigma_{\pi-\pi}$ ($\geq 10(\hbar/m_\pi c)^2$), which seems rather unfavourable.

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Surface Tension of Liquid He⁴ and Liquid He³

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Pollara¹⁾, in a recent communication, has arrived at the following relation connecting vapour pressure and surface tension of liquids

$$T \log p = -aS \left[\frac{M}{D-d} \right]^{1/2} + b, \quad (1)$$

where p is vapour pressure and S is surface tension. T , M , D , and d are temperature, molecular weight and densities of the liquid and vapour phases respectively; a and b are

constants.

We have calculated the surface tension as a function of temperature for liquid He⁴ and liquid He³ by making use of equation (1) with a and b having the values given in Table 1. The calculated values are com-

Table 1. Values of a and b with p expressed in mm. of Hg. and densities gm/c.c.

Liquid.	a	b
He ⁴	5.271	17.680
He ³	5.644	10.522

pared in Figures 1 and 2 with the experimental results of Allen and Misener²⁾ for liquid He⁴ and of Lovefoyn³⁾ for liquid He³. For liquid He⁴, we have made use of the vapour pressure data corresponding to the 'Agreed Temperature'⁴⁾ and the data for the densities as determined by Keesom⁵⁾. While for liquid He³, we have used the vapour pressure data of Abraham et al⁶⁾.

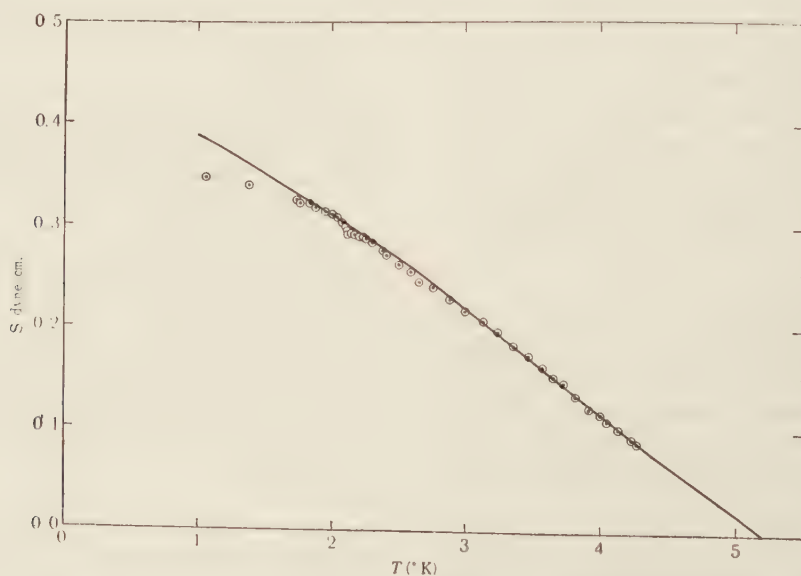


Fig. 1 Surface tension of Liquid He⁴ versus temperature °K. The curve indicates the calculated values, while the points represent experimental results of Allen and Misener²⁾.

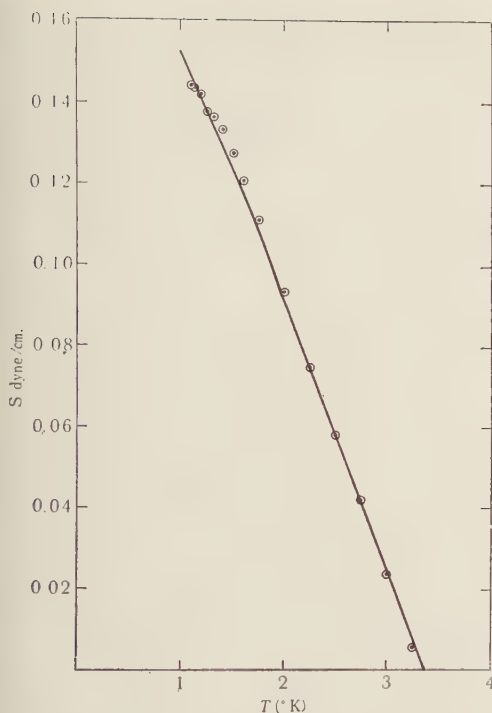


Fig. 2 Surface tension of Liquid He^3 versus temperature $^{\circ}\text{K}$. The curve indicates the calculated values, while the points represent the smoothed experimental results of Lovejoy⁽³⁾.

corresponding to the 'Kistemaker Temperature'⁽⁷⁾ and the data for the densities as determined recently by Kerr⁽⁸⁾. As is evident from Figures 1 and 2, the agreement between calculated and experimental values for the surface tension is fairly good in both the cases. The calculations could not be extended below 1.0 $^{\circ}\text{K}$ because of the lack of the experimental data for densities.

Further, Pollara⁽¹⁾ has also deduced from equation (1) the following expression for Trouton's ratio (see Ref. 1, Equation 4)

$$\frac{\Delta H}{T} = 2.303 R \left(\frac{T_c}{(T_c - T_B)} \right) \cdot \log_{10} (p_c/p_B), \quad (2)$$

in terms of the critical constants T_c and p_c , boiling point T_B and p_B ($=1$ atmosphere).

Here ΔH is the latent heat of evaporation of the liquid at the normal boiling point T_B . Values of $\Delta H/T_B$ for various liquids have been calculated from equation (2) and the results are found to be in good agreement with the Trouton's rule (see Ref. 1, Table 3). This rule states that the quotient of the latent heat of vaporisation per gm. molecule and the absolute temperature of ebullition is about 21. This means that the change in entropy on vaporisation is about the same for all substances, a result ultimately depending on the facts that the molecular volume of all gases is the same and that the entropy of the liquid is negligible compared with that of its vapour. For liquid He^4 and liquid He^3 , the values of $\Delta H/T_B$ as calculated by using equation (2) are found out to be 8.5 and 7.4 respectively. These values are much too low than for other liquids. This is what is expected, because for atoms of low atomic weight, it is necessary to add the zero point energy to the latent heat, since this energy no longer exists in the vapour state, the atoms being then no longer confined. Therefore, for atoms of low atomic weight, for which the zero point energy is appreciable, the modified Trouton's rule is expressed as:

$$(\Delta H + E_0)/T_B = 21, \quad (3)$$

where E_0 is the zero-point energy. The zero point energies for liquid He^4 and liquid He^3 as calculated by using equations (2) and (3) are compared in Table 2 with the corresponding values calculated from Debye theory⁽⁹⁾ and from a consideration of the balance of forces.⁽¹⁰⁾

Our thanks are due to Prof. D. S. Kothari and Dr. V. S. Nanda for taking interest in the above investigations.

Table 2. Zero-point energy of Liquid He⁴ and Liquid He³ Cal/Mole.

Liquid.	Trouton's Rule using		Debye Model (Ref. 9)	Cavity Quantization. (Ref. 10)	
	Eq. (2) and (3)	Value of from Ref. (11)		Slater and Kirk- wood potential	Yntema-Schneider potential
He ⁴	64	68	73	46	29
He ³	44	61	—	—	—

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On the Renormalization Theory of Quantum Electrodynamics

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Although the renormalization theory

has achieved a great success in the realm of quantum electrodynamics, there still remain open questions as to the internal consistency and the prediction of high-energy phenomena. Suppose that when the unrenormalized charge e runs from 0 to ∞ , the renormalized one e_r , being a function of e , runs over a certain domain $N(e_r)$, called the *normal zone* in what follows. In this connection a question immediately arises as to whether the normal zone $N(e_r)$ involves the observed value $1/\sqrt{137}$ as it should be (or a question concerning the magnitude of the Z_3 -factor). If this is not the case, we shall necessarily be faced with a further question of whether the prediction by the theory is still correct in the high-energy region, since the behavior of the renormalized propagator of a photon is intimately connected with the magnitude of the Z_3 -factor. It is the aim of the present note to answer these questions.

Introducing the method of the *renormalization cut-off*⁽¹⁾ we can show that the renormalized propagators of an electron and of a photon and the renormalized vertex function are given as follows :

$$\left. \begin{aligned} S_{Fe}(p) &= S_F(p)/Z_2(i\tilde{p}, e_\Lambda^2), \\ D_{Fe}(k) &= D_F(k)/Z_3(-k_\Lambda^2, e_\Lambda^2), \\ \Gamma_{\mu e}(p, p) &= Z_1(-i\tilde{p}, e_\Lambda^2)\gamma_\mu, \end{aligned} \right\} \quad (1)$$

and asymptotically for large argument

where e_Λ , $Z_1(\Lambda, e_\Lambda^2)$, $Z_2(\Lambda, e_\Lambda^2)$ and $Z_3(\Lambda^2,$

e_A^2) are renormalized charge and renormalization constants as obtained by our cut-off method and Λ the cut-off momentum. This result shows that our two questions as stated above can be reduced to the one of studying the property of the function $Z_3(\Lambda^2, e_A^2)$.

Now, we shall calculate this function, which can be derived from the polarization operator

$$H_{\mu\nu}(k) = e_A^2 Z_3(\Lambda^2, e_A^2) Z_2(\Lambda, e_A^2) \times \int d^4t Tr[S_{Fc}(k-t) \gamma_\mu S_{Fc}(t) \Gamma_{vc}(k-t, t)]. \quad (2)$$

When assuming that the high momentum gives dominant contributions to this integral²⁾ and using (1) to rewrite it, we can neglect k as compared with t in functions Z_2 and Γ_{vc} under the integral and further substitute $Z_1(\Lambda)$ and $Z_2(\Lambda)$ for $Z_1(-i\gamma t)$ and $Z_2(-i\gamma t)$ respectively. Then, in virtue of the Ward identity (2) leads to the following form

$$H_{\mu\nu}(k) = e_A^2 Z_3(\Lambda^2, e_A^2)^{-1} \times \int d^4t Tr[S_F(k-t) \gamma_\mu S_F(t) \gamma_\nu]. \quad (3)$$

This is nothing but the expression familiar in the perturbation theory and we can immediately see that³⁾

$$Z_3(\Lambda^2, e_A^2) = 1 - \frac{e_A^2}{3\pi} \log(\Lambda^2/m^2), \quad (4)$$

$$D_{Fc}(k^2) = D_F(k) \left(1 - \frac{e_A^2}{3\pi} \log(k^2/m^2) \right), \quad (5)$$

where m is the mass of the electron.

From (4), (5) and (1) we are led to the following conclusions:

i) as Λ tends to ∞ , the normal zone $N(e_r)$ shrinks to zero,

ii) it is necessary to cut off at Λ^2 , which should be smaller than

$$\lambda^2 \equiv m^2 \exp(3\pi \times 137), \quad (6)$$

iii) $D_{Fc}(k^2)$ for $-k^2 > \lambda^2$ takes the sign different from the usual one, meaning by it that the Coulomb potential takes the different sign in this energy region.

It is evident that this failure of quantum electrodynamics can never be removed by taking into account effects of other fields, if they are subject to the interactions of the first kind.⁴⁾ The final solution may therefore be obtained by considering the structure of space-time itself (gravitational field)⁵⁾ or by introducing a new fundamental length.

From the similar consideration as above we may expect in meson theory that λ is of the order of nucleon mass and therefore that the contributions from the heavy particles may play an essential role in the discussion. At any rate, the renormalization theory for the system consisting only of meson and nucleon will immediately lead to contradictions in the observable energy region.

Finally, we shall add a remark on the reason why the perturbation calculation could give the correct answers in quantum electrodynamics. The main reason is due to the fact that λ is very large which is a direct consequence of the smallness of e_r^2 as seen from (6). This is not the case in meson theory.

The authors wish to express their appreciation of enlightening discussions by the members of the symposium on field theory held in Nov. 1955 at the Research Institute for Fundamental Physics. Thanks are also due to Professors H. Yukawa, S. Sakata and Z. Kaba for kind interest and encouragement.

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On the Elementarity of the Weak Boson-Fermion Interaction, I*

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In the previous papers^{(1),(2)}, four types of interactions which seem to have certain universal characters and mutual-consistency are discussed. They are: (i) electromagnetic interaction $e "j_\mu A_\mu"$ (e-int.), (ii) strong Baryon-meson interaction $G " \bar{\psi} \psi \phi "$ (G-int.), (iii) weak Fermi-interaction $f " \bar{\psi} \psi \bar{\psi} \psi "$ (f-

int.) and (iv) weak Boson-Fermion interaction $g " \bar{\psi} \psi \phi "$ (g-int.).

The possibility of explaining the g-int. in terms of the other three universal interactions is investigated in this paper. In our discussions the perturbation treatment with regard to G-int. is avoided as much as possible.

(A) π -decay

Let us take the f-int. which consists of two Baryons (a and b) and two leptons (μ, ν) or (e, ν) such as $f(\bar{\psi}_a O_j \psi_b)(\bar{\psi}_\mu O_j \psi_\nu)$ or $f(\bar{\psi}_a O_j \psi_b)(\bar{\psi}_e O_j \psi_\nu)$. First, we want to know whether the following simplest assumption is valid.

"These interactions are completely symmetric with regard to μ -meson and electron⁽³⁾, -process independence of f-int."

The main contributions to π -decay come from the summation of Feynman diagrams in which all vertices are G-int. except the one f-vertex where external lepton lines are drawn. (See Fig. 1). As leptons have

no G-int. we can include every radiative G-correction in our discussions. It is permissible to neglect e-correction. The matrix elements of π -decay due to this mechanism are written in the form,

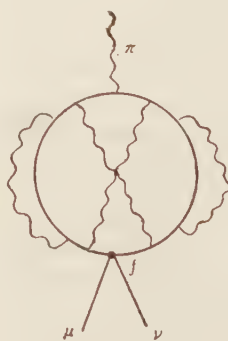


Fig. 1

$$M_j(p_\mu) (\bar{\psi}_{(\mu \text{ or } e)} O_j \psi_\nu) \varphi_\pi(p_\mu), \quad (1)$$

where p_μ is the energy-momentum four vector of external π -meson and $\varphi_\pi(p_\mu)$ is the π -meson wave function. The complicated contribution from various intermediate states is a function of p_μ and the masses of the

* A preliminary account of the present paper was given in "Soryushiron-Kenkyu" (Mimeographed circular in Japanese) **9** (1955), 599.

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intermediate Baryons and Bosons and is denoted by M_j . j shows the transformation property of M_j . From the structure of M_j we have the following results.

$$(\alpha) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_e O_j \psi_\nu)$$

=Scalar (S), Vector (V) and Tensor (T). For these lepton covariants we cannot construct M_j . That is, π -decay cannot occur by this mechanism. (e -int. is neglected).

$$(\beta) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_e O_j \psi_\nu)$$

=Pseudoscalar (P). M_j must be the scalar function of $p_\mu^2 = -m_\pi^2$ and intermediate particle masses. Then (1) is identical with the results given by the following primary interactions

$$M(\bar{\psi}_{(\mu \text{ or } e)} \tilde{\gamma}'_5 \psi_\nu) \varphi_\pi + \text{c. c.} \quad (2)$$

where $\tilde{\gamma}'_5$ is $\tilde{\gamma}_5$ or 1 according to the relative parity of μ (or e) and ν . In the ratio R of the π -decay probabilities $R = W(\pi \rightarrow e + \nu) / W(\pi \rightarrow \mu + \nu)$, M will disappear by the cancellation and we get $R \simeq 5.5$ which is completely unacceptable.

$$(\gamma) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_e O_j \psi_\nu)$$

=Pseudovector (PV). M_j must be of the form $p_\lambda M'$ where M' is a scalar function with similar properties to M . Then (1) is equivalent to the results given by the following elementary interactions which were adopted as g -int. in A.

$$M'(\bar{\psi}_{(\mu \text{ or } e)} \tilde{\gamma}'_5 \gamma'_\lambda \psi_\nu) \partial_\lambda \varphi_\pi + \text{c. c.} \quad (3)$$

In this case R is proportional to $(m_e/m_\mu)^2$ and is $R \simeq 1.3 \times 10^{-4}$ which seems not to be inconsistent with experiments so far.⁴⁾ In the case of (α), however, e -int. correction will be important because otherwise all cases of (α) are forbidden. The inclusion of internal photon lines which contain external μ -meson or electron will make all cases of (α) allowed, though it decreases greatly the probability by the factor $\alpha^2 = (1/137)^2$.

From the general arguments⁵⁾, however, under our assumption, even if we take into account the corrections due to e -int., we cannot expect the factor $\sim (m_e/m_\mu)^2$ in R except in the cases $(\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_e O_j \psi_\nu) = PV$ or V . Thus PV - or V -coupling (V has the drawback that the leading matrix elements contain the factor α) must play essential parts in explaining the ratio R . The analysis of β -decay coupling, however, seems to indicate $S \mp T + pP$ ($p=1$ or 0) and reject PV and V .⁶⁾ Hence it is quite difficult to ascribe $\pi \rightarrow \mu + \nu$ to the above G - and f -mechanism, if we stand firmly on our assumption of process independence of f -int. Thus it is shown that as long as we adhere to the idea of eliminating the universal g -int., we are forced to add more complicated structure to f -int. This problem will be discussed in the following letter.

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On the Elementarity of the Weak Boson-Fermion Interaction, II*

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From the results of the preceding letter¹⁾, we continue the standpoint of eliminating the universal g -int. by making the following minimum relaxation of the universal characters of f -int.

(I) "The value of the coupling constant f is unique but the coupling types are process dependent."

As regards the estimation of the absolute values of the transition probabilities, we have rather little knowledge and we cannot compare with much confidence the transition probabilities calculated for different coupling types of f -vertex. From the knowledge of β -decay coupling, however, we can deduce the following general arguments for the possible coupling types.

(a) $(\bar{\psi}_e O_j \psi_\nu) = S \mp T$. $(\bar{\psi}_\mu O_j \psi_\nu)$ contains at least PV or/and PS .

In this case $\pi \rightarrow e + \nu$ probabilities contain α^2 compared with those of $\pi \rightarrow \mu + \nu$ which will be useful to explain the ratio R .

(b) $(\bar{\psi}_e O_j \psi_\nu) = S \mp T + P$. $(\bar{\psi}_\mu O_j \psi_\nu)$ contains at least PV . This is possible only

if it turns out that $\pi \rightarrow \mu + \nu$ through PV can predominate over $\pi \rightarrow \mu + \nu$ through P .

For every vertex, however, $\pi \rightarrow e + \nu + \gamma$ (also $\pi \rightarrow \mu + \nu + \gamma$) will occur through Gf -process. For the same type of f -vertex $\pi \rightarrow e + \nu + \gamma$ will be more favoured than $\pi \rightarrow \mu + \nu + \gamma$ by the available phase space volume. For $(\bar{\psi}_e O_j \psi_\nu) S \pm T$ and V , $\pi \rightarrow e + \nu + \gamma$ will be more frequent than $\pi \rightarrow e + \nu$ because the transition probabilities contain the factor α for the former cases and α^2 for the latter. The situation is exactly the opposite for P . Thus the experiments on the frequency ratio $R' = W(\pi \rightarrow e + \nu + \gamma) / W(\pi \rightarrow e + \nu)$ are quite interesting. If $R' < 1$, (a) will be completely rejected. On the other hand, if $R' > 1$ (a) will be the case. For (b) it means that $\pi \rightarrow e + \nu$ through P is less frequent than $\pi \rightarrow e + \nu + \gamma$ through S and T which have the factor α in their decay probabilities. So if $R' \gg 1$, (b) seems to be unacceptable. Thus the ratio R' will give an indirect test of β -coupling. If (i) we agree to make such a relaxation as (a) and (b) in the universal and symmetrical characters of f -int. and (ii) in addition, the values of the matrix elements M or M' in (2) or (3) of I turn out to be of the correct order, then we are in a position to be able to deduce $\pi \rightarrow \mu + \nu$ without introducing universal g -int. If neither (i) nor (ii) is the case, the introduction of primary g -int. is the simplest and the most natural procedure. For instance, the assignment $S \mp T$ for process independent f -int. requires the factor α^2 for $\pi \rightarrow \mu(e) + \nu$ and it may be possible that G -, e - and f -mechanism is not responsible for $\pi \rightarrow \mu + \nu$.

(B) Hyperon decays

Let us proceed to other g -reactions, the Hyperon decays, assuming that f -int. have such properties as discussed above and so

* A preliminary account of the present paper was given in "Soryushiron-Kenkyu" (Mimeographed circular in Japanese) 9 (1955), 559.

[†] Now at Department of Theoretical Physics, University of Manchester.

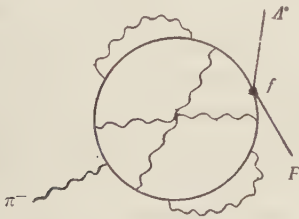
are able to lead to $\pi \rightarrow \mu + \nu$ coupled with G -int. As already pointed out²⁾ we must add a new feature to f -int. for this purpose, namely (II) " f -int. must be extended into four Baryons (not neutrino processes)."

To develop the similar procedure to (A) we make the the following assumption and the following approximation.

Assumption. "Let f -int. containing (μ, ν) and (P, Λ^0) be $(\bar{\psi}_a O_j \psi_b) (\bar{\psi}_u O_j \psi_v)$ and $(\bar{\psi}_a O_j \psi_b) (\bar{\psi}_p O_j \psi_{\Lambda^0})$, where a and b are some sets of Baryons. Among these f -int. there is complete symmetry³⁾ between (μ, ν) and (P, Λ^0) ."

Approximation. " Λ^0 and P in these f -int., as μ and ν , cannot interact strongly with other Baryons."

Under the above assumption and approximation Λ^0 -decay is realized by the processes shown in Fig. 1, which have the same diagrams as Fig. 1 of I for the intermediate states. Then completely similar arguments to (1) of I are possible.



$$(\alpha) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_p O_j \psi_{\Lambda^0})$$

$=S, V$ and T . These are forbidden, but if we do not make the "freezing" approximation $\Lambda^0 \rightarrow P + \pi^-$ is allowed by the inclusion of G^2 -correction (not e^2 -correction).

$$(\beta) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_p O_j \psi_{\Lambda^0}) = P.$$

In place of (2) of I we have for $\Lambda^0 \rightarrow P + \pi^-$,

$$M(\bar{\psi}_p \gamma_5' \psi_{\Lambda^0}) \partial_\lambda \varphi_\pi. \quad (1)$$

$$(\gamma) \quad (\bar{\psi}_\mu O_j \psi_\nu) = (\bar{\psi}_p O_j \psi_{\Lambda^0}) = PV.$$

In place of (3) of I we have

$$M'(\bar{\psi}_p \gamma_5' \gamma_\lambda \psi_{\Lambda^0}) \partial_\lambda \varphi_\pi. \quad (2)$$

Comparing (1) and (2) with (2) of I and (3) of I respectively, we can estimate the decay life-times of $\Lambda^0 \rightarrow P + \pi^-$ from that of $\pi \rightarrow \mu + \nu$ without knowing the details of M and M' . From the results of A , it is shown that $O_j = 1$, $\gamma_5 \gamma_\lambda$ and γ_λ can give the correct order of magnitude within the factor $2 \sim 10$. The best fit is given in the case of γ_λ . This is interesting because it predicts the different space inversion parity for Λ^0 and P . For Σ^- and Ξ^- -decay we can have similar procedure taking their spin values to be $1/2$ and can also expect the workability of G - and f -intermediation, provided that it is the case with $\pi \rightarrow \mu + \nu$. The main part of the G -corrections neglected due to our "freezing" approximation seems to correspond to the G -corrections which are also neglected in the discussion of primary g -int. for $\Lambda^0 \rightarrow P + \pi^-$. It is not certain whether such corrections may be responsible for the discrepancy of the order of 2 to 10. Analogous mechanism, of which details we do not give here, is of course considered for $K_\mu \rightarrow \mu + \nu$.

In conclusion we infer that in order to eliminate the elementary g -int. we must, at least, impose the more complicated structure (I) and the extension (II) to f -int. It is also shown that if we admit these conditions, it is possible to explain the Hyperon decays and possibly the $K_\mu \rightarrow \mu + \nu$ if we can explain $\pi \rightarrow \mu + \nu$ by G -, e - and f -intermediation. More detailed discussions about universal interactions will soon appear in *Nuclear Physics*. We are indebted to Prof. T. Tati and Dr. S. Ogawa for their kind discussions.

- 1) S. Ôneda, S. Hori and A. Wakasa, Prog. Theor. Phys. **15** (1956), 300.
- 2) S. Ogawa, H. Okonogi and S. Ôneda, Prog. Theor. Phys. **11** (1954), 331 and reference (1) of I, § 5. Dallaporta also discussed this problem independently by different method. N. Dallaporta, Nuovo Cimento **1** (1955), 962.
- 3) From the decay schemes $\pi \rightarrow \mu + \bar{\nu}$ and $\Lambda^0 \rightarrow P + \pi^-$, it may be reasonable to assume symmetry between $(\mu + \bar{\nu})$ and $(P + \Lambda^0)$ if some regularity exists for weak interactions.

On K -Meson Decays and the Universal Interactions*

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February 2, 1956

As the continuation of the preceding works¹⁾, we want to discuss here some aspects of K -meson-decays from the standpoint of universal interactions. Recently the existence of the decay modes $K \rightarrow \pi + \pi^0$ has been confirmed. We also have other similar processes, $\theta^0 \rightarrow \pi^+ + \pi^-$ and $\tau \rightarrow 3\pi$. These might be explained by some universal weak Boson-Boson interaction. Here, however, we continue to try to explain these phenomena in terms of the universal interactions mentioned in the preceding papers. $K \rightarrow \pi + \pi^0$ is a rather slow reaction ($\tau \sim 10^{-8}$ -

10^{-9} sec.) and cannot be realized through G -int. only. Thus some weak interaction must take part in its decay mechanism. On the other hand, we have other charged Boson decays, $K_{\mu 3} \rightarrow \mu + ?^0 + ?^0$ and $K_{e 3} \rightarrow e + ?^0 + ?^0$, which seem to be not less frequent than $K \rightarrow \pi + \pi^0$. Here we interpret $K_{e 3}$ -decay as $K \rightarrow \mu$ (and e) $+ \nu + \pi^0$ ²⁾ and investigate the possibility that they are the alternative decay modes of $K \rightarrow \pi + \pi^0$. As two leptons are contained, we may imagine at once the possible G - and f - (neutrino process) intermediation shown in Fig. 1 for its simple mechanism³⁾. Except for the f -vertex which contains leptons and is mainly responsible for its longevity, all other vertices must be G -int. There are some restrictions on this f -vertex.

Consider the vertex which is obtained by replacing (μ, ν) in the f -vertex by a π -meson. If this new vertex has the character of being able to become G -int., $K \rightarrow \pi + \pi^0$ will occur as a fast process unless there are other selection rules. Thus the longevity of $K \rightarrow \mu$ (and e) $+ \nu + \pi^0$ requires that in our mechanism two Baryons which appear in the f -vertex must be those possible combinations which lead to a weak vertex when leptons are replaced by π -meson. Let us call the new weak vertex comprising external π -meson a g -vertex in Fig. 2. We take K -meson to be a scalar meson. Let the energy-momentum four vector of the K -meson, π^0 -meson, in Fig. 1 and Fig. 2

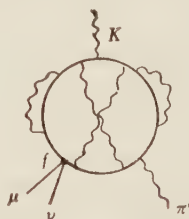


Fig. 1.

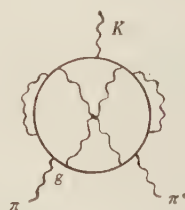


Fig. 2.

* A preliminary account of the present paper was given in "Soryushiron-Kenkyu" (Mimeographed circular in Japanese) **9** (1955), 559.

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be p_μ , q_μ and q'_μ respectively. We assume that both f - and g -int. have Baryon covariance of P type. The transition matrix for each process is given by

$$\begin{aligned} & gM_1(p, q)\varphi_K(p)\varphi_{\pi^0}^*(q)\varphi_\pi(p-q)^* \\ \text{for } & K \rightarrow \pi + \pi^0, \\ & fM_2(p, q')\varphi_K(p)\varphi_{\pi^0}^*(q')(\bar{q}'_\mu\gamma'_5\psi_\nu)(p-q') \\ \text{for } & K \rightarrow \mu + \nu + \pi^0. \end{aligned}$$

$M_1(p, q)$ is a scalar function of $p_\mu^2 = -m_K^2$, $q_\mu^2 = -m_\pi^2$, $(p, q) = -\frac{1}{2}m_K^2$ and the masses of particles which appear in the intermediate states. $M_2(p, q')$ is obtained from $M_1(p, q)$ by replacing only (p, q) by (p, q') . As $-m_K m_\pi \leq (p, q') \leq -\frac{1}{2}(m_K^2 + m_\pi^2 - m_\mu^2)$, the values of the ratio $(p, q')/(p, q)$ only vary from about 0.5 to 1. So if M_1 does not depend radically on (p, q) it will be good approximation⁴⁾ to take $(p, q) \doteq (p, q')$, which means $M_1(p, q) \doteq M_2(p, q')$.

The transition probabilities in the rest system of K -meson are given by ($m_K = 970 m_e$)

$$\begin{aligned} 1/\tau(K \rightarrow \pi + \pi^0) &= 2^{-3}(2\pi)^{-1}g^2M_1^2m_K^{-1} \\ &\times (1 - 4u^2)^{1/2} \\ 1/\tau(K \rightarrow \mu + \nu + \pi^0) &= 2^{-5}(2\pi)^{-1}f^2M_2^2m_K^3 \\ &\times \{(1 + u^2 - 2v^2)A - 2/3B - 2v^4C\}, \end{aligned}$$

where

$$\begin{aligned} A &= (1 + u^2 - v^2) - 4u^2 \log[(1/2u) \\ &\times (1 + u^2 - v^2 + w)], \\ B &= w^3, \\ C &= w - (1 - u^2) \log[\{1 - 2u^2 - v^2 - u^2v^2 \\ &+ u^4 + (1 - u^2)w\}/2uv^2] + (1 + u^2) \\ &\times \log(1/2u)(1 + u^2 - v^2 + w), \\ w &= (1 - 2u^2 - 2v^2 - 2u^2v^2 + u^4 + v^4)^{1/2} \\ u &= m_\pi/m_K, \quad v = m_\mu/m_K. \end{aligned}$$

Here we put $M_1 \doteq M_2$ and assume, for instance, the same frequency for both rea-

ctions. Then taking Fermi constant $f = 1.4 \times 10^{-10}$ erg cm³, we get $g^2/4\pi = 1.8 \times 10^{-10}$. This is strikingly in accord with the value of the coupling constant of pseudoscalar interaction for $\pi \rightarrow \mu + \nu$ decay ($g^2/4\pi = 2.08 \times 10^{-15}$). Putting $v \doteq 0$ we get the decay lifetime of $K \rightarrow e + \nu + \pi^0$. The frequency ratio $R = W(K \rightarrow e + \nu + \pi^0)/W(K \rightarrow \mu + \nu + \pi^0)$ turns out ~ 2.3 for our choice of the interaction type⁵⁾. The prediction of the comparable frequency seems to be not inconsistent with experiments. Thus in the above discussions we can show that by these categories of interactions (G -, f - and g -int.) the two possibilities of $K \rightarrow \pi + \pi^0$ and $K \rightarrow \mu(e) + \nu + \pi^0$ may be expected to have about equal probability. In the above discussions we only assign g -vertex to the outgoing π^\pm meson. If we introduce weak g -int. comprising π^0 -meson and K -meson, g -int. may also appear in other vertices. On the whole, we might say that g -int., in effect, will appear in three external vertices with the same probability. This will make $K \rightarrow \pi + \pi^0$ more frequent than K_S -decay. Similar procedure is also possible for vector K -meson if we confine ourselves to the same f - and g -coupling. We can also expect that the same mechanism will lead to anomalous θ^0 -decay such as $\theta^0 \rightarrow \mu^\pm + \nu + \pi^\mp$, etc. It must be noted that our discussions do not give light on the elementarity of g -int., because there still remains the possibility of deducing g -int. from other universal interactions.

- 1) S. Ôneda, S. Hori and A. Wakasa, Prog. Theor. Phys. 15 (1956), 300, 302.
- 2) Most of the experimental results used in this paper are based on those reported in Padua Conference, Nuove Cimento 12, Supplement No. 2, (1954) and Fifth Rochester Conference 1955.

- 3) S. Oneda, Prog. Theor. Phys. **9** (1953), 327, § 4, G. Costa and N. Dallaporta (Nuovo Cimento **2** (1955), 519) also discussed this problem independently. We may refer to their results as regards the rough order estimation of the absolute value of the transition probabilities.
- 4) This will be the case because p_μ^2 and q_μ^2 will

also appear in the same way as (p, q') and (p, q) . This is also quite a good approximation if S -wave emission of π^0 -meson in K_S^0 -decay is favoured.

- 5) There may be some doubts whether $(\bar{\psi}_e O_3 \psi_\nu)$ can be pseudoscalar. Other types for it will change the ratio R .

Errata

Theory of Antiferromagnetic Resonance in $\text{CuCl}_2 \cdot 2\text{H}_2\text{O}$.

Takeo NAGAMIYA

Progress of Theoretical Physics, **11** (1954), 309

As pointed out by Gerritsen,^{1,2)} there is a simple obvious error in my paper which has been published under the above title in Prog. Theor. Phys. **11** (1954), 309. The eigenfrequencies of the antiferromagnetic system turning in the ab -plane at the critical field applied along the a -axis are given by eq. (5.1); in the limiting case of $\zeta^b=0$ (spin axis parallel to the a -axis) they are given by

$$(\omega/\gamma)^2=0 \quad \text{and} \quad (\omega/\gamma)^2=c_2-c_1+c_1(\alpha^{-1}+2+\alpha), \quad (\text{a})$$

and in the other limiting case of $\psi=\pi/2$ (spin axis parallel to the b -axis) by

$$(\omega/\gamma)^2=c_2-c_1 \quad \text{and} \quad (\omega/\gamma)^2=c_1(\alpha^{-1}-1), \quad (\text{b})$$

as are explicitly stated in the text. The two values of $(\omega/\gamma)^2$ as functions of ζ^b connect the above two sets of the limiting values continuously and monotonously, without crossing with each other, as shown schematically by Fig. 3 which is the result of a careful examination of eq. (5.1). However, the term $-c_1$ in the second formula of (a) has been dropped in Fig. 3 and in the subsequent discussions. In particular, Figs. 4, 10 and 11 contain incorrect features. Gerritsen writes that this error made me "overlook the simple physical meaning of these frequencies"; however, one will clearly see the physical meaning of these frequencies from the statement in the text, only Figures 10 and 11 being not faithful representations of my idea. (Gerritsen remarks further that I called the resonance phenomenon related to these frequencies "orientation resonance", but I have never used these words.)

The uppermost curve of Fig. 4 should be reduced by $-c_1$; the same is applied to Fig. 18 of the review article by Nagamiya, Yosida and Kubo, **Advances in Physics** **4** (1955), p. 1, which is a reproduction of this Fig. 4.

In Fig. 10 (b) and (c), the middle curve should be broken beyond the point where it crosses with the lower curve.

In Fig. 11, the top of each vertical solid line should be cut down to the point where the ascending curve from the left meets it. Further, from the bottom of the same solid lines in Fig. 11 (a) and (b) and from the top of the lower solid line in Fig. 11 (c), a horizontal line going to the right should be added, which were dropped by mistake while drawing the figures.

- 1) H. J. Gerritsen, Thesis (Leiden, 1955).
 2) H. J. Gerritsen, *Physica* **21** (1955), 639.

Absorption Spectra of Unsymmetrical Cyanines

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The absorption maximum of an unsymmetrical cyanine with two different end nuclei is generally of the shorter wave length than a mean value of absorption maxima of two symmetrical cyanines with the respective nuclei. Two different simple models are found for describing this fact. The first is a modification of the free-electron model which is based on an assumption that the influence of end nuclei on the π electrons in the conjugated chain can be represented by the Pöschl-Teller potential. The second is the Herzfeld-Sklar model which is based on the consideration of the interaction between various chemical states of cyanine ions. Both the models can fairly reproduce the observed fact.

Introduction

A characteristic property of linear long conjugated molecules is the so-called vinylenic shift. It is the shift in the absorption maxima, to the longer wave length, resulting from increasing the length of the conjugated chain by adding a vinylenic group. Such vinylenic shifts have been interpreted by assuming a molecular model in which one electron levels in a deep square-well potential are occupied by π electrons belonging to the conjugated chain¹⁾. In certain molecules, the amount of the shift depends on and is convergent with the number of conjugated double bonds. The convergence was shown to be due to interactions between π electrons²⁾.

Brooker³⁾ showed that the absorption maximum of an unsymmetrical cyanine with two different nuclei is generally of the shorter wave length than a mean value of absorption maxima of two symmetrical cyanines with the respective nuclei. It was further shown that the deviation from the mean value varies according as the basicity of nuclei. For example, let us consider three cyanines with two end nuclei: a symmetrical cyanine with the same end nuclei A, another symmetrical cyanine with the same end nuclei B, and an unsymmetrical cyanine with the different nuclei A and B, as is shown in Fig. 1. We denote these cyanines by a, b and c respectively. If the wave lengths of the absorption maxima

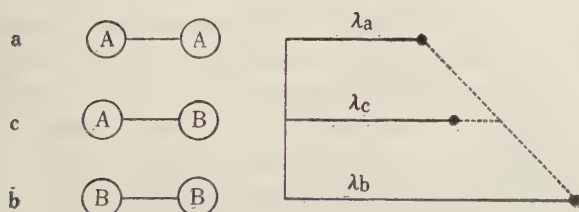


Fig. 1. Deviation of absorption maximum

of respective cyanines are λ_a , λ_b , and λ_c , Brooker's experiment shows that λ_c is generally less than $(\lambda_a + \lambda_b)/2$. We shall refer to the difference between λ_c and the mean value of λ_a and λ_b as the deviation of λ_c , the absorption maximum of the cyanine c. According to Brooker,

if we fix the nucleus A and vary the nucleus B, the deviation of λ_c varies with the basicity of B. The purpose of the present paper is to find the simple model which is capable of reproducing these deviations.

The free-electron model in a square well cannot account for the deviation because this model contains only one parameter which gives a measure of the conjugated chain length and there is no freedom of including the influence of end nuclei. Though the application of the free-electron model to the cyanine dyes was tried by many authors, they did not quantitatively consider the difference of end nuclei. In order to take into account the influence of end nuclei we should modify the model.

The influence may be represented in various ways. The simplest way may be to modify the potential in the end portion. One may consider three ways of the modification (Fig. 2): (a) to make a small well in each end of the bottom of the square well and make their breadth and depth different; (b) to make a terrace in each end of the bottom and make their breadth and height different; (c) to make the end walls of the square well finite and make their height different. We first studied the well and

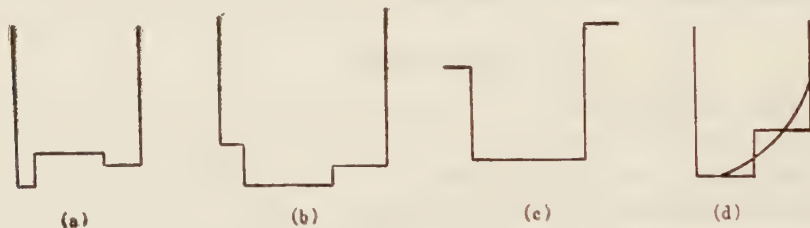


Fig. 2. Modifications of square well
(a) small wells (b) small terraces (c) finite walls (d) extreme case

terrace modifications. The eigenvalue problem could exactly be solved but the numerical computation by the desk machine was a very tedious task. We could not obtain any satisfactory result so far as we introduced only one well or one terrace for each end.

To introduce many terraces means to change the end into the smooth curve. This modification was what we next studied. If we can represent the modified potential in an analytical way and further if we can obtain the eigenvalue in an analytical form, the task of the numerical computation can greatly be reduced. Such an eigenvalue problem was already studied by Pöschl and Teller¹¹. We examined whether this model can reproduce Brooker's experiment. The result was fairly good. This will be reported in the first section.

In the next section the problem will be considered from a different view point according to Herzfeld-Sklar's idea. Herzfeld and Sklar⁵⁾ considered, besides the two states of a cyanine ion in which a nitrogen atom in one end of its conjugated chain is ionic, intermediate states in which one of carbon atoms in the conjugated chain is ionic. They tried to interpret the absorption maxima of cyanines by the interaction of these states, but arrived at the conclusion that the general feature of their result was not satisfactory. Thus they did not compare the theory with experiment. We shall apply their idea to

the present problem in a somewhat different way. It will be found that this model is capable of reproducing the above mentioned deviation of absorption maxima of unsymmetrical cyanines if the appropriate values of parameters are assumed for various nuclei. This is the quantitative version of Brooker's qualitative explanation.³⁾

In both models there yet remain unsatisfactory points in some finer details. The scope of the present models is, however, to account for a general tendency but not a detailed behaviour. The influence of the end nuclei is introduced as the modification of the potential for π electrons in the first model whereas it is taken into account through the stability of the extreme states in the second model, as Brooker did qualitatively. The latter has a direct connection with the basicity of nuclei, but these two may be considered as standing on the same basis if we consider the electron distribution along the conjugated chain.

§ 1. A modified free-electron model

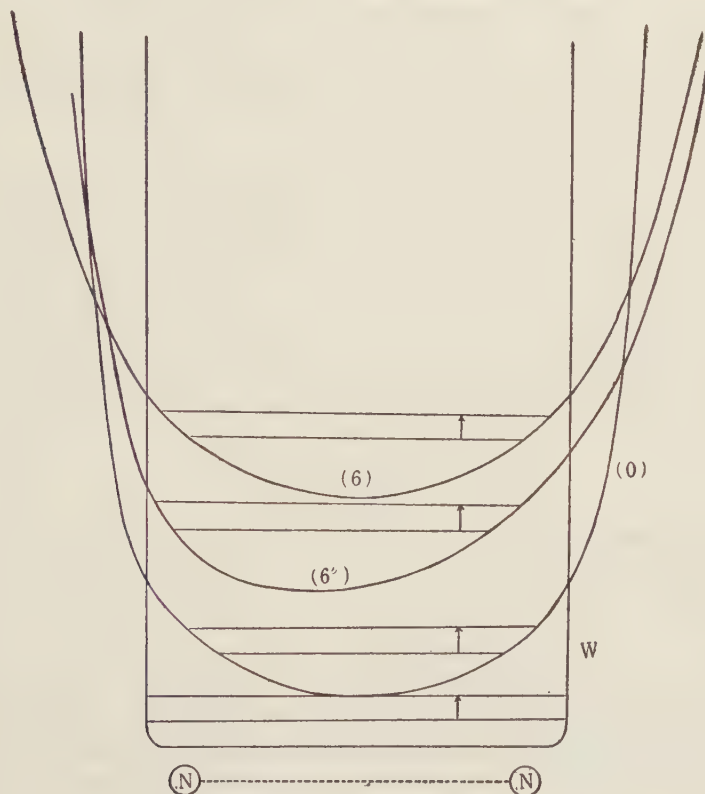


Fig. 3. Pöschl-Teller potential

W :	$\gamma = 1.0$	$\delta \ll 1$
(0):	$\gamma = 3.266$	$\delta = 6.082$
(6):	$\gamma = 7.333$	$\delta = 24.590$
(6'):	a combination of (0) and (6)	

Energy levels giving absorption maxima are indicated. All quantities are measured in atomic units. The numbers (0), (6) and (7') correspond to those in Fig. 4 and Fig. 5.

The Pöschl-Teller one-dimensional potential⁽⁴⁾ is given, in atomic units, by

$$V(x) = \frac{\pi^2}{8b^2} \left\{ \frac{\delta_1(\delta_1+1)}{\sin^2(\pi x/2b)} + \frac{\delta_2(\delta_2+1)}{\cos^2(\pi x/2b)} \right\}, \quad 0 < x < b, \quad 0 < \delta_1, \delta_2. \quad (1.1)$$

When δ_1 and δ_2 are the same, the potential is symmetrical, and, when they become smaller and smaller, the potential approaches to a square well with a breadth of b . Therefore δ_1 and δ_2 give a measure of its deviation from a square well as is shown in Fig. 3. We assume that, in the ground state of a cyanine molecule, lower one-electron levels up to the n_0 -th in this potential are occupied by π electrons which belong to the conjugated chain of the cyanine molecule. We shall refer to n_0 as the Fermi maximum. We further assume that $b = (m + \gamma_1 + \gamma_2)l$ where m is odd and denotes the number of carbon atoms between two nitrogen atoms in the conjugated chain, $l = 1.4\text{\AA}$ is the bond length between two carbon atoms⁽⁶⁾, and γ_1 and γ_2 are parameters, respectively, belonging to one of the end nuclei (see Figs. 3, 4 and 5). In this model the character of end nuclei is represented by γ and δ .

The energy eigenvalue of an electron moving in the potential field given by $V(x)$ was obtained by Pöschl and Teller⁽⁴⁾ as follows:

$$E = (\pi/l)^2 (n + \delta)^2 / [2(m + \gamma_1 + \gamma_2)^2], \quad n = 1, 2, \dots \quad (1.2)$$

where $\delta = (\delta_1 + \delta_2)/2$. In the limit of vanishing δ -value the eigenvalues coincide with those in a square well. If δ is very large the intervals between two adjacent eigenvalues are nearly constant with regard to small values of n . Therefore, when δ becomes larger and larger the spacing of the lower part of the energy spectrum approaches to that of a harmonic oscillator. If we assume that there is no interaction between π electrons we have a first excitation energy of the total system as follows:

$$\Delta E = \left(\frac{\pi}{l} \right)^2 \frac{(2n_0 + \delta_1 + \delta_2 + 1)}{2(m + \gamma_1 + \gamma_2)^2} \quad (1.3)$$

where n_0 is the Fermi maximum, and $2n_0$ is equal to the number of π electrons belonging to the system. If the linear chain of conjugated double bonds is bounded by two nitrogen atoms, for example, as is shown by (6) in Fig. 4, the number of π electrons is equal to $m+1$, where m is the number of carbon atoms between the nitrogen atoms. When the conjugated chain is extending beyond one of nitrogen atoms, for example, as is shown by (5) and (11) of Fig. 4, there is an ambiguity in counting this number. In this case we assume that the number of π electrons is equal to $m+3$. This assumption is somewhat arbitrary. The residual part of the influence of end nuclei is contained in the adjustable parameters, γ_1 , γ_2 , δ_1 and δ_2 . The assumption is justified if the theoretical interpretation of observed facts is satisfactory. In cases of (22) and (24) of Fig. 4 the conjugated chain is bent. Thus the theory is more qualitative in these cases.

We can adjust the values of the parameters γ and δ for each nucleus discussed by Brooker⁽³⁾ so that calculated wave lengths of absorption maxima agree with observed values for both symmetrical and unsymmetrical cyanines in which one definite nucleus is combined

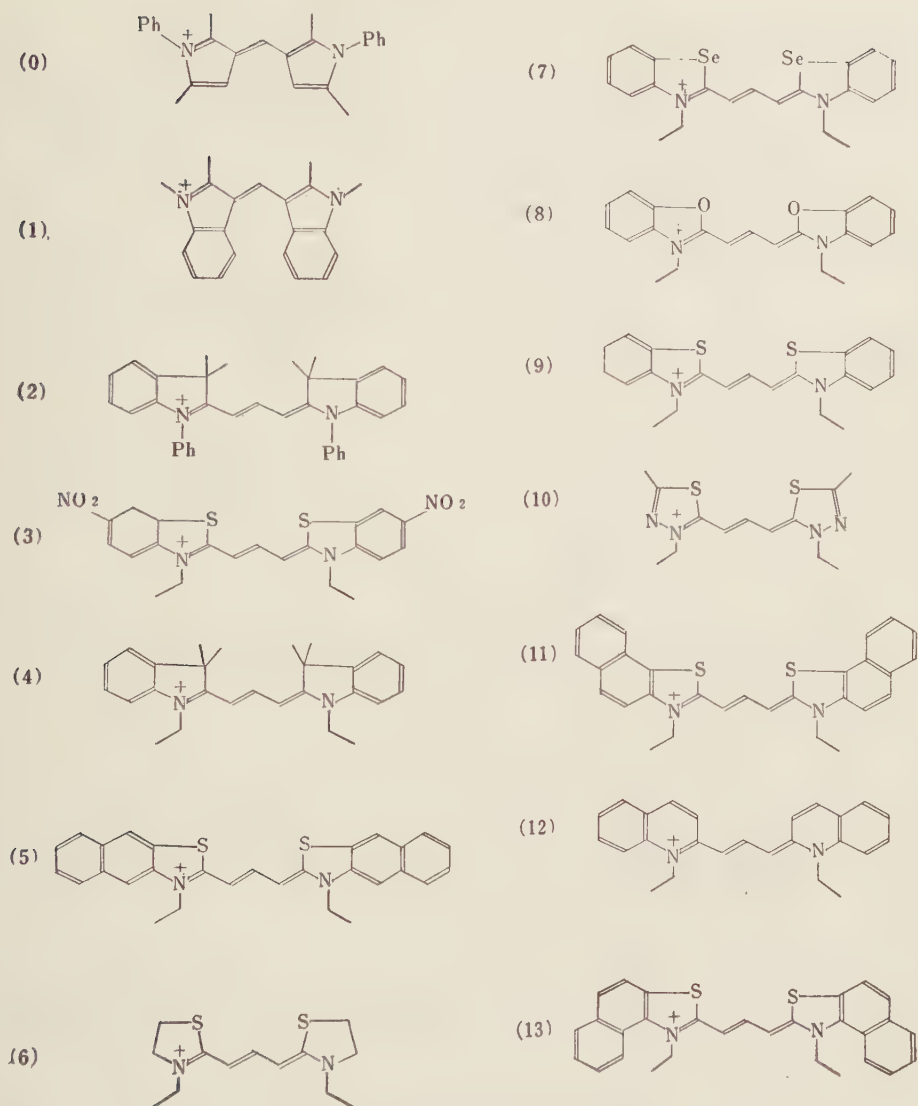


Fig. 4. Symmetrical cyanines

H, C and CH bonds are not written. At corners and ends of bond segments there are carbons. H's should be so added that valences of carbons and positive ions of nitrogens are four and those of neutral nitrogens are three. Therefore open segments mean methyl radicals for example.

with different nuclei (Fig. 5). The definite nucleus is the one contained in (0) of Fig. 4. The adjusted values of the parameters for various nuclei, absorption maxima and deviations are shown in Table 1 and Table 2. In cases of the nuclei contained in the molecules denoted by (0), (3), (8), (9) and (22), there are several observed values to fit for the absorption maxima of the molecules of different length. The number of molecules in each case is respectively 2, 3, 2, 3 and 3. The number of observed data is

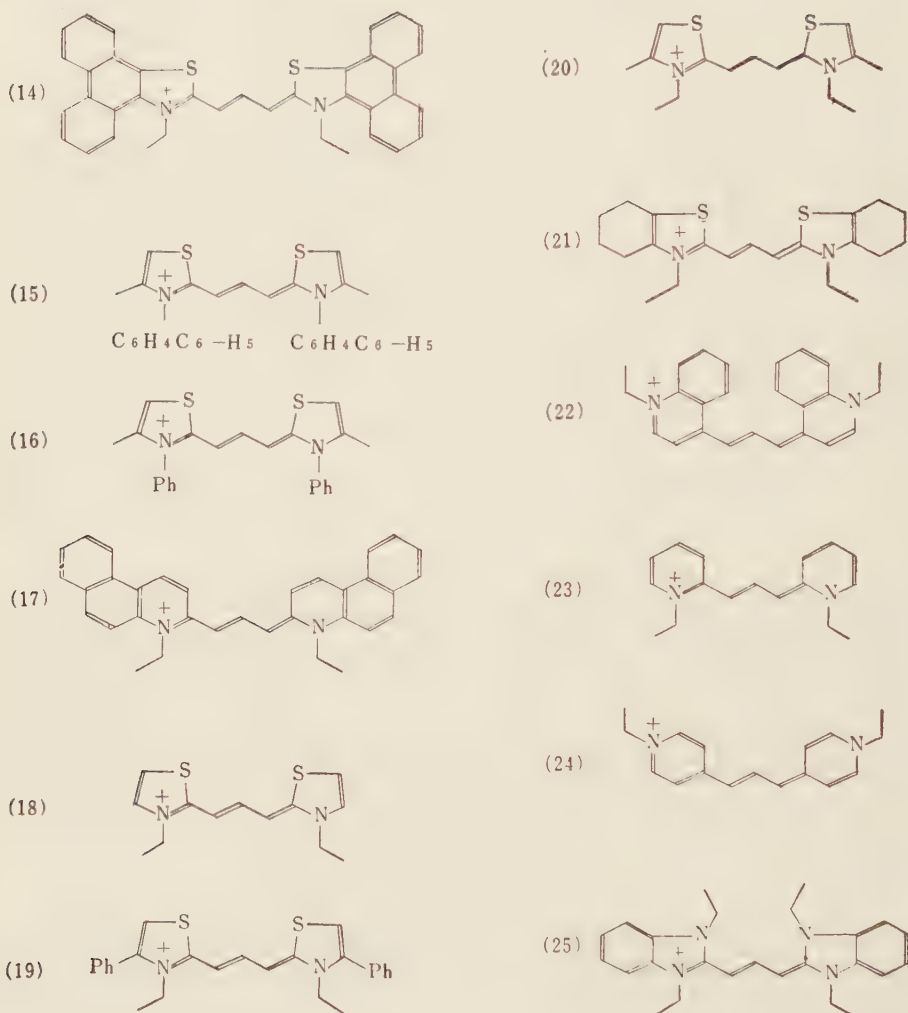


Fig. 4. (continued from the last page)

thus larger than that of adjustable parameters, and some of the calculated values consequently deviate from the observed ones. In the case of (2) we have no value of the parameters to fit the experiment. The calculated values in this case are shown in the parenthesis. The similar differences between calculated and observed values are seen for the molecules of different length. These differences are of the similar magnitude as was seen in the free-electron model without the electronic interaction. Since we are here interested in the deviation we do not discuss the vinylenic shift which was already discussed elsewhere. We see from these tables that a set of γ and δ of the varied nucleus is capable of giving a

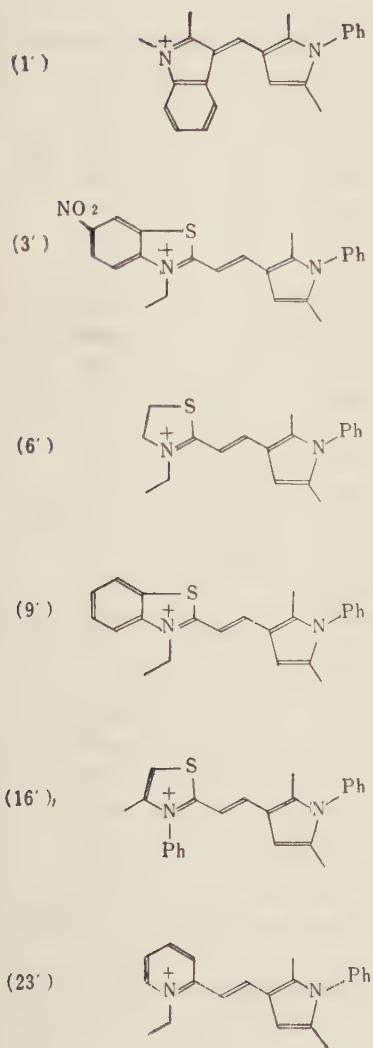


Fig. 5. Unsymmetrical cyanines

measure of the deviation. In fact, the deviation is the greater, the greater the value of δ , but a few exception, as is shown in Table 2.

That the greater δ means the stronger basicity may be seen from a simple consideration. If we consider an extreme case that $\delta_1 \doteq 0$ and $\delta_2 \gg 1$, the Pöschl-Teller potential corresponds to a square-well with a terrace in its right part as is shown in Fig. 2(d). In this case the π -electron density of the molecule is larger in the left part. This means that the nucleus on the right is more basic than the other. From the similar consideration we may infer that the basicity is the stronger, the greater the value of δ . It should be noted here that b itself does not mean the molecular length. Particularly these two are quite different in case of larger γ . Nevertheless the region in which the wave function differs distinctly from zero is located within the real molecule even if the values of γ_1 and γ_2 are very large as is seen from Fig. 3. This owes to large values of δ_1 and δ_2 . In an extreme case that $\gamma_1 = \delta_1 \doteq 0$ and $\delta_2 \gg 1$, the asymmetry of the potential within the molecule is the larger the greater the value of γ_2 . This means that the basicity is the stronger the greater the value of γ . Since thus we see that both γ and δ give a measure of the basicity of nuclei, we may adopt $\gamma\delta$ or $\gamma\delta^2$ as its indicator. As was mentioned above, the latter was seen to be better in fact, but this does not mean a quantitative measure of basicity.

In case of large δ the potential is quite different from the square well. The π electrons in such a potential field are always acted on by a

restoring force but at its minimum point. Therefore the present result means that the π electrons in the shorter cyanines with nuclei of strong basicity cannot move so freely as is expected from the simple free-electron model.

The present model cannot account for the convergence of vinylenic shift because the interaction between π electrons is not taken into account. To calculate the interaction in this model, if it would be taken into account, is very difficult because the eigenfunction is very complicated, as was shown by Pöschl and Teller.⁴⁾ The purpose of the present section is only to see what gives rise to the deviation of the absorption maxima and we do not intend to study finer details. The vinylenic shift itself is somewhat irregular

Table 1. Absorption maxima, and values of parameters for symmetrical cyanines
 λ =wave length of absorption maximum in $m\mu$

molecule	λ	τ	δ
(0)	448.5	3.266	6.082
(1)	490	5.139	11.89
(2)	555(565)	5.700	10.88
(3)	583.5	6.500	13.44
(4)	546.5	7.164	17.59
(5)	569.5	7.655	18.91
(6)	445	7.334	24.59
(7)	570	8.728	24.09
(8)	482.5	7.339	21.43
(9)	557.5	9.258	27.56
(10)	513.5	10.40	38.38
(11)	594	11.40	37.51
(12)	604	12.58	44.19
(13)	595	12.61	45.12
(14)	612.5	13.16	47.22
(15)	564	13.88	57.97
(16)	561	14.02	59.36
(17)	634	15.02	58.08
(18)	542.5	13.84	60.09
(19)	559	14.41	62.61
(20)	556	14.86	66.55
(21)	570	15.58	70.65
(22)	705	18.05	87.76
(23)	562	16.85	82.64
(24)	603	17.33	96.63
(25)	495.5	16.20	86.73

Table 2. Absorption maxima, deviations and values of parameters for unsymmetrical cyanines
 If one of end nuclei of the molecule (0) is replaced by the nucleus of (k) the resultant molecule is denoted by (k') as is shown in Fig. 5.

λ =wave length of absorption maximum in $m\mu$

$\Delta\lambda = \lambda_{max} - \lambda$ =deviation of absorption maximum in $m\mu$

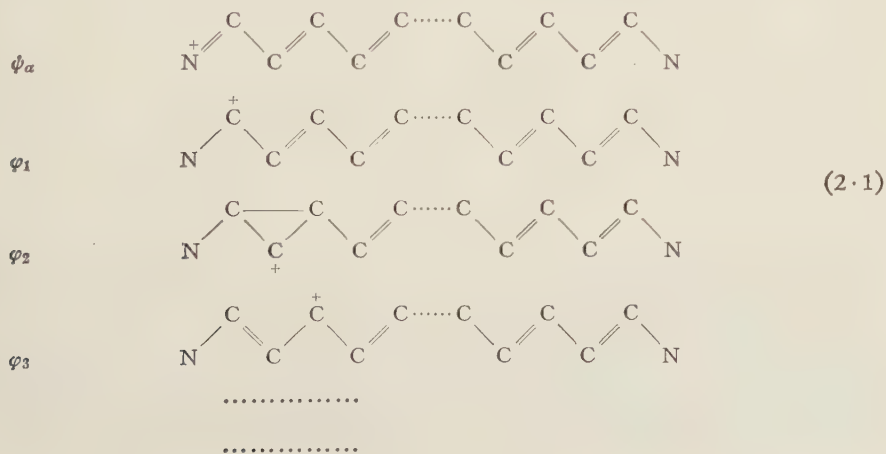
molecule	λ	$\Delta\lambda$	$\tau\delta^2$
(1')	465	4.5	73×10
(2')	488(486)	14.0(21.5)	68×10
(3')	494	22.0	12×10^2
(4')	471	26	22×10^2
(5')	482	27	27×10^2
(6')	417.5	29.5	44×10^2
(7')	476.5	33	51×10^2

molecule	λ	$\Delta\lambda$	$r\delta^2$
(8')	431	34.5	34×10^2
(9')	465.5	37.5	70×10^2
(10')	437.5	43.5	15×10^3
(11')	475	46.5	16×10^3
(12')	474	52.5	25×10^3
(13')	468	54	26×10^3
(14')	476	54.5	29×10^3
(15')	446	60.5	47×10^3
(16')	443	62	49×10^3
(17')	479	62.5	51×10^3
(18')	431.5	64	50×10^3
(19')	439	65	56×10^3
(20')	434	68.5	66×10^3
(21')	439	70.5	78×10^3
(22')	504	73.0	139×10^3
(23')	426	79.5	115×10^3
(24')	440.5	85.5	162×10^3
(25')	380	92	122×10^3

in the shorter cyanines with nuclei of strong basicity. This may show the complex nature of the influence of end nuclei. To account for such a detailed point is beyond the scope of a simple model. Its scope is to give a general tendency.

§ 2. Superposition of chemical structures

We assume that the lower energy states of a cyanine molecule are represented by the superpositions of $m+2$ states which correspond to the following structures:





where only carbons and nitrogens in the conjugated chain are shown, and m is the number of carbon atoms between the nitrogen atoms. The states of the extreme structures are denoted by the normalized wave functions, ψ_a and ψ_b . Those of the intermediate structures are denoted by $\varphi_1, \varphi_2, \dots$, and φ_m , but they are not normalized. The intermediate states, in which one carbon is ionic irrespectively of the position of the ionic carbon, are represented by the superpositions of $\varphi_1, \varphi_2, \dots$, and φ_m . Since these functions are not normalized, we can assume that the deepest state of them is given by*

$$\psi_i = \varphi_1 + \varphi_2 + \varphi_3 + \dots + \varphi_m, \text{ where } \|\psi_i\|^2 = m. \quad (2.2)$$

The superpositions of $\psi_a, \varphi_1, \varphi_2, \dots, \varphi_m$, and ψ_b give $m+2$ levels which are denoted by E_0, E_1, E_2, \dots , and E_{m+1} in the order from the deepest to the highest. The level spectrum is schematically shown by (a) in Fig. 6. The first excitation energy of the molecule is given by $JE = E_1 - E_0$. We shall calculate E_0 and E_1 in an approximate way.

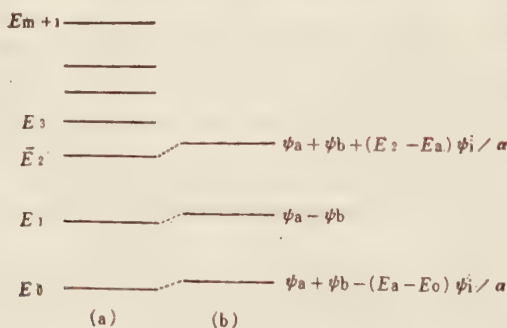


Fig. 6. U

We assume the orthogonality of $\psi_a, \varphi_1, \varphi_2, \dots, \varphi_m$, and ψ_b . This assumption is mathematically correct although it may be technically difficult to make these functions physically significant

*) For example, let us consider all linear combinations of three orthonormal real functions ψ_1, ψ_2 and ψ_3 . If we assume $(\psi_1, H\psi_1) = (\psi_2, H\psi_2) = (\psi_3, H\psi_3) = a$, $(\psi_1, H\psi_2) = (\psi_2, H\psi_1) = b < 0$, and $(\psi_1, H\psi_3) = 0$. The deepest state is $\psi_1 + \sqrt{2} \psi_2 + \psi_3$ and the corresponding deepest level is equal to $E = a + \sqrt{2} b$. In this case we have $m=3$, $\varphi_1 = \sqrt{3} \psi_1/2$, $\varphi_2 = \sqrt{6} \psi_2/2$, $\varphi_3 = \sqrt{3} \psi_3/2$, $\|\varphi_1\|^2 = 3/4$, $\|\varphi_2\|^2 = 3/2$, $\|\varphi_3\|^2 = 3/4$, $\psi_i = \varphi_1 + \varphi_2 + \varphi_3$ and $\|\psi_i\|^2 = 3$.

In this case the mean square norm of φ_k ($k=1, 2, \dots, m$) is equal to unity. If the Hamiltonian of the molecule is denoted by H , and if we neglect its matrix elements between non-adjacent pairs of φ_k 's, the diagonal energy for ψ_i is given by

$$(\psi_i, H\psi_i) = \sum_{k=1}^m (\varphi_k, H\varphi_k) + 2\Re \epsilon \sum_{k=1}^{m-1} (\varphi_k, H\varphi_{k+1}),$$

where $\Re N$ stands for the real part of the number N . If we denote the mean value of $(\varphi_k, H\varphi_k)$ by E_i and that of $\Re \epsilon (\varphi_k, H\varphi_{k+1})$ by γ , we have

$$(\psi_i, H\psi_i) = mE_i + 2(m-1)\gamma \quad (2.3)$$

$$E_i = \sum_{k=1}^m (\varphi_k, H\varphi_k) / m, \quad \gamma = \Re \epsilon \sum_{k=1}^{m-1} (\varphi_k, H\varphi_{k+1}) / (m-1).$$

We further assume that the lower states of the molecule are given by the superpositions of ψ_a , ψ_i , and ψ_b . The total energy of the molecule is then given by the root of the secular equation as follows:

$$\begin{vmatrix} E_a - E & \alpha & 0 \\ \alpha & mE_i + 2(m-1)\gamma - mE & \beta \\ 0 & \beta & E_b - E \end{vmatrix} = 0 \quad (2.4)$$

where the interaction between ψ_a and ψ_b is neglected and

$$\begin{aligned} E_a &= (\psi_a, H\psi_a) & E_b &= (\psi_b, H\psi_b) \\ \alpha &= |(\psi_a, H\psi_i)| & \beta &= |(\psi_b, H\psi_i)|. \end{aligned} \quad (2.5)$$

It may be seen, from the structures (2.1), that the mean diagonal energy E_i is higher than E_a and E_b . The three roots of this secular equation give the approximate values of the first three levels of the energy spectrum. If we neglect, as before, the interactions between the non-adjacent pairs of the structures (2.1), ψ_i is replaced by φ_1 and φ_m respectively in α and β . If we put

$$\begin{aligned} \varepsilon &= E - (E_i + 2\gamma - 2\gamma/m) \\ \varepsilon_a &= E_i - E_a + 2\gamma, & \varepsilon_b &= E_i - E_b + 2\gamma \end{aligned} \quad (2.6)$$

the secular equation can be written as

$$\begin{vmatrix} \varepsilon + \varepsilon_a - 2\gamma/m & \alpha & 0 \\ \alpha & m\varepsilon & \beta \\ 0 & \beta & \varepsilon + \varepsilon_b - 2\gamma/m \end{vmatrix} = 0. \quad (2.7)$$

The difference of two roots of the secular equation (2.4) is equal to the difference of the corresponding two roots of this equation. Therefore the excitation energy is determined by ε_a , ε_b , α , β , γ , and m . We assume that the interaction integrals α , β , and γ are independent of the number, m , of carbon atoms and that γ is independent of the end nuclei. The former two, α and β , depend on the end nuclei A and B respectively. We

further assume that the energy differences ϵ_a and ϵ_b are characteristic of nuclei A and B respectively but independent of m . It may be seen, from the structures given by (2.1), that these assumptions are nearly correct if we neglect finer details.

When the ion is symmetric, the secular equation can easily be solved and the three levels are given by

$$\begin{aligned} E_2 &= E_a + [(\epsilon_a - 2\gamma/m) + \{(\epsilon_a - 2\gamma/m)^2 + 8\alpha^2/m\}^{1/2}]/2 \\ E_1 &= E_a \end{aligned} \tag{2.8}$$

Table 3. Values of parameters for symmetrical cyanines
All quantities are measured in atomic units

molecule	$\gamma = -1$		$\gamma = -0.5$	
	ϵ_a	α^2	ϵ_a	α^2
(0)	0.32	0.2087	0.15	0.1147
(1)	0.34	0.1936	0.17	0.1076
(3)	0.36	0.1636	0.19	0.09138
(6)	0.37	0.2233	0.20	0.1286
(9)	0.38	0.1761	0.21	0.1005
(16)	0.39	0.1769	0.22	0.1018
(23)	0.40	0.1786	0.23	0.1036

Table 4. Symmetrical cyanines
 λ =wave length of absorption maximum in $m\mu$
 m =number of carbon atoms between nitrogen atoms

molecule	m	$\lambda_{obs.}$	$\lambda_{calc.}$	
			$\gamma = -0.5$	$\gamma = -1$
(0)	5	448.5	448.5	448.5
	7	541	527.5	529
(1)	3		393	396
	5	490	490	490
	7		580	580
(3)	3	449.5	465	469.5
	5	583.5	583.5	583.5
	7	682	695.5	693.5
(6)	3		352	356.5
	5	445	445	445
	7		531.5	530
(9)	3	423	440	446
	5	557.5	557.5	557.5
	7	650	668.5	665
(16)	3		441	443
	5	561	561	561
	7		674.5	670.5
(23)	3		440	447
	5	562	562	562
	7		677.5	673

$$E_0 = E_a + [(\varepsilon_a - 2\gamma/m) - \{(\varepsilon_a - 2\gamma/m)^2 + 8\alpha^2/m\}^{1/2}]/2.$$

These are shown by (b) in Fig. 6. The first excitation energy of the symmetrical cyanine is

$$\Delta E = E_1 - E_0 = [\{(\varepsilon_a - 2\gamma/m)^2 + 8\alpha^2/m\}^{1/2} - (\varepsilon_a - 2\gamma/m)]/2. \quad (2.9)$$

We determine the values of the parameters ε_a and α , which are characteristic of the nucleus A, by comparing this equation with observed absorption maxima of symmetrical cyanines in the cases of $\gamma = -0.5$ and -1 atomic units. The result is shown in Table 3. The calculated absorption maxima are shown in Table 4.

Next we calculate the wave lengths and deviations of absorption maxima for unsymmetrical cyanines by using these values of parameters. The secular equation is numerically solved by the method of Newton. The values of parameters are so adjusted that we obtain a uniformly better agreement with experiment, by repeating the computations for both symmetrical and unsymmetrical cyanines (see Table 4). The result is shown in Table 5. We see that the characteristic properties of the experimental result obtained by Brooker are reproduced in a somewhat qualitative and nearly equal way by the present model for both cases of $\gamma = -0.5$ and -1 . Particularly it should be noted that the energy difference parameter ε_a gives a measure of the deviation, the value of the parameter α is nearly independent of nuclei, and the calculated deviation of the absorption maximum is the greater the larger the number of carbons, in qualitative agreement with experiment. The last fact means that the model accounts for the convergence of vinylenes shifts in the case of unsymmetrical cyanines since the vinylenes shifts are non-convergent in the case of symmetrical cyanines.

The large ε_a implies that the extreme structure is more stable, and this means the stronger basicity of the end nucleus A. The detailed account for this circumstance was given by Brooker³¹. The present result shows that Brooker's qualitative expectation is quantitatively correct. As for the meaning of α we may consider as follows. If we assume that ψ'_a and φ_1 are represented by the atomic orbital method, the energy integral α means a kind of the interaction between two π electrons on the N atom of the nucleus A and on the adjacent C atom of the chain under the influence of the other π electrons and the molecular core. In the similar way we can consider that γ represents the mean of the similar interaction energy between two π electrons on the adjacent C atoms in the conjugated chain. This consideration may make plausible the justification of neglecting the interactions between non-adjacent pairs of states. The molecular length has an influence on these energies, but it may be possible to assume the variation of the influence to be small within the small range of m , as is done in the present model.

The present model roughly accounts for the vinylenes shift by the number of intermediate states which gives the measure of the molecular length. The explanation may hold only in a restricted range of the molecular length because the assumption for parameters may not be valid for the broader range of the number of carbons. In this model, when the number m of carbon atoms is very large, the wave length of the absorption maximum of a symmetrical cyanine reduces to

Table 5. Unsymmetrical cyanines

 λ =wave length of absorption maximum in $m\mu$ $\Delta\lambda = \lambda_{m\epsilon\alpha\gamma_1} - \lambda$ =deviation of absorption maximum in $m\mu$

molecule	m	λ_{obs}	$\Delta\lambda_{obs}$	$\gamma = -0.5$		$\gamma = -1$	
				λ_{calc}	$\Delta\lambda_{calc}$	λ_{calc}	$\Delta\lambda_{calc}$
(1')	3	465	4.5	375	3	374.5	5.5
	5			462	7.5	463	6.5
	7			543	11	544	10.5
(3')	3	494	22	401.5	12.5	403	14
	5			491	25	497	19
	7			568	43.5	563.5	48
(6')	3	417.5	29.5	330.5	27	333.5	27
	5			400	47	400.5	46.5
	7			458.5	71	457.5	72
(9')	3	465.5	37.5	369	32.5	371.5	33.5
	5			442	61	440.5	62.5
	7			499.5	98.5	496.5	100.5
(16')	3	443	62	358	44	360	43.5
	5			423	82	421	84
	7			472	129	469	131
(23')	3	426	79.5	344.5	57	347	58.5
	5			402	103.5	400.5	105
	7			444	158.5	400.5	160.5

$$\lambda = hc / \Delta E = hc(m\epsilon_a - 2\gamma) / (2\alpha^2). \quad (2 \cdot 10)$$

We have no real example in which m is so large that this equation is valid and the present model may not be valid in such a case. Nevertheless this equation shows the vinylenic shift in a qualitative way.

Brooker showed that the vinylenic shift is convergent in case of unsymmetrical cyanines whereas it is non-convergent in case of symmetrical cyanines. This may qualitatively be understood in the following way. In the present model the energies E_a and E_b of the extreme states are respectively displaced to E_a' and E_b' by the influence of the intermediate states. The absorption maximum corresponds to the excitation energy $E_b' - E_a'$ when E_b' is higher than E_a' . The influence is the smaller the longer the molecular length, and the excitation energy ultimately converges to the limit $E_b - E_a$ when the molecular length becomes infinitely long. In case of unsymmetrical cyanines $E_b - E_a$ has a finite value owing to the difference of two nuclei while it vanishes for symmetrical cyanines, that is, the wave length of the absorption maximum has an upper bound in the former case while it has no bound in the latter case. This phenomenon results in the difference in question.

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On the Wave Theory of Light in General Relativity, III

— *Electromagnetic Four Potential* —

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The character of the generalized field equation for the electromagnetic four potential in curved space-time is investigated. The main problems treated in this paper are (i) the problems concerning the gauge transformation and the auxiliary condition, and (ii) those concerning the properties of the spherical wave solutions of the field equation in static spherically symmetric space-time.

§ 1. Introduction

In Minkowski space-time the wave character of light has been investigated in detail, while in curved space-time it has scarcely been studied. The main purpose of the two papers published by one of the present writers was to investigate this problem.^{1,2)} In the first paper the relation between the wave front represented by the solution of the wave equation of light and the paths of light, i. e., the null geodesics of the space-time, has been clarified. In the second paper the properties of light wave have been studied from the standpoint of the electromagnetic optics by examining the spherical wave solution of the Maxwell equation generalized in an S (abbreviation of 'static spherically symmetric space-time'). By these investigations it was made clear that the character of the wave solution of the generalized Maxwell equation in curved space-time, especially that of the spherical wave solutions in an S , may be considered as a natural generalization of the corresponding ones in Minkowski space-time.

In this paper we shall investigate the properties of the electromagnetic four potential φ_i in curved space-time by solving its field equation in order to clarify whether the above stated circumstances may hold for φ_i or not. In treating the electromagnetic phenomena the potential φ_i is used frequently together with the field strength F_{ij} (or its physical components \mathbf{E} and \mathbf{H}). Historically the potential φ_i was introduced as an auxiliary quantity for solving the Maxwell equation. Later, however, it has become an essential quantity for constructing the electromagnetic theory, especially when the Lagrangian formalism is concerned. And moreover it is needless to say about its important role in quantum electrodynamics. Taking account of such circumstances it will not be meaningless to investigate the property of the potential φ_i in discussing the electromagnetic phenomena in curved space-time. In [II] we have made clear the properties of the spherical wave solutions for

E and H in an S , but the corresponding solutions for φ_i have not yet been obtained. Therefore we shall obtain its concrete form, and in connection with this we shall treat the gauge transformation and shall study its properties especially in an S .

§ 2. Generalized field equation for four potential

In [II] we have examined the general character of the field equation for four potential in curved space-times. Now we shall investigate the characteristic feature of the gauge transformation in curved space-time and shall obtain some special solutions of the equation, restricting the problems within the case of free radiation field as in [II].

First in this section we shall obtain the concrete form of the field equation in an S . In the absence of four current the generalized Maxwell equation for the field strength F_{ij} is given by

$$\nabla_j F^{ij} = 0, \quad (i, j, k, \dots = 1, \dots, 4) \quad (2.1)$$

$$F_{ij,k} = \nabla_k F_{ij} = 0, \quad (2.2)$$

where ∇_k denotes the covariant derivative with respect to x^k . By (2.2) the existence of the four potential φ_i satisfying*

$$F_{ij} = \nabla_i \varphi_j - \nabla_j \varphi_i = \partial_i \varphi_j - \partial_j \varphi_i, \quad (\partial_i \equiv \partial / \partial x^i) \quad (2.3)$$

is assured. Inserting this into (2.1) we obtain the generalized field equation for four potential:

$$g^{ij} \nabla_j \nabla_i \varphi_l - \nabla_l (\nabla_j \varphi^j) + K^i_l \varphi_i = 0, \quad (2.4)$$

where K_{ij} is the Ricci tensor of the space-time. If we impose the relation

$$\nabla_i \varphi^i = 0, \quad (2.5)$$

the field equation (2.4) becomes simply**

$$g^{ij} \nabla_i \nabla_j \varphi_l + K^i_l \varphi_i = 0. \quad (2.6)$$

The relation (2.5) can be regarded as a generalization of the usual Lorentz condition, therefore we shall call it 'generalized Lorentz condition.'

A gauge transformation for φ_i is given by

$$\varphi_i = \bar{\varphi}_i + \nabla_i A = \bar{\varphi}_i + \partial_i A \quad (2.7)$$

where A is an arbitrary scalar function. Though F_{ij} is invariant under (2.7) for any A , (2.5) and (2.6) are not always invariant. The condition for the invariance of (2.5) and (2.6) under (2.7) is given by

$$\square_s A \equiv g^{ij} \nabla_i \nabla_j A = 0, \quad (2.8)$$

* In this paper the tensor components, not the physical components, of φ_i are used.

** Concrete form of (2.6) is not always simpler than that of (2.4), though the latter seems to be simpler at first sight.

where \square_κ denotes the generalized d'Alembertian for a scalar. This is the natural generalization of the well-known relations of the so-called Lorentz gauge in Minkowski space-time.

On the other hand, if we give up the requirement of the covariancy, we can take another way of restricting the gauge, namely, the generalization of the Coulomb gauge. That is, we can take

$$\nabla_\kappa \varphi^\kappa = 0, \quad (\kappa = 1, 2, 3) \quad (2.5')$$

as the auxiliary condition for φ_i instead of (2.5). If we use this (2.5') and take any time orthogonal coordinate system, the field equation takes the following form instead of (2.6):

$$\begin{aligned} g^{ij} \nabla_i \nabla_j \varphi_\kappa - \nabla_\kappa (\nabla_i \varphi^i) + K_\kappa^i \varphi_i &= 0, \quad (\kappa = 1, 2, 3) \\ g^{i\kappa} \nabla_i \nabla_\kappa \varphi_4 + K_\kappa^i \varphi_i &= 0 \quad (\kappa = 1, 2, 3). \end{aligned} \quad (2.6')$$

We can consider the meaning of these equations analogously to the case of Minkowski space-time. It is also to be remarked that the field equation for φ_i is of the elliptic type by virtue of the fact that the signature of the space-time is given by $(---+)$.

Next we shall obtain the concrete forms of (2.5) and (2.6) in an S taking the canonical coordinate system in order to study the properties of the generalized equation for φ_i . In this coordinate system the line element takes the form:

$$ds^2 = -A(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + C(r)dt^2. \quad (x^i \equiv r, \theta, \phi, t) \quad (2.9)$$

Inserting this into (2.5) and (2.6), we obtain as the generalized field equation for φ_i :

$$\begin{aligned} \square_0 \varphi_1 + \tau_2 \partial_1 \varphi_1 + C^{-1} (2/r - C'/C) \partial_4 \varphi_1 + \tau_3 \varphi_1 &= 0, \\ \square_0 \varphi_2 + \tau_4 \partial_1 \varphi_2 - (2/rA) \partial_2 \varphi_1 + (2 \cot \theta / r^2 \sin^2 \theta) \partial_3 \varphi_3 + (1/r^2 \sin^2 \theta) \varphi_2 &= 0, \\ \square_0 \varphi_3 + \tau_4 \partial_1 \varphi_3 - (2/rA) \partial_3 \varphi_1 + (2 \cot \theta / r^2) \partial_2 \varphi_3 - (2 \cot \theta / r^2) \partial_3 \varphi_2 &= 0, \\ \square_0 \varphi_4 + \tau_5 \partial_1 \varphi_4 - (C'/AC) \partial_4 \varphi_1 &= 0, \end{aligned} \quad (2.10)$$

where

$$\begin{aligned} \square_0 &= -A^{-1} \partial_{11} - (r^2 \sin \theta)^{-1} \partial_2 (\sin \theta \partial_2) - (r \sin \theta)^{-2} \partial_{33} + C^{-1} \partial_{44}, \quad (\partial_{ij} \equiv \partial^2 / \partial x^i \partial x^j) \\ \tau_2 &= (2A)^{-1} (3A'/A - C'/C - 8/r), \\ \tau_3 &= A''/2A^2 - C''/2AC + 3A'/rA^2 + C'^2/2AC^2 + A'C'/2A^2C - A'^2/A^3 \\ &\quad - 2/Ar^2 - C'/ACr, \\ \tau_4 &= (2A)^{-1} (A'/A - C'/C), \quad \tau_5 = (2A)^{-1} (A'/A + C'/C - 4/r), \end{aligned} \quad (2.11)$$

and as the generalized Lorentz condition:

$$-A^{-1} \partial_1 \varphi_1 - r^{-2} \partial_2 \varphi_2 - (r \sin \theta)^{-2} \partial_3 \varphi_3 + C^{-1} \partial_4 \varphi_4 + \tau_1 \varphi_1 - r^{-2} \cot \theta \cdot \varphi_2 = 0, \quad (2.12)$$

where

$$\tau_1 = (2A)^{-1} (A'/A - C'/C - 4/r), \quad (2.13)$$

respectively. Here primes indicate derivatives with respect to $x^1 \equiv r$.

In the next section we shall investigate the properties of the gauge transformation using (2.10) and (2.12).

N. B. We shall give some remarks concerning the auxiliary condition :

(1) As was already discussed in [II] the equation (2.5) is not a unique one which can be used as the auxiliary condition in curved space-time. For instance, we may take another form :

$$K_{;i} \varphi_i - \nabla_i (\nabla_j \varphi^j) = 0 \quad (2.14)$$

instead of (2.5). In this case we obtain

$$\square \varphi_i \equiv g^{ij} \nabla_j \nabla_i \varphi_i = 0 \quad (2.15)$$

as the field equation. This seems to be simpler than (2.6), but in some coordinate system its concrete form is not simpler than that of (2.6). Further when the space-time reduces to the Minkowskian, (2.14) reduces to the form :

$$\partial_i \varphi^i = \text{const.} \quad (2.16)$$

in cartesian coordinate system, which does not coincide with, and is less stringent than, the usual Lorentz condition. Therefore it seems to us that (2.5) is more natural as a generalization of the usual Lorentz condition than (2.14).

(2) We can take

$$r^{-2} \sqrt{C/A} \partial_1 (r^2 / \sqrt{AC} \cdot \varphi_1) + r^{-2} (\sin \theta)^{-1} \{ \partial_2 \sin \theta \varphi_2 + (\sin \theta)^{-1} \partial_3 \varphi_3 \} - C^{-1} \partial_4 \varphi_4 = 0, \quad (2.17)$$

as the auxiliary condition for φ_i in place of (2.5). It is always possible to obtain a solution satisfying (2.17) from any solution φ_i of (2.4) by a suitable gauge transformation (2.7). The field equation and the auxiliary condition (2.17) are invariant under the gauge transformation when the gauge A satisfies the equation :

$$r^{-2} \sqrt{C/A} \partial_1 (r^2 / \sqrt{AC} \cdot \partial_1 A) + r^{-2} (\sin \theta)^{-1} \{ \partial_2 \sin \theta \partial_2 A + (\sin \theta)^{-1} \partial_3 \partial_3 A \} - C^{-1} \partial_4 \partial_4 A = 0. \quad (2.18)$$

In this case the field equation for φ_4 becomes

$$r^{-2} \sqrt{C/A} \partial_1 (r^2 / \sqrt{AC} \cdot \partial_1 \varphi_4) + r^{-2} (\sin \theta)^{-1} \{ \partial_2 \sin \theta \partial_2 \varphi_4 + (\sin \theta)^{-1} \partial_3 \partial_3 \varphi_4 \} - C^{-1} \partial_4 \partial_4 \varphi_4 = 0, \quad (2.19)$$

hence it is clear that there exists a solution in which $\varphi_4 = 0$. Furthermore we can always obtain a solution in which $\varphi_4 = 0$ from any solution by a gauge transformation satisfying (2.18). (Compare this result with those of the next section. By using (2.5) we cannot obtain a solution whose $\varphi_4 = 0$ without some additional condition for the solution.) (2.17) is not tensorial and this formalism is not invariant under arbitrary coordinate transformations.

§ 3. Some properties of the gauge transformation

Taking the cartesian coordinates in Minkowski space-time, the general solution of the field equation (2.6) for φ_i is formed by superposing the plane waves. In this case, as is well known, we can choose A so that φ_4 and the longitudinal wave vanish within the Lorentz gauge.* The corresponding character of the solution of the field equation in curved space-times is not yet known. Therefore it will not be useless to investigate this problem, especially in an S . It is, however, difficult to determine distinctly how to generalize the concept of the plane wave in curved space-times.** Hence in this paper we shall discuss only the problem concerning the elimination of φ_4 in an S . Laying stress on the covariancy of the formalism, we shall use (2.10) and (2.12) as the field equation and the auxiliary condition respectively. As will be seen in due course, it is generally impossible to deduce $\varphi_4=0$ unconditionally. This is a remarkable feature in the solution of the field equation of the electromagnetic four potential in curved space-time compared with that in flat space-time.

By the following series of mathematical lemmas the answer of the above problem is given :

(I) Let φ_i be any solution of FL (abbreviation of 'the generalized field equation (2.10) with the generalized Lorentz condition (2.12)') in an S which is not an S_η . Then a necessary and sufficient condition that φ_i be transformable into the form $\bar{\varphi}_4=0$ by a suitable GL (abbreviation of 'the gauge transformation (2.7) with the condition (2.8)') is that $F_{ij}=\partial_i\varphi_j-\partial_j\varphi_i$ satisfy $F_{14}=0$.

Here the S_η is a special kind of S introduced by one of the present writers, and it is an S in which $C(r)$ of the line element (2.9) is a positive constant in any canonical coordinate system, and its properties have been investigated in detail.³ Minkowski space-time and that of the Einstein universe are both S_η , while both of the space-time given by the exterior solution of Schwarzschild and de Sitter space-time are not S_η ***.

(II) In an S which is not an S_η , if φ_i is any solution of FL in which $\varphi_4=0$, it is always possible to make $\bar{\varphi}_1=\bar{\varphi}_4=0$ by a suitable GL.

Summarizing the above (I) and (II) we have

(III) When an S is not an S_η , a necessary and sufficient condition that we can deduce a solution $\bar{\varphi}_i$ satisfying $\bar{\varphi}_1=\bar{\varphi}_4=0$ from an arbitrary solution φ_i of FL by a suitable GL is $F_{11}=0$.

Next when the space-time is an S_η the following lemmas hold :

(IV) In any S_η it is possible to make $\bar{\varphi}_1=0$ by a suitable GL from any solution φ_i of FL.

(V) Let S be an S_η and φ_i be any solution of FL satisfying $\varphi_4=0$. Then a necessary and sufficient condition that we can make $\bar{\varphi}_1=\bar{\varphi}_4=0$ by a suitable GL is that $F_{11}=0$.

From the above two lemmas it is clear that (III) holds also when the S is an S_η .

* We can make not only φ_4 but also any component of φ_i be zero.

** See Appendix II in [II].

*** S_η plays an important role in the theory of the superposition of S 's.⁴⁾

Therefore, we obtain

(VI) In any S , a necessary and sufficient condition that we can make $\bar{\varphi}_1 = \bar{\varphi}_4 = 0$ by a suitable GL from any solution φ_i of FL is that $F_{14} = 0$.

The proofs of (I), ..., (V) are given in the Appendix. By the above lemmas the relation between the GL which makes $\bar{\varphi}_1 = 0$ and the character of the space-time, and that between the GL and the properties of the TE wave (transverse electric wave: $F_{14} = 0$) solution are clarified. Here we shall repeat the main points:

(i) In order to make $\bar{\varphi}_4 = 0$ by a suitable GL from any solution φ_i of FL, it is necessary that the space-time be a special one, i. e., an S_η .

(ii) If the space-time is not an S_η , we can make $\bar{\varphi}_4 = 0$ by a GL only from a special solution, i. e., the one whose $F_{14} = 0$.

§ 4. Solution of FL

In this section we shall solve FL and shall get φ_i 's which correspond to the three kinds of spherical waves of E and H in S obtained in [II], i. e., TE (transverse electric), TM (transverse magnetic) and TEM (transverse electromagnetic) waves.

(*) TE wave. Let φ_i be the potential of an arbitrary TE wave. Then from the results obtained in the last section we can make $\bar{\varphi}_1 = \bar{\varphi}_4 = 0$ by a suitable GL. If we substitute this relation in (2.10) and (2.12) we have

$$\square_0 \varphi_2 + \tau_4 \partial_1 \varphi_2 + 2 \cot \theta (r \sin \theta)^{-2} \partial_3 \varphi_3 + (r \sin \theta)^{-2} \varphi_2 = 0, \quad (4.1)$$

$$\square_0 \varphi_3 + \tau_4 \partial_1 \varphi_3 + 2r^{-2} \cot \theta \partial_2 \varphi_3 - 2r^{-2} \cot \theta \partial_3 \varphi_2 = 0, \\ \partial_2 \varphi_2 + (\sin \theta)^{-2} \partial_3 \varphi_3 + \cot \theta \varphi_2 = 0, \quad (4.2)$$

where, for simplicity the bar of $\bar{\varphi}_i^*$ is omitted. Eliminating φ_3 from the first of (4.1) and (4.2) we have

$$\square_0 \varphi_2 + \tau_4 \partial_1 \varphi_2 - 2r^{-2} \cot \theta \partial_2 \varphi_2 + r^{-2} (1 - \cot^2 \theta) \varphi_2 = 0. \quad (4.3)$$

Then putting the following assumption of the separation of variables:

$$\varphi_2 = R_2(r, t) \Theta_2(\theta) \Phi_2(\phi), \quad \varphi_3 = R_3(r, t) \Theta_3(\theta) \Phi_3(\phi), \quad (4.4)$$

we have from (4.3)

$$[d_{33} + m^2] \Phi_2 = 0, \\ [d_{22} + 3 \cot \theta d_2 + \{n(n+1) + \cot^2 \theta - 1 - m^2 \sin^{-2} \theta\}] \Theta_2 = 0, \quad (4.5) \\ [-A^{-1} \partial_{11} + \tau_4 \partial_1 + C^{-1} \partial_{44} + n(n+1)r^{-2}] R_2 = 0,$$

* Differentiating the first and the second equations of (4.1) with respect to ϕ and θ respectively, we obtain by subtraction

$$\{\square_0 + \tau_4 \partial_1 + 2r^{-2} \cot \theta \partial_2 - (r \sin \theta)^{-2}\} F_{23} = 0,$$

which coincides with (4.8) i.e., (3.4) in [II].

where m and n are constants of separation, and $d_{\infty}=d^2/d\phi^2$, etc. From (4.5) and the second of (4.1) we have the following equation for φ_3 :

$$d_3\varphi_3=a\varphi_2, \quad \theta_3=-b^{-1}\sin^2\theta(d_2\theta_2+\theta_2\cot\theta), \quad R_3=ba^{-1}R_2, \quad (4.6)$$

where a and b are arbitrary constants.

φ_2 and φ_3 are determined from (4.5) and (4.6) respectively and thus φ_i is completely determined. We can easily show that E and H made from this solution φ_i in the case of $n \neq 0$ coincide with those obtained in [II], and that we can rewrite the φ_i in the form

$$\varphi_2=-\{n(n+1)\sin\theta\}^{-1}(\partial_{\infty}ru), \quad \varphi_3=\sin\theta\{n(n+1)\}^{-1}(\partial_2ru), \quad (4.7)$$

where u is a function satisfying

$$(r\sqrt{AC})^{-1}\partial_1(\sqrt{C/A}\partial_1(ru)) + r^{-2}(\sin\theta)^{-1}\partial_2(\sin\theta\partial_2u) \\ + (r\sin\theta)^{-2}\partial_{\infty}u - C^{-1}\partial_{44}u = 0. \quad (4.8)$$

φ_i for the case of $n=0$ is nothing but the TEM wave which will be treated in the following (ii).

(ii) TEM wave. In this case $E_r=0$ holds again, so we have (4.1), ..., (4.6). Since $H_r=0$ further, however, we have

$$\partial_{\infty}\varphi_2 - \partial_2\varphi_3 = 0. \quad (4.9)$$

Rewriting (4.2) in the form

$$\sin\theta\partial_2(\sin\theta\varphi_2) = -\partial_3\varphi_3, \quad (4.10)$$

we have from (4.9) and (4.10) the following equation:

$$[\partial_{33}\sin\theta + \sin\theta\partial_2\sin\theta\partial_2\sin\theta]\varphi_2 = 0, \\ [\partial_{33} + \sin\theta\partial_2\sin\theta\partial_2]\varphi_3 = 0, \quad (4.11)$$

of which the first concerns φ_2 alone and the second φ_3 alone. Now by putting the same assumption of the separation of variables (4.4), we have from (4.5) and (4.11)

$$[\partial_{\infty} + 3\cot\theta\partial_2 + (\cot^2\theta - 1 - m^2\sin^{-2}\theta)]\theta_2 = 0, \quad (4.12)$$

which is nothing but the second of (4.5) for $n=0$. By substituting this in the remaining equation of (4.5), we can easily prove that there exist dispersionless wave solutions. Of course E and H obtained from this solution coincide with those in [II]. From these considerations we may conclude that TEM waves are TE waves corresponding to $n=0$.

(iii) TM wave. Next we shall obtain φ_i which gives TM wave. Since we have no theorem for TM waves which corresponds to the lemmas obtained in §3 concerning TE waves, it is somewhat complicated to obtain TM solution of FL. In this case, however, if we take (2.4) without the restriction (2.5) in place of (2.6) we can easily solve it. Since (2.4) is invariant under any gauge transformation (2.7), from the relation $H_r=0$, i. e., $\partial_2\varphi_3 - \partial_3\varphi_2 = 0$, we can make $\varphi_2 = \varphi_3 = 0$ by a suitable (2.7). (Bar of $\bar{\varphi}_i$ being

omitted for simplicity as before.) Then, corresponding to (4.7), we construct the φ_i of the form

$$\begin{aligned}\varphi_1 &= -\{n(n+1)\}^{-1} \sqrt{A/C} \partial_4(rv), & \varphi_4 &= -\{n(n+1)\}^{-1} \sqrt{A/C} \partial_1(rv), \\ \varphi_2 &= \varphi_3 = 0,\end{aligned}\tag{4.13}$$

where v is a function satisfying the following equation :

$$\begin{aligned}(r\sqrt{AC})^{-1} \partial_1(\sqrt{C/A} \partial_1(rv)) + r^{-2} (\sin \theta)^{-1} \partial_2(\sin \theta \partial_2 v) \\ + (r \sin \theta)^{-2} \partial_{33} v - C^{-1} \partial_{44} v = 0.\end{aligned}\tag{4.14}$$

As is easily verified this φ_i satisfies (3.4) and E and H made from it coincide with those obtained in [II]. The general TM solution (of course under the assumption of the separation of variables) is given by the summation of a solution of the form (4.13) and the general TEM solution to within an arbitrary gauge transformation. This fact is also seen from the results of [II].

Though φ_i given by (4.13) is a solution of (2.4), it does not satisfy (2.6) since the generalized Lorentz condition (2.5) is not satisfied. Of course we can deduce a solution of (2.6) by making a suitable gauge transformation from (4.13), but we have not the relation $\varphi_2 = \varphi_3 = 0$ any more.

In [II] we have seen remarkable symmetry between TE waves and TM waves in solving the generalized Maxwell equation for F_i . However, we have seen above that this symmetric correspondence between two types of solutions fails in the case of the solution φ_i of FL by virtue of the generalized Lorentz condition.

Lastly it should be noticed that though we have put the assumption of the separation of variables with respect to E and H in [II] and to φ_i in the present paper, both results have given the same field.

§ 5. Discussion of the results

In this paper we have so far investigated some properties of the four potential in curved space-time and made clear their difference from those in Minkowski space-time. Further we have obtained spherical wave solutions of the field equation for four potential φ_i in an S and studied their character. In this section we shall examine some questions arising from the results obtained in the preceding sections. The questions discussed in the following are concerned with the auxiliary condition and the expression of the free radiation field. We shall deal with these problems separately, though their contents have some connection with each other.

First we shall examine the problem of the auxiliary condition. In Minkowski space-time the auxiliary condition is imposed mainly in order to simplify the form of the field equation and to restrict the freedom of the gauge transformation. In curved space-time, however, the condition does not always simplify the concrete form of the field equation, then it is a question whether this condition is to be imposed or not. Moreover since the

possible form of this condition is not unique, we must decide in each case the form of the condition to be adopted. For instance, if the covariance of the formalism is required, the so-called generalized Lorentz condition (2.5) serves this purpose, and perhaps there is no other possible convenient form of the condition as simple as this. Using this form of the condition, however, it does not always follow that we can simplify the concrete form of the field equation. On the other hand in Minkowski space-time if we take the Lorentz condition using the cartesian coordinate system, the field equation for each φ_i , i being fixed, takes the same form and moreover it contains only the same one component φ_i . In curved space-time, however, this is generally not the case, e. g., the field equation for φ_i in an S usually contains other components $\varphi_j (i \neq j)$ as is seen from (2.10). Moreover in this case, comparing the field equation (2.6) (with the generalized Lorentz condition) with (2.4) (without an auxiliary condition), the former is not always simpler than the latter. In addition to this it is clear from the example of the TM wave solution in § 4 that in certain cases we can get the solution of the field equation rather easily without using an auxiliary condition. For these reasons we may conclude that we do not get much advantage by introducing the formal generalization of the Lorentz condition such as (2.5), though it has the advantage of covariance.

As was stated in § 2 there are other possible forms of the auxiliary condition. But all of them have advantages and disadvantages, therefore in each case we must decide the form of the condition so as to serve each purpose. For instance, when the covariance of the formalism is required, we must use the covariant form of the condition such as the generalized Lorentz condition (2.5). On the other hand, if the covariance of the condition is not required, we may impose, for instance, the condition in order to simplify the form of the field equation. In this case we have only to simplify the form of the equation by means of a suitable gauge transformation and restrict the gauge so that the simplified form of the equation is invariant. As a result of this procedure, we shall be able to determine the form of the auxiliary condition. In this manner the problem of the auxiliary condition in curved space-time is not so simple as that in Minkowski space-time.

Secondly, we shall examine the concept of the free radiation field. Strictly speaking, we cannot consider any electromagnetic field in curved space-time as a free field, since in this space-time there exist gravitational fields which interact with the electromagnetic field. Therefore it will be possible to conclude that there is no essentially free electromagnetic field in curved space-time. But in this space-time we tentatively call the electromagnetic field 'free' when the four current vanishes using the terminology for the corresponding one in Minkowski space-time. In Minkowski space-time the free radiation field can always be given by a superposition of the transverse plane wave solutions in which $\varphi_4 = 0$, which is a remarkable property of the free radiation field in this space-time. In curved space-time, however, the circumstances are somewhat different. Namely, in this space-time, φ_4 cannot be made zero in general, when the generalized Lorentz condition is used as an auxiliary condition as was shown in the above sections. (The condition to make $\varphi_4 = 0$ has been obtained in § 3 when the space-time is an S .) Moreover, as was shown in [II], it is generally difficult to define the plane wave in curved space-time.* Therefore it is not easy

to represent simply the characteristics of the solutions of the free radiation fields.

Lastly, we shall touch on the problem of the gauge transformation. In § 3 we have studied some of its properties in S , and have made clear that the gauge transformation in curved space-time has some essentially different properties from those in Minkowski space-time. This fact is worth noticing, so we mention it here especially.

According to considerations given above we may conclude that the introduction of the four potential scarcely brings advantages but rather some complexities arise in the formalism. Though it may be too hasty to draw some conclusion only from the above discussions, it will be clear that the four potential in curved space-time is not so useful as in Minkowski space-time. Recently an attempt to construct the theory of the electromagnetism without potential has been proposed by Infeld and Plebanski.⁵⁾ We are inclined that such an attempt 'without potential' should rather be advocated in curved space-time.

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Appendix

Proofs of the lemmas in § 3.

(I) The condition is necessary: FL is invariant under GL. Now, if $\bar{\varphi}_4=0$, we have $C'\partial_4\bar{\varphi}_1=0$ from (2.10) concerning $\bar{\varphi}_4$. Since $C'\neq 0$, however, the relation $E_r=0$ is obtained.

The condition is sufficient: From (2.10) and $E_r=0$ we have

$$\square_s\varphi_4=0, \quad (\square_s=\square_0+\tau_5\partial_1-(C'/AC)\partial_1). \quad (\text{A} \cdot 1)$$

Evidently there exists a function $M=M(r, \theta, \phi, t)$ such that $\varphi_1=\partial_1M$ and $\varphi_4=\partial_4M$, and it satisfies $\partial_4(\square_sM)=0$ by virtue of (A.1). Hence we can find a function N such that

$$\square_sM=N, \quad (N=N(r, \theta, \phi)). \quad (\text{A} \cdot 2)$$

Let L be a function of r, θ, ϕ satisfying $\square_sL=N$, then by the GL whose $A=M-L$ we have $\bar{\varphi}_4=0$.

(II) From $\varphi_4=0$ we have $\partial_4\varphi_1=0$. Hence there exists a function $M=M(r, \theta, \phi)$ such that $\varphi_1=\partial_1M$. On the other hand, it holds that

$$\partial_1\square_sM=\{\square_s\partial_1-(2/r)\square_s+A^{-1}(A'/A-2/r)\partial_{11}+\tau_3\partial_{11}\}M, \quad (\text{A} \cdot 3)$$

and from the first of (2.10) we have

$$\{\square_s\partial_1+A^{-1}(A'/A-2/r)\partial_{11}+\tau_3\partial_{11}\}M=0. \quad (\text{A} \cdot 4)$$

From these two equations we have

* The spherical wave solutions in S obtained in § 4 can be considered approximately as representing the plane waves at the points of great distance from the origin, and also it can be easily verified that the longitudinal components of the waves, E_r and H_r , decrease rapidly compared with the other transverse components in some space-time such as Schwarzschild's. Therefore in these cases we can consider the transverse plane waves approximately.

$$\square_s M = r^{-2} N, \quad (N = N(\theta, \phi)). \quad (\text{A} \cdot 5)$$

Hence there exists a function L such that

$$\square_s L = r^{-2} N, \quad L = L(\theta, \phi). \quad (\text{A} \cdot 6)$$

Then by the GL whose $A = M - L$ we have $\bar{\varphi}_1 = \bar{\varphi}_4 = 0$.

(IV) Substituting $C' = 0$ in the fourth equation of (2.10) we have $\square_s \varphi_4 = 0$. Let M be any function satisfying $\partial_4 M = \varphi_4$, then we have $\partial_4 \square_s M = 0$, from which we further get

$$\square_s M = N(r, \theta, \phi). \quad (\text{A} \cdot 7)$$

Next if $L = L(r, \theta, \phi)$ be a solution of $\square_s L = N$, then by the GL whose $A = M - L$ we have $\bar{\varphi}_4 = 0$.

(V) Since the proof is similar to that of (I), we shall omit it here.

Next we shall add an interesting fact in connection with the lemmas in § 3.

If we solve (2.10) under the assumption of spherical symmetry and staticness of the vector φ_i , i. e.,

$$\varphi_1 = \varphi_1(r), \quad \varphi_2 = \varphi_3 = 0, \quad \varphi_4 = \varphi_4(r), \quad (\text{A} \cdot 8)$$

we have

$$\varphi_i = (ar^{-2}\sqrt{A/C}, 0, 0, b \int r^{-2}\sqrt{AC} dr) \quad (\text{A} \cdot 9)$$

where a and b are constants, as the general solution. Then if we make the GL with $A = \int ar^{-2}\sqrt{A/C} dr$ we get from (A.9)

$$\bar{\varphi}_i = (0, 0, 0, b \int r^{-2}\sqrt{AC} dr) \quad (\text{A} \cdot 10)$$

which satisfies (2.5) also. This (A.10) is the well known form corresponding to a spherically symmetric electrostatic field, and it satisfies the field equation

$$G_{ij} = -8\pi E_{ij}, \quad (\text{A} \cdot 11)$$

also, where E_{ij} is the electromagnetic energy tensor.⁶⁾

From the lemmas in § 3 we know that a necessary and sufficient condition in order that we get $\bar{\varphi}_4 = 0$ from (A.10) by a suitable GL is that the S be an S_q .

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A Relativistic Field Theory of an Extended Particle, I

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Quantization of motion of a rigid body and its extension to the relativistic form are attempted here. The special theory of relativity is believed that it rules the phenomena in the macroscopic world. But, so far, we do not know the rule which governs the microscopic world as the interior of an elementary particle. Here we assume that the internal space of an elementary particle is Euclidean. The field equation, however, has the relativistically invariant form so that the expectation values of observable quantities become relativistically covariant. The condition that the time components of an internal angular momentum have zero expectation values in the rest system leads to the general linear equation derived by Bhabha⁷⁾ for elementary particles with arbitrary spins.

When interactions are absent, the solutions of the field equations are obtained and quantization of the field of a rigid sphere is done.

§ 1. Introduction

Quantum electrodynamics has achieved many successes in describing atomic spectra. It is, however, subjected to the so-called divergence difficulties. These difficulties can formally be avoided by the self-consistent subtraction method (renormalization), but the subtracted quantities, which should be small, become infinite. Moreover, in the meson theory the subtracted quantities can not in general be determined unambiguously and there remain infinities which seem to be unrenormalizable.

Several authors¹⁾ have considered that these difficulties are caused by localization of fields or interactions. As they have required the non-locality on the field equations, its correspondence with classical mechanics is not clear and arbitrariness has been left undetermined with respect to the form of equations.

In this paper we start from classical mechanics of a rigid sphere, as we start from the one of material point in ordinary quantum mechanics. In classical mechanics the most simply idealized model of a body is a material point and it is a good approximation as far as one is concerned only with phenomena of scales large compared with dimensions of the body under consideration. However, when one is to do with phenomena in such a domain that is of comparable order of magnitude of the scale of the body, its dimension has to be taken into account. The model of the second simplicity in the classical mechanics is a rigid body. It seems to us sufficient to consider that elementary particles are fairly well represented by the rigid body model, for internal motion of the individual points in an elementary particle are considered not to be observed.

Several authors²⁾ have investigated quantum mechanics of a rigid body in the frame of the non-relativistic theory. Unfortunately the relativistic theory has no parallelism to

a rigid body³⁾ and one is not able to construct relativistic quantum mechanics of a rigid body. Though it is believed that the special theory of relativity can well describe phenomena concerning the macroscopic world, yet we do not know the law which governs the microscopic one as the interior of an elementary particle.

When one considers the internal structure of an elementary particle, it may be desirable that a closed invariant domain can be made in the internal space. Thus we assume that the internal motion of an elementary particle can be described by the Euclidean variables. This assumption may also be suggested by the fact that, even in the present relativistic field theory, the internal motion of the bound states of two body system can be described by the Euclidean variables.⁴⁾

To construct a relativistic field theory of a rigid body we put the two assumptions:

- (I) *The internal space of the elementary particle is Euclidean but the field equation, which is directly related to observations, is of a relativistically invariant form.*
- (II) *The expectation values of the time components of an internal angular momentum should vanish in the rest system of the centre of mass, because it seems not to be observed in a non-relativistic energy region.*

With these assumptions we can obtain a wave equation of a rigid sphere. Rigidity and sphere is defined as follows: When internal motion of body is represented by an internal angular momentum only, the body is rigid, and when a moment-of-inertia tensor with respect to a centre of mass of a body reduces to a scalar, the body is sphere.

In this paper the interaction free case is treated and interaction with electromagnetic field will be investigated in the succeeding one.

We first review the non-relativistic mechanics of a rigid sphere (§ 2). Then, we derive the relativistic equation of motion of a rigid sphere in accordance with the above-mentioned two assumptions (§ 3). Wave functions, thus derived, are vectors in representation space of five-dimensional rotation group, representations of which are obtained (§ 4). Invariance of the equations under Lorentz transformation is checked up (§ 5). The relativistic eigenvalue problem and the above-mentioned second assumption for wave functions are considered (§ 6). The field equation can be derived by variational principle (§ 7) and its solutions are obtained (§ 8). Quantization of the field with integral and half-odd-integral spin is done according to Bose-Einstein and Fermi-Dirac statistics, respectively (§ 9).

When one quantizes motion of a rigid sphere, one obtains an internal angular momentum which is invariant under Lorentz transformation of outer space and it might be identified with the isobaric spin operator (§ 2 and § 6).

§ 2. Non-relativistic mechanics of a rigid sphere

a) Classical Mechanics

We investigate the equation of motion of a rigid sphere in the non-relativistic classical mechanics.

The coordinate of a point in the rigid sphere is written as

$$X_i = x_i + \xi_i, \quad (i=1, 2, 3), \quad (2.1)$$

where x_i is the coordinate of the centre of mass of the sphere and ξ_i is the internal coordinate of a point in the sphere with respect to the centre of mass. The internal coordinate $\bar{\xi}_A$ which is the one fixed to the sphere is connected with ξ_i by the following orthogonal transformations :

$$\bar{\xi}_A = a_{Ai} \xi_i, \quad (2.2)$$

$$\xi_i = a_{Ai} \bar{\xi}_A, \quad (2.3)$$

$$\xi_i \xi_i = \bar{\xi}_A \bar{\xi}_A = \rho^2, \quad (2.4)$$

$$a_{Ai} a_{Aj} = \delta_{ij}, \quad a_{Ai} a_{Bi} = \delta_{AB}. \quad (2.5)$$

$\bar{\xi}_A$'s are invariant with respect to coordinate transformations in space. When differentiated by time, (2.5) gives

$$a_{Ai} \dot{a}_{Aj} + \dot{a}_{Ai} a_{Aj} = a_{Ai} \dot{a}_{Bi} + \dot{a}_{Ai} a_{Bi} = 0, \quad (2.6)$$

where the dots indicate the differentiation with respect to time. Thus, we can define the angular velocity tensor ω_{ij} with the relations

$$\omega_{ij} = \dot{a}_{Ai} a_{Aj} = (1/2) (\dot{a}_{Ai} a_{Aj} - a_{Ai} \dot{a}_{Aj}), \quad (2.7)$$

$$\omega_{ij} + \omega_{ji} = 0. \quad (2.8)$$

As $\bar{\xi}_A$'s are constants in time, the time derivative of (2.3) yields

$$\begin{aligned} \dot{\xi}_i &= \dot{a}_{Ai} \bar{\xi}_A = \dot{a}_{Ai} a_{Aj} \xi_j \\ &= \omega_{ij} \xi_j. \end{aligned} \quad (2.9)$$

$$L = m/2 \int_V \dot{\xi}_i \dot{\xi}_i \mu(\rho) dV, \quad (2.10)$$

where m is the total mass and $\mu(\rho)$ is the mass distribution density at position $\rho = \sqrt{(\xi_i \xi_i)}$. It is normalized as

$$\int_V \mu(\rho) dV = 1. \quad (2.11)$$

$\int_V dV$ indicates the integration over the whole sphere whose volume is V . Using the relations (2.1) and (2.9), (2.10) can be rewritten in the following form :

$$L = (m/2) \dot{x}_i \dot{x}_i + (ml^2/4) \omega_{ij} \omega_{ij}, \quad (2.12)$$

where

$$l^2 = 8\pi/3 \int_V \mu(\rho) \rho^4 d\rho \quad (2.13)$$

and l has the dimension of length.

From the Lagrange function (2.12) Euler's equations for free motion are derived with the aid of the variational principle, taking account of (2.5) as a subsidiary condition. Adding a term

$$\lambda_{ij}(a_{Ai}a_{Aj}-\delta_{ij})$$

to (2.12), we have the equations of motion

$$m\ddot{x}_i = ml^2\dot{\omega}_{ij} = 0, \quad (2.14)$$

in course of derivation of which we have used the antisymmetry property of ω_{ij} .

Now we define the canonical conjugate momenta of x_i 's and a_{Ai} 's as

$$p_i = \partial L / \partial \dot{x}_i = m\dot{x}_i \quad (2.15)$$

and

$$b_{Ai} = \partial L / \partial \dot{a}_{Ai} = (\partial L / \partial \omega_{ij}) a_{Aj} = ml^2 a_{Aj} \omega_{ij} / 2. \quad (2.16)$$

Then the Hamilton function is given by

$$\begin{aligned} H_{NR} &= \dot{x}_i p_i + \dot{a}_{Ai} b_{Ai} - L \\ &= (1/2m) p_i p_i + (1/ml^2) b_{Ai} b_{Ai}. \end{aligned} \quad (2.17)$$

With the aid of Poisson's brackets

$$\begin{cases} [x_i, p_j] = \delta_{ij}, \\ [a_{Ai}, b_{Bj}] = \partial_{AB} \delta_{ij}, \\ [x_i, x_j] = [p_i, p_j] = [a_{Ai}, a_{Bj}] = [b_{Ai}, b_{Bj}] = 0, \end{cases} \quad (2.18)$$

the canonical equation is obtained as

$$\dot{F}(x_i, p_i, a_{Ai}, b_{Ai}) = [F, H_{NR} - \lambda_{ij}(a_{Ai}a_{Aj} - \delta_{ij})]. \quad (2.19)$$

As it is inconvenient to treat b_{Ai} 's as independent quantities because of appearance of the subsidiary condition (2.5) and suffix A which does not indicate a tensor component, we introduce the intrinsic angular momentum

$$\begin{aligned} \mu_{ij} &= a_{Ai} b_{Aj} - a_{Aj} b_{Ai} = 2a_{Ai} b_{Aj} \\ &= ml^2 \omega_{ij}. \end{aligned} \quad (2.20)$$

Poisson's bracket for μ_{ij} 's is derived from (2.18):

$$[\mu_{ij}, \mu_{kl}] = \delta_{ik} \mu_{jl} + \delta_{jl} \mu_{ik} - \delta_{il} \mu_{jk} - \delta_{jk} \mu_{il}. \quad (2.21)$$

Then H_{NR} is reduced, with the use of μ_{ij} 's instead of b_{Ai} 's, to

$$H_{NR} = 1/(2m) \cdot p_i p_i + 1/(4ml^2) \cdot \mu_{ij} \mu_{ij} \quad (2.22)$$

and the canonical equation of motion is written as

$$\dot{F}(x_i, p_i, \mu_{ij}) = [F, H_{NR}], \quad (2.23)$$

in which the term $-\lambda_{ij}(a_{Ai}a_{Aj} - \delta_{ij})$ is eliminated, because Poisson's bracket for μ_{ij} and $a_{Ai}a_{Aj}$ vanishes:

$$[\mu_{ij}, a_{Ak}a_{Al}] = 0.$$

An alternative way of deriving the canonical equation (2.23) results from the

variation with respect to x_i , p_i , a_{Ai} , b_{Ai} and λ_{ij} of the functional

$$L + \lambda_{ij}(a_{Ai}a_{Aj} - \delta_{ij}) = \dot{x}_i p_i + 1/2 \omega_{ij} \mu_{ji} - H_{NR} + \lambda_{ij}(a_{Ai}a_{Aj} - \delta_{ij}). \quad (2.24)$$

This modified Lagrange function gives us the Euler's equations

$$\begin{cases} \dot{x}_i = \partial H_{NR} / \partial p_i = [x_i, H_{NR}], \\ \dot{p}_i = -\partial H_{NR} / \partial x_i = [p_i, H_{NR}], \\ \dot{\mu}_{ij} = 2 \{ (\partial H_{NR} / \partial \mu_{ki}) \mu_{kj} - (\partial H_{NR} / \partial \mu_{kj}) \mu_{ki} \} = [\mu_{ij}, H_{NR}]. \end{cases} \quad (2.25)$$

The internal angular velocity ω_{ij} and the internal angular momentum μ_{ij} defined by (2.7) and (2.20), respectively, are referred to the coordinate system fixed to space. Further, there are the proper angular velocity $\bar{\omega}_{AB}$ and the proper angular momentum τ_{AB} with respect to the coordinate system fixed to the rigid sphere under consideration. They are connected with ω_{ij} and μ_{ij} through the following relations:

$$\bar{\omega}_{AB} = a_{Ai} a_{Bj} \omega_{ij} = a_{Ai} \dot{a}_{Bi}, \quad (2.26)$$

$$\begin{aligned} \tau_{AB} &= a_{Ai} a_{Bj} \mu_{ij} = 2 a_{Ai} b_{Bi} \\ &= m l^2 \bar{\omega}_{AB}. \end{aligned} \quad (2.27)$$

Poisson's brackets for τ_{AB} 's are derived from (2.18) and (2.27) as

$$[\tau_{AB}, \tau_{CD}] = \partial_{AC} \tau_{BD} + \partial_{BD} \tau_{AC} - \partial_{AD} \tau_{BC} - \partial_{BC} \tau_{AD}, \quad (2.28)$$

$$[\tau_{AB}, \mu_{ij}] = 0. \quad (2.29)$$

$\bar{\omega}_{AB}$ and τ_{AB} are invariant under the coordinate transformations in the space. The relations

$$\bar{\omega}_{AB} \bar{\omega}_{AB} = \omega_{ij} \omega_{ij} \quad (2.30)$$

and

$$\tau_{AB} \tau_{AB} = \mu_{ij} \mu_{ij} \quad (2.31)$$

are derived from (2.26) and (2.27), respectively.

b) The classical field theory (The first quantization)

As in the case of the quantization of motion of a material point, we translate Poisson's brackets into the commutation relations:

$$\begin{cases} [x_i, p_j]_- = i\hbar \delta_{ij}, \\ [\mu_{ij}, \mu_{kl}]_- = i\hbar (\delta_{ik} \mu_{jl} + \delta_{jl} \mu_{ik} - \delta_{il} \mu_{jk} - \delta_{jk} \mu_{il}), \\ [\tau_{AB}, \tau_{CD}]_- = i\hbar (\partial_{AC} \tau_{BD} + \partial_{BD} \tau_{AC} - \partial_{AD} \tau_{BC} - \partial_{BC} \tau_{AD}), \\ [x_i, x_j]_- = [p_i, p_j]_- = [\tau_{AB}, \mu_{ij}]_- = \dots = 0, \end{cases} \quad (2.32)$$

in which the bracket symbol is defined by

$$[A, B]_{\pm} = AB \pm BA.$$

The equation of motion in the Heisenberg representation, using the Hamilton

function (2.22), is given by

$$i\hbar F(x_i, p_i, \mu_{ij}, \tau_{AB}) = [F, H_{NR}]_- . \quad (2.33)$$

Alternatively, representing the wave function in the Schroedinger representation as $\phi(x, t)$, we have Schroedinger equation

$$i\hbar \partial \phi / \partial t = H_{NR} \phi, \quad (2.34)$$

where p_i 's in H_{NR} are replaced by $-i\hbar \partial / \partial x_i$'s:

$$p_i \rightarrow -i\hbar \partial / \partial x_i,$$

and ϕ is a vector in the representation space of μ_{ij} and τ_{AB} .

§ 3. A relativistic equation of motion of a rigid sphere

We try to extend the equation of motion obtained in the previous section to a relativistic form, taking the fourth coordinate

$$x_4 = ix_0 = ict$$

into account in a covariant way.

a) Classical mechanics

In the special theory of relativity the "invariant Hamilton function" of a point particle is given by^{*)}

$$H = 1/(2m) p_\mu p_\mu, \quad (\mu = 1, \dots, 4) \quad (3.1)$$

or

$$H = 1/(2m) (p_\mu p_\mu + m^2 c^2), \quad (3.2)$$

which is derived from the non-relativistic one by addition of the term $1/(2m) p_i^2$, where $p_i = i/cE$ and E is energy of the particle, or of the terms $1/(2m) (p_i^2 + m^2 c^2)$, respectively. (3.1) and (3.2) give the same Hamilton's equation of motion. Corresponding to this extension, we have the relativistic Hamilton function

$$H = (1/2m) (p_\nu p_\nu + (1/2l^2) \mu_{\nu\rho} \mu_{\nu\rho}) \quad (3.3)$$

or

$$H = (1/2m) \{ p_\nu p_\nu + m^2 c^2 + (1/2l^2) (\mu_{\nu\rho} \mu_{\nu\rho} + 2\beta_\nu \beta_\nu + g) \}, \quad (3.4)$$

which is derived from the non-relativistic one (2.22) by addition of the terms

* Greek suffices run from 1 to 4, while Latin ones run from 1 to 3 except in § 4.

$$(1/2m) (p_4^2 + (1/l^2) \mu_{i4} \mu_{i4})$$

or of the terms

$$(1/2m) \{ p_4^2 + m^2 c^2 + (1/l^2) (\mu_{i4} \mu_{i4} + \beta_v \beta_v) + 1/2g \},$$

respectively. μ_{i4} 's are the fourth components of the internal angular momentum, β_v is a four vector which have similar property to the angular momentum and g is an arbitrary scalar quantity.

We take the second type of the Hamilton function (3.2) or (3.4) after Fock⁵⁾ and consider β_μ 's as the fifth components of the five-dimensional angular momentum. As μ_{i4} 's, β_μ 's and g are completely a new concept appearing in the relativistic theory, in the non-relativistic energy region they should be negligibly small or scalar quantities which are absorbed in mass. The relativistic vanishing conditions of μ_{i4} 's and β_i 's in the rest system are

$$\mu_{vp} p_p = 0 \quad (3.5)$$

and

$$\beta_v p_v p_p = \beta_p p_v p_v. \quad (3.6)$$

In fact, in the rest system $p_i = 0$ (3.5) and (3.6) lead the relations

$$\mu_{i4} p_4 = 0$$

and

$$\beta_i p_4^2 = 0.$$

Thus, when $p_4 \neq 0$, we have

$$\mu_{i4} = \beta_i = 0.$$

The equations of motion and conditions (3.5) and (3.6) are derived from the following variational principle

$$\delta I = 0$$

with the relativistic version of (2.24)*)

$$\begin{aligned} I = \int \{ & \dot{x}_v p_v + 1/2 \omega_{vp} \mu_{vp} - \omega_v \beta_v - H + \lambda_{vp} (a_{\lambda v} a_{\lambda p} + a_v a_p - \delta_{vp}) \\ & + \lambda (\bar{a}_\lambda \bar{a}_\lambda + a^2 - 1) + 2\lambda_v (a_{\lambda v} \bar{a}_\lambda - a_v a) - \rho_v \mu_{vp} p_p \\ & - \sigma_v (\beta_p p_p p_v - \beta_v p_p p_p) \} d\tau, \\ & d\tau^2 = -dx_\mu dx_\mu, \end{aligned} \quad (3.7)$$

where H is given by (3.4) and the variation is taken with respect to x_v , p_v , a_{vp} , a_v , \bar{a}_v , a , b_{vp} , \bar{b}_v , b , λ_{vp} , λ_v , λ , ρ_v and σ_v . ω_{vp} , ω_v , μ_{vp} and β_v are defined as

* In this section dots indicate differentiation by proper time.

$$\begin{cases} \omega_{vp} = \dot{a}_{av} a_{ap} + \dot{a}_v a_p, \\ \omega_v = \dot{a}_{av} \bar{a}_a + \dot{a}_v a, \\ \mu_{vp} = 2(a_{av} b_{ap} + a_v b_p), \\ \beta_v = 2(a_{av} \bar{b}_a + a_v b). \end{cases} \quad (3.8)$$

The equations of motion for x_v , p_v , μ_{vp} and β_v are obtained as

$$\dot{F}(x_v, p_v, \mu_{vp}, \beta_v) = [F, H + \rho_v \mu_{vp} p_p + \sigma_v (\beta_p p_p p_v - \beta_v p_p p_p)]. \quad (3.9)$$

With the aid of Poisson's brackets

$$\begin{cases} [x_v, p_p] = \delta_{vp}, \\ [\mu_{vp}, \mu_{\lambda\sigma}] = \delta_{v\lambda} \mu_{p\sigma} + \delta_{p\sigma} \mu_{v\lambda} - \delta_{v\sigma} \mu_{p\lambda} - \delta_{p\lambda} \mu_{v\sigma}, \\ [\mu_{vp}, \beta_\lambda] = \delta_{v\lambda} \beta_p - \delta_{p\lambda} \beta_v, \\ [\beta_v, \beta_\lambda] = \mu_{v\lambda} \end{cases} \quad (3.10)$$

and the condition (3.5) we have

$$\begin{cases} \dot{x}_v = 1/mp_v + \rho_\lambda \mu_{\lambda v} + \sigma_\lambda \beta_v p_\lambda + \sigma_v \beta_\lambda p_\lambda - 2\sigma_\lambda \beta_\lambda p_v, \\ \dot{p}_v = 0, \\ \dot{\mu}_{v\lambda} = \rho_\sigma \mu_{v\sigma} p_\lambda - \rho_\sigma \mu_{\lambda\sigma} p_v + \sigma_p \beta_\lambda p_v p_p - \sigma_p \beta_v p_\lambda p_p - \sigma_v \beta_\lambda p_p p_p + \sigma_\lambda \beta_v p_p p_p, \\ \dot{\beta}_v = \rho_\lambda \beta_\lambda p_\lambda - \rho_v \beta_\lambda p_\lambda - \sigma_\lambda \mu_{v\lambda} p_p p_p. \end{cases} \quad (3.11)$$

Connecting $\dot{\mu}_{v\lambda}$ with the proper time derivative of the orbital angular momentum

$$\begin{aligned} \dot{m}_{v\lambda} &= \dot{x}_v p_\lambda - \dot{x}_\lambda p_v \\ &= \rho_\sigma \mu_{\sigma v} p_\lambda - \rho_\sigma \mu_{\sigma \lambda} p_v + \sigma_p \beta_v p_\lambda p_p - \sigma_p \beta_\lambda p_v p_p \\ &\quad + \sigma_v \beta_p p_p p_\lambda - \sigma_\lambda \beta_p p_p p_v, \end{aligned} \quad (3.12)$$

we have

$$\dot{\mu}_{v\lambda} + \dot{m}_{v\lambda} = 0 \quad (3.13)$$

by the use of (3.6).

The equations (3.11) can be reduced to simple form as will be shown in what follows. Since the relations (3.5) and (3.6) should be independent of time, there hold

$$\dot{\mu}_{vp} p_p = \{ \rho_\sigma \mu_{v\sigma} + (\sigma_\lambda \beta_v - \sigma_v \beta_\lambda) p_\lambda \} p_p p_p = 0 \quad (3.5)'$$

and

$$\dot{\beta}_v p_v p_p - \dot{\beta}_p p_v p_v = (\rho_p \beta_\lambda p_\lambda - \rho_v \beta_v p_p + \sigma_\lambda \mu_{p\lambda} p_v p_v) p_p p_p = 0. \quad (3.6)'$$

(3.5), (3.6), (3.5)', (3.6)' and (3.11) give

$$\dot{x}_v p_p p_p = 1/mp_v p_p p_p,$$

$$\dot{p}_v = 0,$$

$$\dot{\mu}_{\nu\lambda} p_\rho p_\rho = 0,$$

$$\dot{\beta}_\nu p_\rho p_\rho = 0$$

Thus, if $p_\rho p_\rho \neq 0$, we have the equations of motion

$$\begin{cases} \dot{x}_\nu = (1/m) p_\nu, \\ \dot{p}_\nu = 0, \\ \dot{\mu}_{\nu\lambda} = 0, \\ \dot{\beta}_\nu = 0, \end{cases} \quad (3.11)'$$

and from (3.12) or (3.13) we obtain

$$\dot{m}_{\nu\lambda} = 0. \quad (3.12)'$$

From the condition that (3.5) has a significant solution for p_ρ 's we have

$$\mu_{\nu\rho} \tilde{\mu}_{\nu\rho} = 0,$$

where $\tilde{\mu}_{\nu\rho}$'s are the components of the dual tensor of $\mu_{\nu\rho}$:

$$\tilde{\mu}_{\nu\rho} = 1/2 \varepsilon_{\nu\rho\lambda\sigma} \mu_{\lambda\sigma}, \quad (3.14)$$

$$\varepsilon_{\nu\rho\lambda\sigma} = \begin{cases} 1 & \text{if } \nu\rho\lambda\sigma \text{ is given by even permutation of } 1234, \\ -1 & \text{if } \nu\rho\lambda\sigma \text{ is given by odd permutation of } 1234, \\ 0 & \text{if } \nu\rho\lambda\sigma \text{ has same numbers in it.} \end{cases} \quad (3.15)$$

From (3.6) we know that β_ν is proportional to p_ν :

$$\beta_\nu = k p_\nu, \quad (3.16)$$

where k is an arbitrary scalar.

b) Classical field theory

The relativistic wave equation for a rigid particle is obtained by putting

$$H\psi(x) = (1/2m) \{p_\nu p_\nu + m^2 c^2 + (1/2l^2) (\mu_{\nu\rho} \mu_{\nu\rho} + 2\beta_\nu \beta_\nu + g)\} \psi(x) = 0 \quad (3.17)$$

with Hamilton function (3.4). In this equation p_ν 's are the differential operators:

$$p_\nu = -i\hbar \partial / \partial x_\nu \quad (3.18)$$

and $\mu_{\nu\rho}$'s together with β_ν 's constitute the generating operators of the rotational group in the five-dimensional Euclidean space, so that they satisfy the commutation relations

$$\begin{cases} [\mu_{\nu\rho}, \mu_{\lambda\sigma}]_- = i\hbar (\partial_{\nu\lambda} \mu_{\rho\sigma} + \partial_{\rho\sigma} \mu_{\nu\lambda} - \partial_{\nu\sigma} \mu_{\rho\lambda} - \partial_{\rho\lambda} \mu_{\nu\sigma}), \\ [\mu_{\nu\rho}, \beta_\lambda]_- = i\hbar (\partial_{\nu\lambda} \beta_\rho - \partial_{\rho\lambda} \beta_\nu), \\ [\beta_\nu, \beta_\rho]_- = i\hbar \mu_{\nu\rho}. \end{cases} \quad (3.19)$$

From the four-dimensional point of view the last relation is not necessary⁶⁾ but, if we require that $\mu_{\nu\rho}$ and β_ν are commutable with H , it and the relations

$$[\mu_{\nu\lambda}, g]_- = [\beta_\nu, g]_- = 0 \quad (3 \cdot 20)$$

are necessary. We assume all the relations (3·17) and (3·19).

The equation (3·17) is so separable as to be decomposed into

$$(p_\nu p_\nu + \kappa^2 c^2) \psi(x) = 0 \quad (3 \cdot 21)$$

and

$$(1/2l^2) (\mu_{\nu\rho} \mu_{\nu\rho} + 2\beta_\nu \beta_\nu + g) \psi(x) = (\kappa^2 - m^2) c^2 \psi(x) \quad (3 \cdot 22)$$

with $\kappa^2 c^2$ to be a separation constant. The magnitude of $\kappa^2 c^2$ is given by solving the eigenvalue problem (3·22) with the aid of the representation of a Lie ring defined by the commutation relations (3·19).

For further investigation algebraical knowledge about $\mu_{\nu\rho}$ and β_ν is necessary. So in the following two sections we shall study it briefly.

§ 4. The Unitary representation of the rotational group in the five-dimensional Euclidean space

We shall obtain the representation of the five-dimensional rotational group according to the method of Racah⁸⁾. As the five-dimensional rotational group has ten parameters, eigenvalues of ten mutually commutable operators are necessary to establish the representation. There are two mutually commutable parameter groups P_1 and P_2 which are isomorphic with the rotational group. They are generated by the Lie rings μ_{ab} and τ_{ab} ($a, b = 1, \dots, 5$) which satisfy the commutation relation

$$[\mu_{ab}, \mu_{cd}]_- = i\hbar (\delta_{ac} \mu_{bd} + \delta_{bd} \mu_{ac} - \delta_{ad} \mu_{bc} - \delta_{bc} \mu_{ad}), \quad (4 \cdot 1)$$

$$[\mu_{ab}, \tau_{cd}]_- = 0 \quad (4 \cdot 2)$$

$$(a, b, c, d = 1, \dots, 5)$$

and the same relation to (4·1) for τ_{ab} . Between μ_{ab} and τ_{ab} there are two relations,

$$2\hbar^2 G_1 = \mu_{ab} \mu_{ab} = \tau_{ab} \tau_{ab} \quad (4 \cdot 3)$$

and

$$2\hbar^4 G_2 = \mu_{ab} \mu_{bc} \mu_{cd} \mu_{da} = \tau_{ab} \tau_{bc} \tau_{cd} \tau_{da} \quad (4 \cdot 4)$$

which are numbers in an irreducible representation.

As μ_{ab} and τ_{ab} have algebraically the same structure, it is sufficient to investigate μ_{ab} only. β_ν in the preceding section is connected with μ_{ab} by the relation

$$\beta_\nu = \mu_{\nu 5}, \quad (4 \cdot 5)$$

Since the number of the elements which are commutable with each other is two, we make the representation matrices of μ_{15} and $\mu_{15} = \beta_4$ diagonal. Now, we define the ten linearly independent operators in place of μ_{ab} 's:

$$\begin{cases} H_1 = 1/\hbar \mu_{45}, \\ H_2 = 1/\hbar \mu_{12}, \end{cases} \quad (4.6)$$

$$\begin{cases} E_{\pm 1} = (1/\sqrt{2}\hbar) (\mu_{23} \pm i\mu_{31}), \\ E_{\pm 2} = (1/2\hbar) (\mu_{14} \pm i\mu_{15} \mp i\mu_{24} + \mu_{25}), \\ E_{\pm 3} = (1/\sqrt{2}\hbar) (\mu_{34} \pm i\mu_{35}), \\ E_{\pm 4} = (1/2\hbar) (\mu_{14} \pm i\mu_{15} \pm i\mu_{24} - \mu_{25}). \end{cases} \quad (4.7)$$

These operators are the eigenvectors of an operator

$$A = 3H_1 + H_2. \quad (4.8)$$

That is to say, they satisfy the eigenvalue problem of the form

$$\begin{cases} [A, E_\alpha]_- = \alpha E_\alpha, \\ [A, H_i]_- = 0, \end{cases} \quad (4.9)$$

$$(\alpha = \pm 1, \dots, \pm 4), (i = 1, 2).$$

The commutation relations for H_i 's and E_α 's are derived from (4.1), (4.6) and (4.7) in the following form:

$$\begin{cases} [H_i, H_j]_- = 0, \\ [H_i, E_\alpha]_- = \alpha_i E_\alpha, \\ [E_\alpha, E_{-\alpha}]_- = \sum_{i=1}^2 \alpha_i H_i, \\ [E_\alpha, E_\beta]_- = C_{\alpha\beta} E_{\alpha+\beta}, \text{ if } \alpha + \beta \neq 0. \end{cases} \quad (4.10)$$

From (4.9) and (4.10) we have the relation

$$\alpha = 3\alpha_1 + \alpha_2. \quad (4.11)$$

1	0								
-1		0							
2	1	0	0						
-2	0	-1		0					
3	-1	1	0	-1	0				
-3	-1	1	1	0		0			
4	0	-1	0	0	0	1	0		
-4	1	0	0	0	-1	0		0	
$\beta \backslash \alpha$	1	-1	2	-2	3	-3	4	-4	

Table I
The values of $C_{\alpha\beta}$.

As α is determined when α_1 and α_2 are given and vice versa, we can regard α as the vector in the two-dimensional space, the components of which are α_1 and α_2 . Thus, if we write as

$$\alpha = (\alpha_1, \alpha_2), \quad (4.12)$$

we obtain

$$\begin{cases} \pm 1 = (0, \pm 1), \\ \pm 2 = (\pm 1, \mp 1), \\ \pm 3 = (\pm 1, 0), \\ \pm 4 = (\pm 1, \pm 1). \end{cases} \quad (4.13)$$

$C_{\alpha\beta} = -C_{\beta\alpha}$ is given in Table I.

From here on the letter α or the term *root* will be used to denote either the form (4.11) or the vector with components α_i in the two-dimensional space.

Let u_m be a vector in the representation space such that

$$H_i u_m = m_i u_m. \quad (4.14)$$

Thus, u_m is a simultaneous eigen-vector of the two matrices H_i . The set of eigenvalues m_1 and m_2 form the components of a vector in the two-dimensional space, which is denoted as

$$m = (m_1, m_2). \quad (4.15)$$

We call this vector the *weight* of u_m . From (4.10) and (4.14) we have

$$H_i E_\alpha u_m = [H_i, E_\alpha] u_m + E_\alpha H_i u_m = (m_i + \alpha_i) E_\alpha u_m. \quad (4.16)$$

Thus, if u_m is an eigenvector, $E_\alpha u_m$ is also an eigenvector. Correspondingly, if m is a weight, $m + \alpha$ is a weight, too.

As μ_{ab} 's are the real operators, from definition (4.7) we know that

$$E_\alpha^* = E_{-\alpha}, \quad (4.17)$$

where E_α^* is the complex conjugate of E_α . When we consider the Casimir operator

$$G_1 = (1/2b^2) \mu_{ab} \mu_{ab}, \quad (a, b = 1, \dots, 5) \quad (4.18)$$

and call the eigenvalue of G_1 in an irreducible representation λ , we have the following relations:

$$G_1 u_m = \lambda u_m, \quad (4.19)$$

$$\begin{cases} \sum_{\alpha > 0} E_\alpha E_{-\alpha} = 1/2 (G - H_1^2 - H_2^2 + 3H_1 + H_2), \\ \sum_{\alpha > 0} E_{-\alpha} E_\alpha = 1/2 (G - H_1^2 - H_2^2 - 3H_1 - H_2), \end{cases} \quad (4.20)$$

$$\begin{cases} \sum_{\alpha > 0} E_\alpha E_{-\alpha} u_m = 1/2 \{ \lambda + 5/2 - (m_1 - 3/2)^2 - (m_2 - 1/2)^2 \} u_m, \\ \sum_{\alpha > 0} E_{-\alpha} E_\alpha u_m = 1/2 \{ \lambda + 5/2 - (m_1 + 3/2)^2 - (m_2 + 1/2)^2 \} u_m. \end{cases} \quad (4.21)$$

From (4.17) we know that the eigenvalues of $\sum_{\alpha > 0} E_{\pm\alpha} E_{\mp\alpha}$ are positive numbers, and so m_1 and m_2 have the maximum and the minimum values. We call the highest value of m_1 L_1^{\max} and the highest value, which can be taken by m_2 when $m_1 = L_1^{\max}$, L_2^{\max} . The dominant weight is defined by $L(\max) = (L_1^{\max}, L_2^{\max})$. Similarly we can define the lowest weight $L(\min) = (L_1^{\min}, L_2^{\min})$, where L_1^{\min} is the lowest value of m_1 and L_2^{\min} is the lowest value of m_2 which can be taken when $m_1 = L_1^{\min}$. When $\alpha > 0$, (4.16) leads to the relations

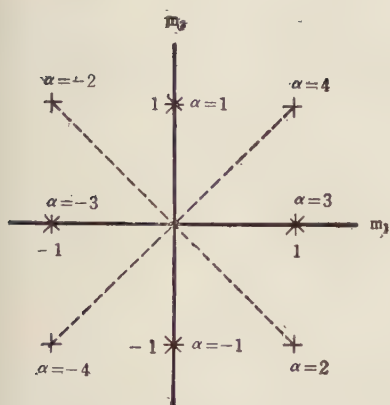
$$E_{-\alpha} u_{L(\min)} = E_\alpha u_{L(\max)} = 0. \quad (4.22)$$

Hence, from (4.21) and (4.22), we have

$$L(\max) = -L(\min) = L = (L_1, L_2) \quad (4.23)$$

and

$$\lambda = L_1(L_1 + 3) + L_2(L_2 + 1). \quad (4.24)$$



From the relations (4.16) and (4.23) we know that L_1 , L_2 and all m_i 's are simultaneously integers or simultaneously half odd integers.

When we regard the components of a root or a weight as the components in the two-dimensional Cartesian coordinate, all possible weights belong to a lattice which is invariant under the group S generated by the reflections with respect to the lines through the origin perpendicular to the roots (See Fig. 1). Weights which can be obtained from one another by operation of S have the same multiplicity. When we represent a two-dimensional vector by a two row and one column matrix, the element of S are written down in matrix form as

$$\begin{aligned} S_0 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, S_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, S_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, S_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ S_4 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, S_5 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, S_6 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, S_7 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \quad (4.25)$$

and

$$\det(S_i) = (-1)^i. \quad (4.26)$$

If we introduce the vectors

$$R = 1/2 \sum_{\alpha > 0} \alpha \quad (4.27)$$

and

$$K = L + R, \quad (4.28)$$

λ can be written as

$$\lambda = (L \cdot L) + 2(R \cdot L) = (K \cdot K) - (R \cdot R), \quad (4.29)$$

where $(A \cdot B)$ is the scalar product of two-dimensional vectors A and B .

A unitary representation matrix of the rotational group in the five-dimensional Euclidean space, for example, corresponding to the rotation of an angle φ between ν - and ρ -axis is given by

$$\exp(i\varphi\mu_{\nu\rho}/\hbar), \quad (\nu, \rho = 1, \dots, 5). \quad (4.30)$$

The trace of the matrices of an irreducible representation corresponding to two rotations

of angles φ_1 and φ_2 which are commutable each other, i.e., the character of an irreducible representation is given by the formula⁹⁾

$$\chi(L; \varphi) = \zeta(K) / \zeta(R), \quad (4.31)$$

where

$$\zeta(K) = \sum_S \partial_S e^{i(SK \cdot \varphi)}, \quad (4.32)$$

$$\varphi = (\varphi_1, \varphi_2)$$

and ∂_S takes plus or minus sign according to the parity of the element S_i of S :

$$\partial_S = \det(S_i) = (-1)^i, \quad (i=0, 1, \dots, 7).$$

The dimension of an irreducible representation space is given by

$$\chi(L; 0) = \prod_{\alpha > 0} (\alpha \cdot K) / (\alpha \cdot R). \quad (4.33)$$

We can write down (4.31) and (4.33) in the explicit form:

$$\chi(L; \varphi) = \frac{\sin \{ (L_1 + 3/2) \varphi_1 \} \sin \{ (L_2 + 1/2) \varphi_2 \} - \sin \{ (L_2 + 1/2) \varphi_1 \} \sin \{ (L_1 + 3/2) \varphi_2 \}}{\sin(3/2 \varphi_1) \sin(1/2 \varphi_2) - \sin(1/2 \varphi_1) \sin(3/2 \varphi_2)}, \quad (4.34)$$

$$\chi(L; 0) = 1/6 (2L_1 + 3) (2L_2 + 1) (L_1 + L_2 + 2) (L_1 - L_2 + 1). \quad (4.35)$$

$\chi(L; \varphi)$ has the form

$$\chi(L; \varphi) = \sum_m \zeta_m e^{i(m \cdot \varphi)}, \quad (4.36)$$

where ζ_m is the multiplicity of the weight m . Especially the multiplicity of the eigenvalue L_1 is given by

$$\sum_{m_2 = -L_2}^{L_2} \zeta_{(L_1, m_2)} = 2L_2 + 1, \quad (4.37)$$

which is equal to the number of the eigenvalues that can be taken by H_2 when the eigenvalue of H_1 is L_1 . In the state in which H_1 takes the eigenvalue L_1 the highest eigenvalue that can be taken by H_2 is L_2 and in this state the eigenvalues of H_1 are not degenerate.

The complete set of the commutable and linearly independent operators and their eigenvalues are given by

$$\left\{ \begin{array}{l} G_1 = (1/2\hbar^2) \mu_{ab} \mu_{ab}, \quad G_1 u = \{L_1(L_1 + 3) + L_2(L_2 + 1)\} u, \\ G_2 = (1/2\hbar^2) \mu_{ab} \mu_{bc} \mu_{cd} \mu_{da}, \\ G_2 u = \{L_1(L_1 + 3) (L_1^2 + 3L_1 + 3) + L_2(L_2 + 1) (L_2^2 + L_2 - 1)\} u, \\ H_1 = (1/\hbar) \mu_{43}, \quad H_1 u = m_1 u, \\ H_2 = (1/\hbar) \mu_{12}, \quad H_2 u = m_2 u, \\ K_1 = (1/\hbar^2) (\mu_{23}^2 + \mu_{31}^2 + \mu_{12}^2), \quad K_1 u = l_2(l_2 + 1) u, \\ K_2 = I_1^2 + I_2^2, \quad K_2 u = k u, \end{array} \right. \quad (4.38)$$

where

$$\begin{cases} I_1 = 1/b^2 (\mu_{23}\mu_{14} + \mu_{31}\mu_{24} + \mu_{12}\mu_{34}), \\ I_2 = 1/b^2 (\mu_{23}\mu_{15} + \mu_{31}\mu_{25} + \mu_{12}\mu_{35}), \end{cases} \quad (4.39)$$

and they satisfy the commutation relations

$$\begin{aligned} [I_1, H_1]_- &= -iI_2, & [I_2, H_1] &= iI_1, \\ [I_\alpha, \mu_{ij}]_- &= 0, & (\alpha &= 1, 2; i, j = 1, 2, 3). \end{aligned} \quad (4.40)$$

These eigenvalues obey the limitations

$$\begin{cases} L_1 \geq L_2, \\ L_1 \geq m_1 \geq -L_1, \\ L_1 \geq l_2 \geq 0, \\ l_2 \geq m_2 \geq -l_2. \end{cases} \quad (4.41)$$

When $m_1 = \pm L_1$, l_2 and k become as

$$\begin{aligned} l_2 &= L_2, \\ k &= \{L_1(L_2 + 2) - L_2 \pm (2L_2 - L_1 + 1)\} L_2. \end{aligned}$$

In general relation between k and the other eigenvalues is not easily obtained. In order to complete representation one should let the quantities

$$\begin{aligned} \lambda_1 &= \mu_{14}^2 + \mu_{24}^2 + \mu_{15}^2 + \mu_{25}^2 + \mu_{12}^2 + \mu_{45}^2, \\ \lambda_2 &= \mu_{24}\mu_{15} + \mu_{41}\mu_{25} + \mu_{12}\mu_{45} \end{aligned}$$

be diagonal instead of K_1 and K_2 . Then, we have

$$\begin{aligned} \lambda_1 u &= \{l_1(l_1 + 2) + l_2^2\} u, \\ \lambda_2 u &= (l_1 + 1)l_2 u, \end{aligned}$$

where l_1 and l_2 are simultaneously integers or half odd integers together with L_1 and L_2 . l_1, l_2 obey the limitations

$$\begin{aligned} L_1 &\geq l_1 \geq l_2 \geq 0, \\ l_1 &\geq m_1, m_2 \geq -l_1. \end{aligned}$$

With λ_1 and λ_2 we can construct a representation and then by a similarity transformation we can transform it to the representation in which the complete set (4.38) is diagonal. It is the reason why we do not use λ_1 and λ_2 but K_1 and K_2 that the latters can be written in the forms of world scalars (Cf. § 6) but the formers can not.

For the second parameter group P_2 , as shown by (4.3) and (4.4), the eigenvalues of G_1 and G_2 equal to the ones for P_1 , the commutable operators, which are linearly independent, are H_1, H_2, K_1 and K_2 for the corresponding τ_{ab} 's.

§ 5. The Lorentz transformations

In order that equations, which contain β_ν or $\mu_{\nu\rho}$ as a vector or a tensor, respectively, are invariant or covariant under the Lorentz transformation of the space coordinate

$$x'_\mu = A_{\mu\nu} x_\nu, \quad (5.1)$$

it is sufficient that there exists the similarity transformation matrix A which satisfies the relation

$$A^{-1} \beta_\mu A = A_{\mu\nu} \beta_\nu. \quad (5.2)$$

With A $\psi(x)$ and its Hermitian conjugate $\psi^*(x)$ are transformed as

$$\psi' = A\psi \quad (5.3)$$

$$\psi'^* = \psi^* A^\dagger \quad (5.4)$$

where A^\dagger is the Hermitian conjugate of A .

Introducing the adjoint function of $\psi(x)$

$$\bar{\psi} = \psi^* \eta \quad (5.5)$$

and letting $\bar{\psi}(x)$ be transformed for (5.1) as

$$\bar{\psi}' = \pm \bar{\psi} A^{-1}, \quad (5.6)$$

we have, from (5.4) to (5.6)

$$A^\dagger \eta = \pm \eta A^{-1}. \quad (5.7)$$

This defines η .

The matrix A is defined by (5.2) only to within a constant factor. We shall limit this factor to the roots of unity by the additional requirement

$$\det A = 1. \quad (5.8)$$

When the transformation is infinitesimal, it can be written as

$$x'_\mu = x_\mu + \varepsilon_{\mu\nu} x_\nu, \quad (5.9)$$

$$\varepsilon_{\mu\nu} + \varepsilon_{\nu\mu} = 0.$$

Then A is expressed by

$$A = I + (i/2\hbar) \varepsilon_{\mu\nu} \mu_{\mu\nu}, \quad (5.10)$$

where I is the unit matrix.

For finite transformations connection between $A_{\mu\nu}$ and A is not so simple as (5.10). Let us first take the special case of the rotation through an angle θ about the x_3 -axis in ordinary space. The transformation is represented by

$$\begin{cases} x'_1 = x_1 \cos \theta + x_2 \sin \theta, \\ x'_2 = -x_1 \sin \theta + x_2 \cos \theta, \end{cases} \quad (5.11)$$

$$\begin{cases} x_3' = x_3, \\ x_4' = x_4. \end{cases} \quad (5.11)$$

This yields¹⁰⁾

$$A_{12} = \exp(i\theta\mu_{12}/\hbar). \quad (5.12)$$

In fact, owing to the formula

$$e^{-a}be^a = b + [b, a]_- + 1/2! [[b, a]_-, a]_- + \dots, \quad (5.13)$$

we get

$$\begin{cases} A_{12}^{-1}\beta_1 A_{12} = \beta_1 \cos \theta + \beta_2 \sin \theta, \\ A_{12}^{-1}\beta_2 A_{12} = -\beta_1 \sin \theta + \beta_2 \cos \theta, \\ A_{12}^{-1}\beta_3 A_{12} = \beta_3 \\ A_{12}^{-1}\beta_4 A_{12} = \beta_4 \end{cases}$$

and, as the eigenvalues of μ_{12} take plus and minus signs symmetrically,

$$\det A_{12} = 1.$$

Another typical transformation is a rotation through a hyperbolic angle θ in the x_1 - x_0 -plane. The transformation is now represented by

$$\begin{cases} x_1' = x_1 \cosh \theta + x_0 \sinh \theta, \\ x_2' = x_2, \\ x_3' = x_3, \\ x_0' = x_1 \sinh \theta + x_0 \cosh \theta, \end{cases} \quad (5.14)$$

where

$$x_0 = -ix_4.$$

In this case, one can as well verify that the equations (5.2) and (5.8) hold with

$$A_{14} = \exp(\theta\mu_{14}/\hbar). \quad (5.15)$$

If two transformations are made consecutively, we have simply to multiply the corresponding A 's to get the A for the resultant transformation. Any finite proper Lorentz transformation can be built up from two types of rotations we have considered.

A matrix

$$A_\nu = \exp(i\pi\beta_\nu/\hbar) \quad (5.16)$$

induces, by the use of (5.13),

$$A_\nu^{-1}\beta_\lambda A_\nu = \begin{cases} -\beta_\lambda & \text{if } \nu \neq \lambda, \\ \beta_\lambda & \text{if } \nu = \lambda \end{cases} \quad (5.17)$$

(no summation with respect to ν)

and

$$A_\nu^{-1} A_\lambda A_\nu = \begin{cases} A_\lambda^* = A_\lambda^{-1} & \text{if } \nu \neq \lambda \\ A_\lambda & \text{if } \nu = \lambda. \end{cases} \quad (5.18)$$

The eigenvalues of A_ν are given by

$$\exp(i\pi n) = i^{2n},$$

where $n\hbar$ is a corresponding eigenvalue of β_ν and n is an integer or half an odd integer. Hence we have

$$B_\nu = A_\nu^2 = \begin{cases} 1 & \text{for an integral representation} \\ -1 & \text{for a half-odd-integral representation} \end{cases} \quad (5.19)$$

and

$$A_\nu^* = A_\nu^{-1} = B_\nu A_\nu. \quad (5.20)$$

From the relations (5.17) and (5.18) we can construct the transformation matrices for coordinate reflections and η as follows:

$$A_s = A_4 \quad \text{for space reflection,} \quad (5.21)$$

$$A_t = A_1 A_2 A_3 \quad \text{for time reflection,} \quad (5.22)$$

$$A_r = A_1 A_2 A_3 A_4 \quad \text{for space-time reflection} \quad (5.23)$$

and

$$\eta = (-i)^{2L_1} A_4. \quad (5.24)$$

The factor $(-i)^{2L_1}$ in the definition of η is chosen so that it satisfies the relation

$$\eta^2 = (-1)^{2L_1} B_4 = 1. \quad (5.25)$$

Now the transformed adjoint wave function $\bar{\psi}'$ is connected with $\bar{\psi}$ as

$$\bar{\psi}' = \bar{\psi} (\gamma_i A^\dagger \gamma_i).$$

The transformation matrix for an improper Lorentz transformation can be constructed by the multiplication of the matrix of a rotation and that of a reflection. With these definitions of the reflection matrices (5.21) to (5.23) and η we can determine the sign in (5.6):

When L_1 is an integer (an integral representation) always

$$\bar{\psi}' = \bar{\psi} A^{-1}. \quad (5.26)$$

When L_1 is half an odd integer (a half-odd-integral representation)

$$\bar{\psi}' = \bar{\psi} A^{-1} \quad \text{for fixed sense of time} \quad (5.27)$$

and

$$\bar{\psi}' = -\bar{\psi} A^{-1} \quad \text{for the reversal of sense of time.} \quad (5.28)$$

§ 6. The relativistic eigenvalue problems

All the operators of the complete set (4.38) except H_2 can be replaced by the relativistically invariant ones. Introducing a four-vector p_μ which satisfies the relation

$$p_\mu p_\mu + \kappa^2 c^2 = 0, \quad (6.1)$$

we define the following scalar- and vector-operators:

$$\begin{cases} a = -i / (\hbar \kappa c) p_\nu \beta_\nu, \\ b_\nu = -i / (\hbar \kappa c) \tilde{\mu}_{\nu\rho} p_\rho, \\ c_\nu = -i / (\hbar \kappa c) \mu_{\nu\rho} p_\rho, \\ d_\nu = 1 / (\hbar \kappa^2 c^2) (p_\nu \beta_\rho - p_\rho \beta_\nu) p_\rho, \end{cases} \quad (6.2)$$

where $\tilde{\mu}_{\nu\rho}$ is defined by (3.14). They satisfy the commutation relations

$$\begin{cases} [a, b_\nu]_- = 0, \\ [a, c_\nu]_- = i d_\nu, \\ [a, d_\nu]_- = -i c_\nu, \\ [b_\nu, b_\lambda]_- = [c_\nu, c_\lambda]_- = [d_\nu, d_\lambda]_- = i / \hbar \mu_{\nu\lambda} + 1 / (\kappa c) (p_\nu c_\lambda - p_\lambda c_\nu), \\ [b_\nu, c_\lambda]_- = -1 / (\kappa c) \varepsilon_{\nu\lambda\rho\sigma} p_\rho c_\sigma, \\ [b_\nu, d_\lambda]_- = -1 / (\hbar \kappa c) \varepsilon_{\nu\lambda\rho\sigma} p_\rho \beta_\sigma, \\ [c_\nu, d_\lambda]_- = i (\delta_{\nu\lambda} + p_\nu p_\lambda / (\kappa c)^2) a. \end{cases} \quad (6.3)$$

These operators are equivalent to $\mu_{\nu\rho}$'s and β_ν 's. In fact, in the rest system

$$p_1 = p_2 = p_3 = 0, \quad p_4 = i \kappa c, \quad (6.4)$$

they are reduced to $\mu_{\nu\rho}$'s and β_ν 's:

$$\begin{aligned} a &\rightarrow \beta_4 / \hbar = H_1, \\ b_i &\rightarrow \mu_{jk} / \hbar, \quad (i, j, k): \text{cyclic}, \\ c_i &\rightarrow \mu_{44} / \hbar, \\ d_i &\rightarrow \beta_i / \hbar, \\ b_4 = c_4 = d_4 &= 0. \end{aligned}$$

Further, if we introduce the two scalars

$$\begin{aligned} e &= b_\nu b_\nu = 1 / (2 \hbar^2) \mu_{\nu\rho} \mu_{\nu\rho} - c_\nu c_\nu, \\ f &= (b_\nu c_\nu)^2 + (b_\nu d_\nu)^2, \end{aligned} \quad (6.5)$$

they are reduced to K_1 and K_2 of (4.38), respectively, in the rest system (6.4):

$$\begin{aligned} f &\rightarrow K_1, \\ e &\rightarrow K_2. \end{aligned}$$

With the aid of the commutation relations (3·17), we can show that G_1 , G_2 , a , b , e and f are mutually commutable. Thus, the complete set (4·38) can be replaced by

$$\left\{ \begin{array}{ll} G_1, & G_1\psi = \{L_1(L_1+3) + L_2(L_2+1)\}\psi, \\ G_2, & G_2\psi = \{L_1(L_1+3)(L_1^2+3L_1+3) + L_2(L_2+1)(L_2^2+L_2-1)\}\psi, \\ a, & a\psi = m_1\psi, \\ b_3, & b_3\psi = m_2\psi, \\ e, & e\psi = l_2(l_2+1)\psi, \\ f, & f\psi = k\psi, \end{array} \right. \quad (6\cdot6)$$

in which p_v 's are to be understood as the differential operators (3·18) operating on ψ . From transformation property of $\mu_{\nu\mu}$'s and β_ν 's under a Lorentz transformation which is given in § 5, we know that the numbers L_1 , L_2 , m_1 , l_2 and k are world scalars and m_2 depends on the reference system and is transformed as the third component of a pseudo-vector.

Thus, if a wave function of an elementary particle is a representation which is irreducible with respect to the Lorentz transformation, the internal degree of freedom is one and it corresponds to rotation about an axis in the three-dimensional space.

If ψ is an eigenvector of a belonging to an eigenvalue m_1 , then the expectation values by ψ of $\mu_{\nu 4}$ and β_i vanish in the rest system. This is proved as follows:

The expectation value of an operator F in the state $\psi(x)$ is given by

$$\int \bar{\psi}(x) \beta_4 F \psi(x) d^3V, \quad (6\cdot7)$$

in which $\int d^3V$ represents three-dimensional volume integral. In the rest system β_4 is diagonal and

$$\beta_4 \psi^{(0)} = \hbar a \psi^{(0)} = m_1 \hbar \psi^{(0)}. \quad (6\cdot8)$$

If we put $F = \mu_{\nu 4}$, the integrand of (6·6) in the rest system becomes as

$$\begin{aligned} \psi^{(0)*} \gamma_i \beta_4 \mu_{\nu 4} \psi^{(0)} &= i/b \psi^{(0)*} \gamma_i \beta_4 (\beta_{ii} \beta_\nu - \beta_{\nu i} \beta_i) \\ &= i/b (\beta_{ii} \psi^{(0)*} \gamma_i \beta_4 \beta_\nu \psi^{(0)} \\ &\quad - i/b \psi^{(0)*} \gamma_i \beta_4 \beta_{\nu i} \psi^{(0)}) \\ &= i m_1 (\bar{\psi}^{(0)} \beta_{ii} \beta_\nu \psi^{(0)} - \bar{\psi}^{(0)} \beta_{\nu i} \psi^{(0)}) \\ &= 0 \end{aligned}$$

with the aid of (5·5), (3·19) and (5·24). Similarly there results

$$\bar{\psi}^{(0)} \beta_{ii} \beta_i \psi^{(0)} = 0.$$

Q.E.D.

Thus, if we take one of representations which are irreducible with respect to the Lorentz transformation as a wave function of an elementary particle, it satisfies the assumption (II) of § 1. Here we make choice of a special representation which satisfies the subsidiary condition corresponding to (3.5) and (3.6), which are, by the use of the definitions (6.2), written as

$$c_v = d_v = 0.$$

In classical mechanics two sets of conditions (3.5) and (3.6) are necessary, but, in the field theory the assumption (II) of § 1 is satisfied by only one condition. Under these circumstances we require ψ to satisfy the condition which is a linear combination of (3.5) and (3.6):

$$(c_v + kd_v)\psi = 0, \quad (6.9)$$

where k is a constant which will be determined in the following.

With the aid of the commutation relations (6.3) we have

$$[a, c_v + kd_v]_- = i(d_v - kc_v).$$

So, if we put $k = \pm i$, it becomes as

$$[a, c_v \pm id_v]_- = \pm (c_v \pm id_v). \quad (6.10)$$

If ψ is an eigenfunction which satisfies the equation

$$a\psi = m_1\psi,$$

there results

$$a(c_v \pm id_v)\psi = (m_1 \pm 1)(c_v \pm id_v)\psi \quad (6.11)$$

for the sake of (6.10). Thus, $(c_v + id_v)\psi$'s are eigenfunctions of a belonging to the eigenvalue $m_1 + 1$. When we put

$$m_1 = \pm L_1,$$

where L_1 is the highest eigenvalue of a , we get

$$(c_v \pm id_v)\psi = 0. \quad (6.12)$$

This equation is the required one. In order that ψ satisfies the equation (6.12) it is sufficient that ψ is the eigenfunction of a belonging to the eigenvalue $\pm L_1$. Therefore, we regard the relation

$$a\psi = L_1\psi \quad (6.13)$$

as a subsidiary condition (it is equivalent to take $-L_1$ instead of L_1 , because the eigenvalue of p_1 defined by the equation (3.21) can take both signs).

The angular momentum τ_{ab} ($a, b = 1, \dots, 5$), which is measured in the body system, satisfies the following commutation relations:

$$[\tau_{ab}, \tau_{cd}]_- = i\hbar (\partial_{ac}\tau_{bd} + \partial_{bd}\tau_{ac} - \partial_{ad}\tau_{bc} - \partial_{bc}\tau_{ad}), \quad (6.14)$$

and we write

$$\pi_I = \tau_{I5} = -\tau_{5I}, \quad (I=1, \dots, 4).$$

The meaning of τ_{AB} in classical mechanics is obvious; for example, the angular momentum of the earth is represented by μ_{ij} in the solar system and by τ_{AB} in the terrestrial system. But in quantum mechanics the meaning of τ_{AB} is not so obvious. The direction of rotation in the reference system fixed to the particle is so special as to be invariant under the coordinate transformation in the space system. Corresponding to this speciality, it seems natural to specialize the diagonalized quantities, e.g., τ_{12} and π_1 , which generate the two independent and commutable rotation in the five-dimensional space, in quantum mechanics. In order to represent this speciality we shall introduce two constant tensors $h_{\mu\nu}$ and $k_\mu = h_{\mu 5}$, which have the vanishing components except

$$h_{12} = k_4 = h_{45} = 1.$$

The complete set for $\tau_{\Gamma\Delta}$ and π_I is given by

$$2\hbar^2 G_1 = \tau_{ab} \tau_{ab}, \quad (4.3)$$

$$2\hbar^4 G_2 = \tau_{ab} \tau_{bc} \tau_{cd} \tau_{da}, \quad (4.4)$$

$$(a, b, c, d = 1, \dots, 5)$$

$$\begin{cases} \bar{a} = 1/\hbar k_I \pi_I, \\ \bar{b} = (1/2\hbar) h_{\Gamma\Delta} \tau_{\Gamma\Delta}, \\ \bar{c} = 1/\hbar^2 (\tau_{\Gamma\Delta} \tau_{\Gamma\Delta} / 2 - \tau_{\Gamma\Delta} \tau_{I\Gamma} k_\Delta k_\Delta), \\ \bar{f} = 1/\hbar^4 (\tau_{\Gamma\Delta} \tau_{I\Gamma} k_\Delta k_\Delta)^2 + 1/\hbar^4 \{ \tau_{\Gamma\Delta} (k_I \pi_\Delta - k_\Delta \pi_I) k_\Delta k_\Delta \}^2, \end{cases} \quad (6.15)$$

$$(I, J, A = 1, \dots, 4),$$

and their eigenvalues are obtained by the similar manner to (6.6). Lorentz invariance requires that G_1 and G_2 are the strict constants of motion. As the transformation of the coordinate fixed to the particle has no physical correspondence, \bar{a} , \bar{b} , \bar{c} and \bar{f} are unnecessary to be constants of motion. However, if one requires invariance of some kind of interactions under a coordinate transformation of the body system, one may obtain the corresponding conservative quantities which lead to selection rules; e.g., Pais' even-odd rule⁽¹¹⁾ or its extension.^{(12), (13)}

Charge and mass are what we know as such universal quantities that prescribe the property or the behavior of elementary particles and are invariant under the coordinate transformations. Accordingly it would be natural to connect the eigenvalue of τ_{12} or π_4 with the charge and/or the mass of a particle. Moreover, the isobaric spin matrix introduced in the case of interactions between nucleons and the particles with a strong interaction with those has algebraically a similar structure to $\tau_{\Gamma\Delta}$'s and is considered as a conserving quantity based upon various experimental evidences. Extending the concept of the isobaric spin we may put the following expression for the charge of a particle.

$$\begin{aligned} Q &= e/\hbar (h_{\Gamma\Delta}\tau_{\Gamma\Delta} + k_{\Gamma}\pi_{\Gamma}) \\ &= e/\hbar (\tau_{12} + \pi_4). \end{aligned} \quad (6.16)$$

This theory of the isobaric spin cannot afford the theory of Nishijima¹²⁾ and Gell-Mann¹³⁾, unless we abandon relations (4.3) and (4.4).

If we assume the expression (6.16) as charge, the arbitrary scalar term g in (3.22) or (7.3) is to commute with Q . As $\tau_{\Gamma\Delta}$'s and π_{Γ} 's are commutable with $\mu_{\nu\rho}$'s and β_{ν} 's, we can choose g to be an arbitrary function of $\tau_{\Gamma\Delta}$'s and π_{Γ} 's commutable with Q :

$$\kappa_L^2 \psi_L = \{m^2 + (1/4l^2 c^2) (\mu_{\nu\rho} \mu_{\nu\rho} + 2\beta_{\nu} \beta_{\nu}) + g(\tau_{\Gamma\Delta}, \pi_{\Gamma})\} \psi_L, \quad (6.17)$$

$$[g(\tau_{\Gamma\Delta}, \pi_{\Gamma}), Q]_- = 0. \quad (6.18)$$

§ 7. The c-number field theory

The equation of motion of an elementary particle (3.21) and the subsidiary condition (6.13) can be written in the following form

$$(\square - \kappa_L^2 c^2 / \hbar^2) \psi_L(x) = 0 \quad (7.1)$$

and

$$(\beta_{\nu} \partial / \partial x_{\nu} + L_1 \kappa_L c) \psi_L(x) = 0, \quad (7.2)$$

in which

$$\square = \partial / \partial x_{\nu} \cdot \partial / \partial x_{\nu}$$

and the suffix L means that ψ_L is a vector of an irreducible representation labeled by $L = (L_1, L_2)$. κ_L satisfies the relation (3.22). The latter becomes as

$$\kappa_L^2 \psi_L(x) = [m^2 + \hbar^2 / (2l^2 c^2) \{L_1(L_1 + 3) + L_2(L_2 + 1)\} + 1 / (4l^2 c^2) g] \psi_L(x) \quad (7.3)$$

with the aid of (6.6). (7.1) and (7.2) lead the relation (6.12) with the upper sign or

$$(\mu_{\nu\rho} \partial / \partial x_{\rho} - i \kappa_L c / \hbar \beta_{\nu} - i L_1 \hbar \partial / \partial x_{\nu}) \psi_L(x) = 0. \quad (7.4)$$

The Lagrange function which deduces the equations (7.1) and (7.2) by the variational principle is given by

$$\mathcal{L} = \sum_L \mathcal{L}_L, \quad (7.5)$$

$$\begin{aligned} \mathcal{L}_L &= 1/2 \{ \partial \bar{\psi}_L(x) / \partial x_{\nu} \cdot \partial \psi_L(x) / \partial x_{\nu} + \kappa_L^2 c^2 / \hbar^2 \bar{\psi}_L(x) \psi_L(x) \} \\ &\quad + 1/L_1 \{ \bar{\lambda}_L(x) \beta_{\nu} \cdot \partial \psi_L(x) / \partial x_{\nu} - \partial \bar{\psi}_L(x) / \partial x_{\nu} \cdot \beta_{\nu} \lambda_L(x) \} \\ &\quad + \kappa_L c \{ \bar{\lambda}_L(x) \psi_L(x) + \bar{\psi}_L(x) \lambda_L(x) \}, \end{aligned} \quad (7.6)$$

where $\lambda_L(x)$ and $\bar{\lambda}_L(x)$ are Lagrange's undetermined multipliers.

The Euler equations for variation

$$\delta I = \delta \int \mathcal{L} d^4x = 0, \quad (7.7)$$

$$d^4x = dx_1 dx_2 dx_3 dx_0$$

with respect to $\phi_L(x)$, $\bar{\psi}_L(x)$, $\lambda_L(x)$ and $\bar{\lambda}_L(x)$ are

$$\left\{ \begin{array}{l} (\square - \kappa_L^2 c^2 / \hbar^2) \phi_L(x) - (1/L_1) \beta_v \partial / \partial x_v + \kappa_L c \lambda_L(x) = 0, \\ (\beta_v \partial / \partial x_v + L_1 \kappa_L c) \phi_L(x) = 0, \end{array} \right. \quad (7 \cdot 8)$$

$$\left\{ \begin{array}{l} (\square - \kappa_L^2 c^2 / \hbar^2) \bar{\psi}_L(x) + 1/L_1 \partial \bar{\lambda}_L(x) / \partial x_v \cdot \beta_v - \kappa_L c \bar{\lambda}_L(x) = 0, \\ \partial \bar{\psi}_L(x) / \partial x_v \cdot \beta_v - L_1 \kappa_L c \bar{\psi}_L(x) = 0. \end{array} \right. \quad (7 \cdot 8)^*$$

$$\left\{ \begin{array}{l} (\square - \kappa_L^2 c^2 / \hbar^2) \bar{\psi}_L(x) + 1/L_1 \partial \bar{\lambda}_L(x) / \partial x_v \cdot \beta_v - \kappa_L c \bar{\lambda}_L(x) = 0, \\ \partial \bar{\psi}_L(x) / \partial x_v \cdot \beta_v - L_1 \kappa_L c \bar{\psi}_L(x) = 0. \end{array} \right. \quad (7 \cdot 2)^*$$

Operating $(\beta_v \partial / \partial x_v + L_1 \kappa_L c)$ on (7·8) from the left side, we have

$$(\beta_v \partial / \partial x_v + L_1 \kappa_L c)^2 \lambda_L(x) = 0 \quad (7 \cdot 9)$$

on account of (7·2). Similarly from (7·8)* and (7·2)* we have

$$\bar{\lambda}_L(x) (\beta_v \partial / \partial x_v - L_1 \kappa_L c)^2 = 0, \quad (7 \cdot 9)^*$$

where $\partial / \partial x_v$ means an operator to a function standing left of it. The second order differential equations (7·9) and (7·9)* result in

$$\lambda_L(x) = \bar{\lambda}_L(x) = \partial \lambda_L(x) / \partial x_v = 0 \quad (7 \cdot 10)$$

in all space-time, if they hold on any three dimensional space-like surface. As this assumption makes no contradiction with the equations (7·8), (7·2) and (7·8)*, (7·2)*, we assume the relations (7·10). Then (7·8) and (7·8)* become as follows:

$$(\square - \kappa_L^2 c^2 / \hbar^2) \phi_L(x) = 0 \quad (7 \cdot 11)$$

and

$$(\square - \kappa_L^2 c^2 / \hbar^2) \bar{\psi}_L(x) = 0. \quad (7 \cdot 11)^*$$

From variation (7·7) with respect to $\phi_L(x)$'s and their arguments we have the conservative quantities:

current vector

$$j_v^{(L)}(x) = ie / (2\hbar) (\partial \bar{\psi}_L(x) / \partial x_v \cdot \phi_L(x) - \bar{\psi}_L(x) \cdot \partial \phi_L(x) / \partial x_v) \quad (7 \cdot 12)$$

canonical energy momentum tensor

$$\begin{aligned} T_{\nu\lambda}^{(L)}(x) &= T_{\lambda\nu}^{(L)}(x) \\ &= 1/2 \{ \partial \bar{\psi}_L(x) / \partial x_v \cdot \partial \phi_L(x) / \partial x_\lambda + \partial \bar{\psi}_L(x) / \partial x_\lambda \cdot \partial \phi_L(x) / \partial x_v \} \\ &\quad - 1/2 \delta_{\nu\lambda} \{ \partial \bar{\psi}_L(x) / \partial x_p \cdot \partial \phi_L(x) / \partial x_p + \kappa_L^2 c^2 / \hbar^2 \cdot \bar{\psi}_L(x) \phi_L(x) \}, \end{aligned} \quad (7 \cdot 13)$$

angular momentum tensor

$$\begin{aligned} M_{\nu\lambda, \rho}^{(L)}(x) &= x_\nu T_{\lambda\rho}^{(L)}(x) - x_\lambda T_{\nu\rho}^{(L)}(x) \\ &\quad + i / (2\hbar) \{ \partial \bar{\psi}_L(x) / \partial x_p \cdot \mu_{\nu\lambda} \phi_L(x) - \bar{\psi}_L(x) \mu_{\nu\lambda} \cdot \partial \phi_L(x) / \partial x_p \}, \end{aligned} \quad (7 \cdot 14)$$

in the course of derivation of which we used (7·10). It is found with the aid of (7·11) and (7·11)* that (7·12) to (7·14) satisfy the continuity equations

$$\partial j_\nu^{(L)}(x) / \partial x_\nu = \partial T_{\nu\lambda}^{(L)}(x) / \partial x_\lambda = \partial M_{\nu\lambda, \rho}^{(L)}(x) / \partial x_\rho = 0.$$

The conservative quantities (7.12) to (7.14) are simplified owing to the relation (7.10), but, when there is an interacting field, $\lambda_L(x)$ and $\bar{\lambda}_L(x)$ can not be simply eliminated and these definitions (7.12) to (7.14) are not convenient to treat it. The alternative definition of these quantities can be made by using the equations (7.2) and (7.2)* which do not contain either $\lambda_L(x)$ or $\bar{\lambda}_L(x)$. For this purpose, as we know that (7.8), (7.2), (7.8)* and (7.2)* are consistent relations, it is convenient to use the Lagrange function

$$\begin{aligned} \tilde{\mathcal{L}}_L = & (\kappa_L c / 2 L_1 \hbar^2) \{ \bar{\psi}_L(x) \beta_v \cdot \partial \psi_L(x) / \partial x_v - \partial \bar{\psi}_L(x) / \partial x_v \cdot \beta_v \psi_L(x) \} \\ & + (\kappa_L^2 c^2 / \hbar^2) \bar{\psi}_L(x) \psi_L(x), \end{aligned} \quad (7.15)$$

with auxiliary conditions (7.11) and (7.11)*.

Variation

$$\delta \int \tilde{\mathcal{L}}_L(x) d^4x = 0$$

leads the conservative quantities

$$\tilde{j}_v^{(L)}(x) = i e \kappa_L c / (L_1 \hbar^3) \bar{\psi}_L(x) \beta_v \psi_L(x), \quad (7.16)$$

$$\tilde{T}_{\nu\lambda}^{(L)}(x) = \kappa_L c / (2 L_1 \hbar^2) \{ \bar{\psi}_L(x) \beta_\lambda \cdot \partial \psi_L(x) / \partial x_\nu - \partial \bar{\psi}_L(x) / \partial x_\nu \cdot \beta_\lambda \psi_L(x) \}, \quad (7.17)$$

$$\begin{aligned} \tilde{M}_{\nu\lambda, \rho}^{(L)}(x) = & x_\nu \tilde{T}_{\lambda\rho}^{(L)}(x) - x_\lambda \tilde{T}_{\nu\rho}^{(L)}(x) \\ & + i \kappa_L c / (2 L_1 \hbar^3) \{ \bar{\psi}_L(x) \beta_\rho \mu_{\nu\lambda} \psi_L(x) + \bar{\psi}_L(x) \mu_{\nu\lambda} \beta_\rho \psi_L(x) \} \end{aligned} \quad (7.18)$$

and the symmetric energy-momentum tensor can be introduced as

$$\begin{aligned} \tilde{\theta}_{\nu\lambda}^{(L)}(x) = & 1/2 \{ \tilde{T}_{\nu\lambda}^{(L)}(x) + \tilde{T}_{\lambda\nu}^{(L)}(x) \} - \kappa_L^2 c^2 / (2 \hbar^4) \bar{\psi}_L(x) (\beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu) \psi_L(x) \\ & - \kappa_L c / (4 L_1 \hbar^4) \{ \bar{\psi}_L(x) \beta_\rho (\beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu) \partial \psi_L(x) / \partial x_\rho \\ & - \partial \bar{\psi}_L(x) / \partial x_\rho (\beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu) \cdot \beta_\rho \psi_L(x) \}. \end{aligned} \quad (7.19)$$

It is meant by the equivalence of the quantities (7.12) to (7.14) and (7.16) to (7.19) that the differences among the corresponding quantities are expressed by divergences of some quantities. The equivalence can be shown, owing to the relations (7.4) and its adjoint form, i.e.,

$$\partial \psi_L(x) / \partial x_\nu = i / (L_1 \hbar) \mu_{\rho\nu} \partial \psi_L(x) / \partial x_\rho - \kappa_L c / (L_1 \hbar^2) \beta_\nu \psi_L(x) \quad (7.4)'$$

and

$$\partial \bar{\psi}_L(x) / \partial x_\nu = -i / (L_1 \hbar) \partial \bar{\psi}_L(x) / \partial x_\rho \cdot \mu_{\rho\nu} + \kappa_L c / (L_1 \hbar^2) \bar{\psi}_L(x) \beta_\nu \quad (7.4)''$$

derived from (7.2), (7.11), (7.2)* and (7.11)*, as, for example,

$$j_v^{(L)}(x) - \tilde{j}_v^{(L)}(x) = e / (2 L_1 \hbar^2) \cdot \partial / \partial x_\rho \cdot \{ \bar{\psi}_L(x) \mu_{\rho\nu} \psi_L(x) \},$$

and so on.

§ 8. Solutions of the field equations

It is convenient to treat half-odd-integral and integral fields separately. And we shall call the wave function for a half-odd-integral field $\psi(x)^{*)}$ and the one for an integral field $\varphi(x)$ instead of $\psi(x)$.

a) A half-odd-integral field

As the eigenvalues of β_ν/\hbar take all the half-odd-integral numbers between L_1 and $-L_1$, we have the following identity:

$$\prod_{n=1}^{L_1+1/2} \{ (a_\nu \beta_\nu)^2 - (n-1/2)^2 \hbar^2 a_\nu a_\nu \} \equiv 0, \quad (8.1)$$

where a_ν 's are components of a four vector, any two of which commute with each other. If we take $\partial/\partial x_\nu$ as a_ν and operate (8.1) on $\psi(x)$, we have, owing to (7.2),

$$\prod_{n=1}^{L_1+1/2} \left\{ \square - \frac{L_1^2}{(n-1/2)^2} \cdot \frac{\kappa^2 c^2}{\hbar^2} \right\} \psi(x) = 0. \quad (8.2)$$

Thus, every solution of (7.2) satisfies the equation (8.2). We can decompose $\psi(x)$ into such functions that satisfy the second order differential equations:

$$\psi(x) = \sum_{n=1}^{L_1+1/2} \eta_n^{(L_1)} \psi^{(n-1/2)}(x), \quad (8.3)$$

$$\psi^{(n-1/2)}(x) = \frac{1}{(2L_1-1)!} \prod_{m=1}^{L_1+1/2} \left\{ L_1^2 - (m-1/2)^2 \frac{\hbar^2}{\kappa^2 c^2} \square \right\} \psi(x), \quad (8.4)$$

$$(n=1, 2, \dots, L_1+1/2),$$

$$\left\{ \square - \frac{L_1^2}{(n-1/2)^2} \frac{\kappa^2 c^2}{\hbar^2} \right\} \psi^{(n-1/2)}(x) = 0, \quad (8.5)$$

where $\prod^{(n)}$ indicates absence of the n -th factor. $\eta_n^{(L_1)}$'s are defined by

$$\begin{aligned} \eta_n^{(L_1)} &= \frac{(2L_1-1)!}{L_1^{2L_1-1} \prod_{m=1}^{L_1+1/2} \{ 1 - (2m-1)^2 / (2n-1)^2 \}} \\ &= (-1)^{L_1+1/2-n} \frac{(2n-1)^{2L_1} (2L_1-1)!}{(2L_1)^{2L_1-1} (L_1+n-1/2)! (L_1-n+1/2)!} \end{aligned} \quad (8.6)$$

having the following properties

$$\sum_{n=1}^{L_1+1/2} \eta_n^{(L_1)} = (2L_1-1)! / L_1^{2L_1-1}, \quad (8.7)$$

$$\sum_{n=1}^{L_1+1/2} \eta_n^{(L_1)} / (2n-1)^{2k} = 0, \quad (k=1, 2, \dots, L_1-1/2). \quad (8.8)$$

* As we shall be concerned with a certain irreducible field $\psi_L(x)$, we shall omit suffix L after this section.

According to (7.11) there result

$$\psi^{(n-1/2)}(x) = 0, \quad n=1, \dots, L_1-1/2, \quad (8.9)$$

and

$$\psi^{(L_1)}(x) = \psi(x). \quad (8.10)$$

Now we can express the solution of (5.11) as

$$\psi(x) = a^{+,r}(k) e^{ik_\nu x_\nu} \quad (8.11a)$$

and

$$\psi(x) = a^{-,r}(k) e^{-ik_\nu x_\nu}, \quad (8.11b)$$

where k_ν 's follow, due to (7.11),

$$k_\nu k_\nu + \kappa^2 c^2 / \hbar^2 = 0. \quad (8.12)$$

As for the time component of k_ν the positive value should be adopted:

$$k_0 = -ik_i = \sqrt{k_i k_i + \kappa^2 c^2 / \hbar^2}, \quad (i=1, 2, 3).$$

Then (7.2) results in

$$(i/L_1 k_\nu \beta_\nu + \kappa c) a^{+,r}(k) = 0 \quad (8.13a)$$

and

$$(i/L_1 k_\nu \beta_\nu - \kappa c) a^{-,r}(k) = 0. \quad (8.13b)$$

$a^{+,r}(k)$'s and $a^{-,r}(k)$'s denote the independent solutions of (8.13a) and (8.13b) respectively, and the number of the independent solutions is equal to the number of multiplicity of the eigenvalue $L_1 \hbar$ of β_ν , namely, $2L_2+1$ as given by (4.37).

If $\psi(x)$ and $\bar{\psi}(x)$ are suitably normalized, we may write

$$\sum_\alpha a_\alpha^{*,+,r}(k) a_\alpha^{+,s}(k) = \delta_{rs}, \quad (8.14)$$

$$\sum_\alpha a_\alpha^{*,-,r}(k) a_\alpha^{-,s}(k) = \delta_{rs}, \quad (8.15)$$

$$\sum_\alpha a_\alpha^{*,+,r}(k) a_\alpha^{-,s}(k) = \sum_\alpha a_\alpha^{*,-,r}(k) a_\alpha^{+,s}(k) = 0, \quad (8.16)$$

where $a_\alpha^{\pm,r}(k)$ and $a_\alpha^{\pm*,r}(k)$ are the components of $a^{\pm,r}(k)$ and its complex conjugate, respectively. The number of components is found to be

$$1/6(2L_1+3)(2L_2+1)(L_1+L_2+2)(L_1-L_2+1)$$

by (4.35). Defining the adjoint function of $a^{\pm,r}(k)$ as

$$\bar{a}^{\pm,r}(k) = a^{*\pm,r}(k) \eta, \quad (8.17)$$

there hold

$$\bar{a}^{+,r}(k) (i/L_1 k_\nu \beta_\nu + \kappa c) = 0, \quad (8.18a)$$

$$\bar{a}^{-,r}(k) (i/L_1 k_\nu \beta_\nu - \kappa c) = 0. \quad (8.18b)$$

As the expression $\bar{a}^{\pm, r}(k) a^{\pm, s}(k)$ is a world scalar, this can be evaluated in the rest system. From (5.24) and (8.14) to (8.16) we have

$$\sum_{\alpha} \bar{a}_{\alpha}^{+, r}(k) a_{\alpha}^{+, s}(k) = \delta_{rs}, \quad (8.19)$$

$$\sum_{\alpha} \bar{a}_{\alpha}^{-, r}(k) a_{\alpha}^{-, s}(k) = -\delta_{rs}, \quad (8.20)$$

$$\sum_{\alpha} \bar{a}_{\alpha}^{+, r}(k) a_{\alpha}^{-, s}(k) = \sum_{\alpha} \bar{a}_{\alpha}^{-, r}(k) a_{\alpha}^{+, s}(k) = 0. \quad (8.21)$$

In order to obtain the explicit form of solutions, it is convenient to introduce an operator

$$F^{(L_1)}(\pm ik) = \frac{(-1)^{L_1+1/2}}{2(2L_1-1)!} \left(\frac{\hbar}{\kappa c}\right)^{2L_1} \left(\frac{\pm i}{L_1 \hbar} k_v \beta_v - \frac{\kappa c}{\hbar}\right) \cdot \sum_{n=1}^{L_1+1/2} \sum_{m=0}^{L_1+1/2-n} A_{n+m}^{(L_1)} (-L_1^2 \kappa^2 c^2 / \hbar^2)^m (1/\hbar k_v \beta_v)^{2n-2} (k_v k_v)^{L_1-n-m-1/2}, \quad (8.22)$$

where $A_n^{(L_1)}$ is the coefficient of x^n in the expression

$$\prod_{n=1}^{L_1+1/2} \{x - (n-1/2)^2\} = \sum_{n=0}^{L_1+1/2} A_n^{(L_1)} x^n. \quad (8.23)$$

When we multiply the operator $(i/(L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar)$ on the right or left side of (8.22), the identity (8.1) and the relation (8.23) allow us to get*

$$\begin{aligned} (i/(L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) F(\pm ik) &= F(\pm ik) (i/(L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) \\ &= \frac{\{(2L_1)!\}^2}{2^{L_1+1/2} \{(L_1-1/2)!\}^2 L_1^2 (2L_1-1)!} \left(\frac{\hbar}{\kappa c}\right)^{2L_1} \prod_{n=1}^{L_1+1/2} \left\{k_v k_v + \frac{L_1^2 \kappa^2 c^2}{(n-1/2)^2 \hbar^2}\right\}. \end{aligned} \quad (8.24)$$

This expression vanishes due to (8.12):

$$(i/(L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) F(\pm ik) = F(\pm ik) (i/(L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) = 0. \quad (8.25)$$

Hence, we have

$$\begin{aligned} F(\pm ik) F(\pm ik) &= (-1)^{L_1-1/2} / (2L_1-1)! (\hbar/\kappa c)^{2L_1-1} \cdot \sum_{n=1}^{L_1+1/2} \sum_{m=0}^{L_1+1/2-n} A_{n+m} L_1^{2(n+m-1)} (-\kappa^2 c^2 / \hbar^2)^{L_1-1/2} F(\pm ik) \\ &= 1/(2L_1-1)! \sum_{m=1}^{L_1+1/2} m A_m L_1^{2(m-1)} F(\pm ik) \\ &= 1/(2L_1-1)! \lim_{x \rightarrow L_1^2} d/dx \prod_{n=1}^{L_1+1/2} \{x - (n-1/2)^2\} F(\pm ik) \\ &= F(\pm ik), \quad (8.26) \\ F(\pm ik) F(\mp ik) &= 0. \quad (8.27) \end{aligned}$$

* In what follows hereafter we omit the suffix (L_1) of $F^{(L_1)}(\pm ik)$.

Similarly from (8.13), (8.18) and (8.22) we deduce the following relations:

$$\begin{cases} F(ik) a^{+,r}(k) = a^{+,r}(k), \\ F(ik) a^{-,r}(k) = 0, \\ F(-ik) a^{+,r}(k) = 0, \\ F(-ik) a^{-,r}(k) = a^{-,r}(k), \end{cases} \quad (8.28)$$

$$\begin{cases} \bar{a}^{+,r}(k) F(ik) = \bar{a}^{+,r}(k), \\ \bar{a}^{-,r}(k) F(ik) = 0, \\ \bar{a}^{+,r}(k) F(-ik) = 0, \\ \bar{a}^{-,r}(k) F(-ik) = \bar{a}^{-,r}(k). \end{cases} \quad (8.28)^*$$

With the relations (8.24) to (8.28) one sees that the operators $F(ik)$ and $F(-ik)$ are the projection operators to the positive and the negative frequency parts respectively. As is easily seen from (8.19) to (8.21) $\sum_{\alpha} a_{\alpha}^{\pm,s}(k) \bar{a}_{\alpha}^{\pm,r}(k)$ satisfies the same relations as $\pm F_{\alpha\beta}(\pm ik)$. We may thus put

$$\sum_{\alpha} a_{\alpha}^{+,r}(k) \bar{a}_{\alpha}^{+,r}(k) = F_{\alpha\beta}(ik), \quad (8.29a)$$

$$\sum_{\alpha} a_{\alpha}^{-,r}(k) \bar{a}_{\alpha}^{-,r}(k) = -F_{\alpha\beta}(-ik). \quad (8.29b)$$

The general solutions of the simultaneous equations (7.2), (7.11) and (7.2)*, (7.11)* can now be written down in terms of Fourier series as

$$\psi_{\alpha}(x) = V^{-1/2} \sum_k \sum_r (\hbar c/k_0)^{1/2} \{ \psi_{+}^{r,*}(k) a_{\alpha}^{+,r}(k) e^{ik_{\nu}x_{\nu}} + \psi_{-}^{r,*}(k) a_{\alpha}^{-,r}(k) e^{-ik_{\nu}x_{\nu}} \}, \quad (8.30)$$

$$\bar{\psi}_{\alpha}(x) = V^{-1/2} \sum_k \sum_r (\hbar c/k_0)^{1/2} \{ \psi_{+}^{r,*}(k) \bar{a}_{\alpha}^{+,r}(k) e^{-ik_{\nu}x_{\nu}} + \psi_{-}^{r,*}(k) \bar{a}_{\alpha}^{-,r}(k) e^{ik_{\nu}x_{\nu}} \}, \quad (8.30)^*$$

where V is the normalization volume, and \sum_k and \sum_r denote the summation over momenta and over all the states of the same momentum respectively. The coefficients $\psi_{\pm}^{r,*}(k)$ is the complex conjugate of $\psi_{\pm}^r(k)$.

By substituting (8.30) and (8.30)* into (7.12) and (7.13) or (7.16) and (7.17) we have for the total energy, the total momentum and the total charge

$$E = - \int T_{44}(x) dV = \hbar c \sum_k \sum_r |k_0| [\psi_{+}^{r,*}(k) \psi_{+}^r(k) - \psi_{-}^{r,*}(k) \psi_{-}^r(k)], \quad (8.31)$$

$$G_{\nu} = (1/ic) \int T_{i4}(x) dV = \hbar \sum_k \sum_r k_i [\psi_{+}^{r,*}(k) \psi_{+}^r(k) - \psi_{-}^{r,*}(k) \psi_{-}^r(k)], \quad (8.32)$$

($i=1, 2, 3$)

$$\mathcal{E} = (1/ic) \int j_4(x) dV = e \sum_k \sum_r [\psi_{+}^{r,*}(k) \psi_{+}^r(k) + \psi_{-}^{r,*}(k) \psi_{-}^r(k)]. \quad (8.33)$$

The indefiniteness of the sign of the total energy can be avoided in reference to the hole theory as in the case of Dirac's electron theory.

b) Integral field

As the eigenvalue of $(1/\hbar)\beta_\nu$ takes every integer between L_1 and $-L_1$, we have the identity

$$a_\nu \beta_\nu \prod_{n=1}^{L_1} \{ (a_\nu \beta_\nu)^2 - n^2 \hbar^2 a_\nu a_\nu \} \equiv 0, \quad (8.34)$$

where a_ν 's are the components of a four vector and are commutable with each other. By substituting $\partial/\partial x_\nu$ into a_ν and operating on $\varphi(x)^*$, from (7.2) we get

$$\prod_{n=1}^{L_1} \{ \square - L_1^2 \kappa^2 c^2 / (n^2 \hbar^2) \} \varphi(x) = 0,$$

in which $L_1 \kappa \neq 0$ is assumed. $\varphi(x)$ can be decomposed into the functions which are solutions of the second order differential equations:

$$\varphi(x) = \sum_{n=1}^{L_1} \zeta_n^{(L_1)} \varphi^{(n)}(x), \quad (8.35)$$

$$\varphi^{(n)}(x) = L_1 / (2L_1 - 1)! \prod_{m=1}^{L_1} (L_1^2 - (m^2 \hbar^2 / \kappa^2 c^2) \square) \varphi(x) \quad (8.36)$$

$$(n=1, \dots, L_1)$$

$$\{ \square - (L_1^2 \kappa^2 c^2 / n^2 \hbar^2) \} \varphi^{(n)}(x) = 0, \quad (8.37)$$

$$\zeta_n^{(L_1)} = \frac{(2L_1 - 1)!}{L_1^{2L_1-1} \prod_{m=1}^{L_1} (1 - m^2/n^2)} = (-1)^{L_1-n} \frac{2(2L_1 - 1)! n^{2L_1}}{L_1^{2L_1-1} (L_1 + n)! (L_1 - n)!}. \quad (8.38)$$

$\zeta_n^{(L_1)}$'s have the properties

$$\sum_{n=1}^{L_1} \zeta_n^{(L_1)} = (2L_1 - 1)! / L_1^{2L_1-1},$$

$$\sum_{n=1}^{L_1} \zeta_n^{(L_1)} / n^{2k} = 0, \quad (k=1, \dots, L_1 - 1).$$

From equation (7.11) we have

$$\varphi^{(n)}(x) = 0, \quad (n=1, \dots, L_1 - 1), \quad (8.39)$$

$$\varphi^{(L_1)}(x) = \varphi(x). \quad (8.40)$$

The momentum representation is obtained by putting

$$\varphi(x) = b^{+,r}(k) e^{i k_\nu x_\nu} \quad (8.41a)$$

and

$$\varphi(x) = b^{-,r}(k) e^{-i k_\nu x_\nu}, \quad (8.41b)$$

$$r=1, \dots, 2L_2 + 1,$$

* We call the wave function for an integral field $\varphi(x)$ instead of $\psi(x)$.

where k_v 's, by (7.11), satisfy the relation (8.12) and we take positive value for k_0 . From (7.2) $b^{\pm,r}(k)$'s satisfy the equations

$$(i/L_1 k_v \beta_v + \kappa c) b^{+,r}(k) = 0 \quad (8.42a)$$

and

$$(i/L_1 k_v \beta_v - \kappa c) b^{-,r}(k) = 0. \quad (8.42b)$$

If $b^{\pm,r}(k)$'s are suitably normalized, we may write

$$\sum_{\alpha} b_{\alpha}^{*+,r}(k) b_{\alpha}^{+,s}(k) = \delta_{rs}, \quad (8.43)$$

$$\sum_{\alpha} b_{\alpha}^{*- ,r}(k) b_{\alpha}^{-,s}(k) = \delta_{rs}, \quad (8.44)$$

$$\sum_{\alpha} b_{\alpha}^{*+,r}(k) b_{\alpha}^{-,s}(k) = \sum_{\alpha} b_{\alpha}^{*- ,r}(k) b_{\alpha}^{+,s}(k) = 0, \quad (8.45)$$

where the number of components $b_{\alpha}^{\pm}(k)$'s is given by

$$1/6(2L_1+3)(2L_2+1)(L_1+L_2+2)(L_1-L_2+1).$$

From (5.24) and (8.43) to (8.45) we have

$$\sum_{\alpha} \bar{b}_{\alpha}^{+,r}(k) b_{\alpha}^{+,s}(k) = \delta_{rs}, \quad (8.46)$$

$$\sum_{\alpha} \bar{b}_{\alpha}^{-,r}(k) b_{\alpha}^{-,s}(k) = \delta_{rs}, \quad (8.47)$$

$$\sum_{\alpha} \bar{b}_{\alpha}^{+,r}(k) b_{\alpha}^{-,s}(k) = \sum_{\alpha} \bar{b}_{\alpha}^{-,r}(k) b_{\alpha}^{+,s}(k) = 0. \quad (8.48)$$

The projection operators are given by

$$\begin{aligned} G^{(L_1)}(\pm ik) &= (1/2)(-1)^{L_1} L_1! / (2L_1^2 - 1)! (\hbar/\kappa c)^{2L_1} \\ &\cdot \left\{ (\pm 1/\hbar k_v \beta_v + iL_1 \kappa c/\hbar) \sum_{n=1}^{L_1} \sum_{m=1}^{L_1-n} B_{n+m}^{(L_1)} (-L_1^2 \kappa^2 c^2/\hbar^2)^m (k_v k_v)^{L_1-n-m} (\pm 1/\hbar k_v \beta_v)^{2n-1} \right. \\ &\quad \left. + (-1)^{L_1} (L_1!)^2 \prod_{n=1}^{L_1} (k_v k_v + L_1^2 \kappa^2 c^2/(\kappa^2 \hbar^2)) \right\}, \end{aligned} \quad (8.49)$$

where $B_n^{(L_1)}$ is the coefficient of x^n in the expression

$$\prod_{n=1}^{L_1} (x - n^2) = \sum_{n=1}^{L_1} B_n^{(L_1)} x^n. \quad (8.50)$$

By the similar manner with the previous subsection we have the following relations^{*}

$$\begin{aligned} ((i/L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) G(\pm ik) &= G(\pm ik) ((i/L_1 \hbar) k_v \beta_v \pm \kappa c/\hbar) \\ &= (1/2) L_1 (L_1!)^2 / (2L_1 - 1)! (\hbar/\kappa c)^{2L_1-1} \prod_{n=1}^{L_1} \{k_v k_v + L_1^2 \kappa^2 c^2/(n^2 \hbar^2)\} \\ &= 0, \quad (\text{according to (6.42)}) \end{aligned} \quad (8.51)$$

* In the followings we omit the suffix (L_1) of $G^{(L_1)}$.

$$G(\pm ik)G(\pm ik) = G(\pm ik), \quad (8 \cdot 52)$$

$$G(\pm ik)G(\mp ik) = 0, \quad (8 \cdot 53)$$

$$\begin{cases} G(\pm ik)b^{\pm,r}(k) = b^{\pm,r}(k), \\ G(\pm ik)b^{\mp,r}(k) = 0, \end{cases} \quad (8 \cdot 54)$$

$$\begin{cases} \bar{b}^{\pm,r}(k)G(\pm ik) = \bar{b}^{\pm,r}(k), \\ \bar{b}^{\mp,r}(k)G(\pm ik) = 0. \end{cases} \quad (8 \cdot 54)^*$$

Thus we may put

$$\sum_r b_{\alpha}^{+,r}(k)b_{\beta}^{+,r}(k) = G_{\alpha\beta}(ik), \quad (8 \cdot 55a)$$

$$\sum_r b_{\alpha}^{-,r}(k)b_{\beta}^{-,r}(k) = G_{\alpha\beta}(-ik). \quad (8 \cdot 55b)$$

The general solutions of the simultaneous equations (7.2), (7.11) and (7.2)*, (7.11)* can be written down by the Fourier series as

$$\begin{aligned} \varphi_{\alpha}(x) = V^{-1/2} \sum_k \sum_r (\hbar c/k_0)^{1/2} \{ \varphi_+^r(k) b_{\alpha}^{+,r}(k) e^{ik_{\nu}x_{\nu}} \\ + \varphi_-^{*r}(k) b_{\alpha}^{-,r}(k) e^{-ik_{\nu}x_{\nu}} \}, \end{aligned} \quad (8 \cdot 56)$$

$$\begin{aligned} \bar{\varphi}_{\alpha}(x) = V^{-1/2} \sum_k \sum_r (\hbar c/k_0)^{1/2} \{ \varphi_+^{*r}(k) \bar{b}_{\alpha}^{+,r}(k) e^{-ik_{\nu}x_{\nu}} \\ + \varphi_-^r(k) \bar{b}_{\alpha}^{-,r}(k) e^{ik_{\nu}x_{\nu}} \}. \end{aligned} \quad (8 \cdot 56)^*$$

From (7.12) and (7.13) or (7.16) and (7.17) we have for the total energy, the total momentum and the total charge

$$E = - \int T_{4i} dV = \hbar c \sum_k \sum_r |k_0| [\varphi_+^{*r}(k) \varphi_+^r(k) + \varphi_-^r(k) \varphi_-^{*r}(k)], \quad (8 \cdot 57)$$

$$\begin{aligned} G_i = 1/(ic) \int T_{i4} dV \\ = \hbar \sum_k \sum_r k_i [\varphi_+^{*r}(k) \varphi_+^r(k) + \varphi_-^r(k) \varphi_-^{*r}(k)], \end{aligned} \quad (8 \cdot 58)$$

$$\mathcal{E} = (1/ic) \int j_4 dV = e \sum_k \sum_r [\varphi_+^{*r}(k) \varphi_+^r(k) - \varphi_-^r(k) \varphi_-^{*r}(k)]. \quad (8 \cdot 59)$$

§ 9. The q-number field theory

We need not use the indefinite metrics on account of the simultaneous equations (7.2) and (7.11) which guarantee the vanishing condition for μ_{i1} in the rest system. We can quantize the half-odd-integral field according to Fermi-Dirac statistics and the integral field according to Bose-Einstein statistics.

a) Half-odd-integral field

We regard Fourier coefficients $\psi'_{\pm}(k)$ of $\psi(x)$ in the previous section as quantized

quantities and put the following bracket relation according to Pauli's exclusion principle :

$$[\psi_+^r(k), \psi_+^{*s}(k')]_+ = [\psi_-^r(k), \psi_-^{*s}(k')]_+ = \delta_{rs} \delta_{kk'}, \quad (9.1)$$

and the other anticommutation relations vanish. On account of (8.29) and (8.30), therefore, there holds

$$[\psi_\alpha(x), \bar{\psi}_\beta(x')]_+ = 1/V \sum_k \hbar c / k_0 \{ F_{\alpha\beta}(ik) e^{ik_\nu(x_\nu - x'_\nu)} - F_{\alpha\beta}(-ik) e^{-ik_\nu(x_\nu - x'_\nu)} \}.$$

If the summation is replaced by the integral, we have

$$\begin{aligned} [\psi_\alpha(x), \bar{\psi}_\beta(x')]_+ &= i\hbar c F_{\alpha\beta}(\partial/\partial x) D(x-x') \\ &= i\hbar c \mathcal{S}_{\alpha\beta}(x-x'), \end{aligned} \quad (9.2)$$

where

$$\begin{aligned} D(x) &= 1/(2\pi)^3 \int d^3k e^{ik_i x_i} \sin(|k_0| x_0) / |k_0|, \quad (i=1, 2, 3) \\ &= -i/(2\pi)^3 \int d^4k e^{ik_\nu x_\nu} \mathcal{E}(k) \delta(k^2 + \kappa^2 c^2 / \hbar^2), \end{aligned} \quad (9.3)$$

$$\mathcal{S}(x) = F(\partial/\partial x) D(x), \quad (9.4)$$

$$\begin{aligned} d^4k &= dk_1 dk_2 dk_3 dk_0, \\ \mathcal{E}(k) &= k_0 / |k_0|, \end{aligned} \quad (9.5)$$

and $F(\partial/\partial x)$ is given by (8.22), in which ik_ν is substituted by $\partial/\partial x_\nu$. As is easily seen from the relations (8.24), $\mathcal{S}(x)$ satisfies the following equations :

$$\{ (1/L\hbar) \gamma_\nu \partial/\partial x_\nu + \kappa c / \hbar \} \mathcal{S}(x) = 0, \quad (9.6)$$

$$(\square - \kappa^2 c^2 / \hbar) \mathcal{S}(x) = 0. \quad (9.7)$$

Let F be any Hermitian operator of the c -number theory, then

$$\psi^* F \psi$$

is, after Heisenberg, to be replaced by

$$1/2 (\psi^* F \psi - \psi F^* \psi^*)$$

in the quantum theory. By this rule, (8.31) to (8.33) in quantum theory can be written as

$$\begin{aligned} E &= \hbar c \sum_k \sum_r |k_0| [1/2 (\psi_+^{*r}(k) \psi_+^r(k) - \psi_+^{r*}(k) \psi_+^{*r}(k)) \\ &\quad + 1/2 (-\psi_-^r(k) \psi_-^{*r}(k) + \psi_-^{r*}(k) \psi_-^r(k))], \end{aligned}$$

$$\begin{aligned} G_i &= \hbar \sum_k \sum_r k_i [1/2 (\psi_+^{*r}(k) \psi_+^r(k) - \psi_+^{r*}(k) \psi_+^{*r}(k)) \\ &\quad + 1/2 (-\psi_-^r(k) \psi_-^{*r}(k) + \psi_-^{r*}(k) \psi_-^r(k))], \end{aligned}$$

$$\begin{aligned} \mathcal{E} &= e \sum_k \sum_r [1/2 (\psi_+^{*r}(k) \psi_+^r(k) - \psi_+^{r*}(k) \psi_+^{*r}(k)) \\ &\quad + 1/2 (\psi_-^r(k) \psi_-^{*r}(k) - \psi_-^{r*}(k) \psi_-^r(k))]. \end{aligned}$$

Or, with (9.1) and the definition

$$\begin{cases} N_r^+(k) = \psi_{+}^{*r}(k) \psi_{+}^r(k), \\ N_r^-(k) = \psi_{-}^{*r}(k) \psi_{-}^r(k), \end{cases} \quad (9.8)$$

they can be reduced to

$$E = \hbar c \sum_k \sum_r |k_0| [N_r^+(k) + N_r^-(k) - 1], \quad (9.9)$$

$$G_i = \hbar \sum_k \sum_r k_i [N_r^+(k) + N_r^-(k) - 1], \quad (9.10)$$

$$\mathcal{E} = c \sum_k \sum_r [N_r^+(k) - N_r^-(k)]. \quad (9.11)$$

On account of the bracket relations (9.1) the eigenvalues of both $\psi_{+}^{*r} \psi_{+}^r$ and $\psi_{-}^{*r} \psi_{-}^r$ are 0 or 1 and they can, therefore, serve as the definitions of the number of particles. The choice (9.8) is made so that the energy is the smallest when all N vanish; thus this corresponds to the case of the vacuum. As a consequence we obtain a negative zero-point energy of the vacuum which amounts to a half quantum per eigenvibration.

b) Integral field

For the integral field, according to Einstein-Bose statistics, we put the relations

$$[\varphi_r^+(k), \varphi_{+}^{*s}(k')]_- = [\varphi_r^-(k), \varphi_{-}^{*s}(k')]_- = \delta_{rs} \delta_{kk'}, \quad (9.12)$$

and all other commutators vanish. Then from (8.55) and (8.56) we have

$$\begin{aligned} [\varphi_\alpha(x), \bar{\varphi}_\beta(x')]_- &= \hbar c / V \sum_k (1/2k_0) \{ G_{\alpha\beta}(ik) e^{ik_\nu(x_\nu - x'_\nu)} \\ &\quad - G_{\alpha\beta}(-ik) e^{-ik_\nu(x_\nu - x'_\nu)} \} \end{aligned}$$

or

$$\begin{aligned} [\varphi_\alpha(x), \bar{\varphi}_\beta(x')]_- &= i\hbar c G_{\alpha\beta}(\partial/\partial x) D(x - x') \\ &= i\hbar c T_{\alpha\beta}(x - x'). \end{aligned} \quad (9.13)$$

$G(\partial/\partial x)$ is given by (8.49) and

$$T(x) = G(\partial/\partial x) D(x). \quad (9.14)$$

$T(x)$ satisfies the equations

$$((1/L)\hbar) \beta_\nu \partial/\partial x_\nu + \kappa c / \hbar) T(x) = 0, \quad (9.15)$$

$$(\square - \kappa^2 c^2 / \hbar^2) T(x) = 0. \quad (9.16)$$

We define the number operators by

$$\begin{cases} N_r^+(k) = \varphi_{+}^{*r}(k) \varphi_{+}^r(k), \\ N_r^-(k) = \varphi_{-}^{*r}(k) \varphi_{-}^r(k). \end{cases} \quad (9.17)$$

When we replace the c-number relation

$$\varphi^* F \varphi$$

by the q -number relation

$$1/2(\varphi^*F\varphi + \varphi F^*\varphi^*),$$

(8.57) to (8.59) can be rewritten as

$$E = \hbar c \sum_k \sum_r |k_0| [N_r^+(k) + N_r^-(k) + 1],$$

$$G_i = \hbar \sum_k \sum_r k_i [N_r^+(k) + N_r^-(k) + 1],$$

$$\mathcal{E} = c \sum_k \sum_r [N_r^+(k) - N_r^-(k)].$$

The energy is the smallest when all N vanish and this corresponds to the case of the vacuum. We obtain a positive zero-point energy of the vacuum which amounts to a half quantum per eigenvibration.

§ 10. Concluding remarks

The relativistic quantum field theory of a free rigid sphere has been obtained by extending its non-relativistic mechanics. The representation of the full Lorentz group is given by unitary trick from the representation of the rotational group of the five-dimensional Euclidean Space. As the representation thus obtained is reducible with regard to Lorentz transformation, it is necessary to pick up an irreducible subspace by a condition as (6.13). This condition guarantees requirement (II) of § 1 and avoids the difficulty of negative probability or negative energy. (6.13) is equivalent to the condition of vanishing divergence

$$\partial/\partial x_\nu \cdot T_{\nu\lambda\rho} \dots = 0$$

in the case of tensor formalism.

Requirement (II) of § 1 is satisfied by taking eigenfunctions of a belonging to an arbitrary eigenvalues m , but, when one chooses (6.13), one can reduce the differences between (7.12) to (7.14) and (7.16) to (7.19) to divergences of some functions. Besides, as this selection leads (7.14) and its adjoint form, it enables us to use canonical formalism, namely, the fourth component of (7.14) and its adjoint give

$$\partial\psi/\partial t = -c/(L_1\hbar) \{ \mu_{p4} \partial/\partial x_p + i\kappa c/\hbar \beta_4 \} \psi, \quad (10.1)$$

$$\partial\bar{\psi}/\partial t = c/(L_1\hbar) \bar{\psi} \{ \mu_{p4} \tilde{\partial}/\partial x_p + i\kappa c/\hbar \beta_4 \}. \quad (10.1)^*$$

With the aid of (10.1) and (10.1)* together with (7.2) and (7.2)* it can be shown that the expectation value of

$$-ic/(L_1\hbar^2) \beta_{i4} \beta_4 \quad (10.2)$$

is equal to the time derivative of the expectation value of x_i :

$$\begin{aligned} & \partial/\partial t \int \bar{\psi} \beta_{i4} x_i \psi dV \\ &= \int (\partial\bar{\psi}/\partial t \cdot \beta_{i4} x_i \psi + \bar{\psi} \beta_{i4} x_i \partial\psi/\partial t) dV \end{aligned}$$

$$= c/(L_1 \hbar) \int (\partial \bar{\psi} / \partial x_p \cdot \mu_{p4} \beta_4 x_i \psi - \bar{\psi} \beta_4 x_i \mu_{p4} \partial \psi / \partial x_p) dV$$

(by (10.1) and (10.1)*)

$$= -ic/(L_1 \hbar^2) \int (\partial \bar{\psi} / \partial x_p \cdot \beta_{p4} \beta_4^2 x_i \psi + \bar{\psi} \beta_4^2 x_i \beta_p \partial \psi / \partial x_p$$

$$- \partial \bar{\psi} / \partial x_p \beta_{4i} \beta_p \beta_4 x_i \psi - \bar{\psi} \beta_{4i} \beta_p \beta_4 x_i \partial \psi / \partial x_p) dV$$

(by (3.19))

$$= -ic/(L_1 \hbar^2) \int \bar{\psi} \beta_4 (\beta_{4i} \beta_p) \psi dV, \quad (\text{by (7.2), (7.2)* and integration by parts}).$$

Thus, (10.2) can be interpreted as the velocity operator in the same way as the case of Dirac's electron theory.

The subsidiary condition causes some complications when interactions are introduced. In this case variation using Lagrangian (7.15) is convenient. The interaction with the electromagnetic field will be discussed in the succeeding paper.

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A Statistical Theory of Linear Dissipative Systems, II*

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The statistical theory based on the probability of a given succession of non-equilibrium states of a linear dissipative systems given by Onsager and Machlup is proved by direct calculation to be equivalent to the theory of Brownian motion discussed by Wang and Uhlenbeck based on the Fourier series method of Rice. A function, which determines the above-mentioned probability and is constructed from the dissipation functions and the rate of entropy production, is shown to increase on the average with the lapse of time and remain constant (i.e., have the minimum value zero) when and only when the system obeys the phenomenological linear relations given by the thermodynamics of irreversible processes. This theorem is an extension of the second law of thermodynamics. When the system is characterized by a single relaxation time, the expression of the above-mentioned probability is a little simplified.

The method of fluctuating distribution function is developed. When it is applied to the theory of the shape of collision broadened spectral line, the new strong collision treatment proposed by Gross is proved to give practically the same result as that obtained by the theory of Brownian motion of harmonic oscillator. Finally, this method is applied to the fluctuation of electronic distribution in metals (the Johnson noise). The correlation function of electric current thus obtained is in accord with that given by Bakker-Heller and by Spenke. The correlation of heat flow and the cross correlation of electric current and heat flow are also given.

§ 1. Introduction

In a previous paper,²⁾ to which we shall refer as Part I, we could generalize the theory of Brownian motion so as to include the systems treated in "thermodynamics of irreversible processes" based on Onsager's statistical interpretation of the principle of the least dissipation of energy. Recently Onsager and Machlup³⁾ have proposed a more complete formulation of the same problem by confining themselves to the case of Gaussian random variables. These theories give the joint probability of a given succession of non-equilibrium states of the system in terms of a time integral of some function constructed from the dissipation functions and the rate of entropy production, and suggest that this function has a statistical significance similar to that of the entropy in the theory of equilibrium. In the present paper we shall first examine the relation of these theories to the method of Rice in the theory of Brownian motion^{4),5)} (§ 2), and then treat the problem of analogy between the function mentioned above and the entropy (§ 3). The principal aim of the

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present paper lies, however, in the application or extension of our theory to such variables that are not thermodynamical but can be treated in a sense thermodynamically. As an example we shall take up the number distribution function in the kinetic theory of gaseous substances. This distribution function describes the state of the system more precisely than the thermodynamical variables, whereas its equilibrium value is determined by the thermodynamical condition of entropy maximum. We shall illustrate the method of fluctuating distribution function in the case of collision broadening of spectral lines (§ 4), and show its application somewhat in detail in the case of fluctuations of electronic distribution in metals (§ 5).

For the sake of referring convenience we shall summarize the results of the previous theories.

The *linear dissipative system* is the system for which the rate of temporal change of thermodynamical state variables is linearly related to the corresponding thermodynamical forces. For simplicity let us consider an isolated system, and let us specify its macroscopic state by a set of macroscopic (thermodynamical) variables $\alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_N)$. If $S(\alpha)$ denotes the entropy of the system at a state α , the gradient of entropy with respect to α is the thermodynamical force (divided by temperature) corresponding to α . Then the rate of change of α is determined by the phenomenological relations

$$d\alpha_i/dt = \sum_{j=1}^N G_{ij} \cdot \partial S(\alpha) / \partial \alpha_j, \quad (1.1)$$

where G 's denote the kinetic coefficients, to which Onsager-Casimir's reciprocal theorem is applied.^(6,7) In the following, including §§ 2 and 3, we shall write down the expressions with the symmetric G . The extension to the case of general coefficients is formal and easy (see Part I), and we shall make use of such extended expressions in §§ 4 and 5.

In classical thermodynamics the equilibrium state α^0 is characterized by the equilibrium condition

$$S(\alpha) = \max. \quad (1.2)$$

(with respect to α), which is an expression of the second law. The classical thermodynamics is the first approximation to the truth. In the second approximation we should incorporate fluctuations of α into it, and we can not get a correct formulation of the second law itself, till we do so.⁽⁸⁾ Boltzmann's statistical interpretation of the second law includes the statement⁽⁹⁾ (Boltzmann's principle) that an arbitrary non-equilibrium state α may occur with the probability

$$W(\alpha) \propto \exp \{S(\alpha) - S(\alpha^0)\} / k \quad (1.3)$$

as far as the conservation laws of mass, energy, momentum, and so on are not violated. k is Boltzmann's constant.

Boltzmann's statistical interpretation of the second law contains furthermore the statement⁽⁹⁾ that the entropy may increase as well as decrease, though it must increase on the average. The process described by the solution of equation (1.1) always increases the value of entropy. Thus the thermodynamics of irreversible processes is again no more

than the first approximation. As was shown in Part I, we can get the second approximation based on the linear relations (1.1) and Boltzmann's principle (1.3). The equations of motion (1.1), which determine an average course of the process, are derived from the variation principle, *the principle of the least dissipation of energy*,

$$dS/dt - \Phi(\dot{\alpha}, \dot{\alpha}) = \max. \quad (1.4)$$

(with respect to $\dot{\alpha} = d\alpha/dt$ leaving α constant), where Φ denotes the dissipation function (divided by temperature)

$$\Phi(\dot{\alpha}, \dot{\alpha}) \equiv 1/2 \cdot \sum_{i,j=1}^N (G^{-1})_{ij} \cdot d\alpha_i/dt \cdot d\alpha_j/dt, \quad (1.5)$$

G^{-1} being the reciprocal of G . Onsager's statistical interpretation of the principle of the least dissipation of energy (1.4) states that any modified form of the equations of motion

$$d\alpha_i/dt = \sum_{j=1}^N G_{ij} \cdot \partial S(\alpha) / \partial \alpha_j + \mathcal{F}_i(t) \quad (1.6)$$

may occur with a certain probability. Here $\mathcal{F}_i(t)$ denotes any continuous one-valued function of time t , whose time average is zero. Since in the principle (1.4) the variation is taken with respect to $\dot{\alpha}$, the probability mentioned above will be a conditional one, by which the probability of a change $\Delta\alpha$ of α during a macroscopically small time interval Δt is determined. If we denote this conditional probability from a state α to another $\alpha + \Delta\alpha$ by $\Psi(\alpha, \Delta t, \alpha + \Delta\alpha)$, Onsager's statement is equivalent to

$$\Psi(\alpha, \Delta t, \alpha + \Delta\alpha) \propto \exp[1/(2k) \cdot \{\Delta S - \Phi(\Delta\alpha, \Delta\alpha)/\Delta t\}], \quad (1.7)$$

where $\Phi(\Delta\alpha, \Delta\alpha)$ is obtained from $\Phi(\dot{\alpha}, \dot{\alpha})$ by replacing $\dot{\alpha}$ with $\Delta\alpha$. This transition probability (density) gives in fact the phenomenological relations (1.1), in the limit of $\Delta t \rightarrow 0$, as the most probable course of the process. If an explicit form of the function $\mathcal{F}_i(t)$ is once given, we can integrate in principle the equations of motion (1.1), and get a graph $\alpha(t)$ in the (α, t) -plane, which we shall call *path*. This path, of course, starts from a given initial state α' at a time t' . Since we may give other forms for $\mathcal{F}_i(t)$, we obtain in general a bundle of paths starting from the given initial point (α', t') . For each path of this bundle we can ascribe a probability, the conditional probability of path, provided that the time series described by the path $\alpha(t)$ is Markoffian. If we divide the time axis into successive time intervals of length Δt , the increase $\Delta\alpha$ at an interval is independent of those at other intervals, and has the probability (1.7). Thus the conditional probability of a path will be given by the product of (1.7) for the successive intervals. As was shown in Part I, the transition probability (1.7) is sufficient to give Boltzmann's principle (1.3) as the equilibrium distribution of α . Thus in the following, we shall call (1.7) [and (1.11) or (1.12)] *Onsager's principle*.

Onsager and Machlup⁽³⁾ gave a closed expression for the probability of path. In the expression (1.7) the normalization factor is in general a function of α and Δt , and this value of α changes from interval to interval mentioned above. Therefore it is more convenient to write some part of the normalization factor in the exponent. Remembering

the fact²⁾ that (1.7) is a Gaussian distribution with regard to $\mathcal{J}\alpha$, let us introduce a function

$$\mathcal{L}(\alpha, \dot{\alpha}) \equiv \Phi(\dot{\alpha}, \dot{\alpha}) + \Phi^{-1}(\partial S/\partial \alpha, \partial S/\partial \alpha) - dS/dt, \quad (1.8)$$

where

$$\Phi^{-1}(\partial S/\partial \alpha, \partial S/\partial \alpha) \equiv 1/2 \cdot \sum_{i,j=1}^N G_{ij} \cdot \partial S/\partial \alpha_i \cdot \partial S/\partial \alpha_j \quad (1.9)$$

is a kind of dissipation function made up of the thermodynamical forces. The explicit expression of (1.8) is as follows:

$$\mathcal{L}(\alpha, \dot{\alpha}) = 1/2 \cdot \sum_{i,j=1}^N (G^{-1})_{ij} \{d\alpha_i/dt - \sum_{l=1}^N G_{il} \cdot \partial S/\partial \alpha_l\} \{d\alpha_j/dt - \sum_{m=1}^N G_{jm} \cdot \partial S/\partial \alpha_m\}. \quad (1.10)$$

We see at once the exponent of (1.7) (including a part of the normalization factor) being given by the trapezoidal approximation of the time integral of (1.10) over the time interval $\mathcal{J}t$. Thus the product of (1.7) can be written, after taking the limit $\mathcal{J}t \rightarrow 0$, in the form

$$\mathcal{P}[\alpha(t); t' \leq t \leq t''] \propto \exp \left\{ -1/(2k) \cdot \int_{t'}^{t''} \mathcal{L}(\alpha, \dot{\alpha}) dt \right\}. \quad (1.11)$$

Here the product is replaced with a continuous product. Mathematically speaking, as was emphasized by Onsager,^{1),2)} this type of expression contains many difficulties such that Dirac's δ -function in quantum mechanics has encountered in the theory of functions of real variables. These difficulties may be avoided for instance by making use of Kolmogoroff's slalom gate description as was done by Onsager and Machlup.³⁾ We shall, however, not touch on these mathematical points in the present paper, and reserve the more intuitive form of expressions such as (1.11). We should therefore bear in mind that we must always come back to the original discrete $(\mathcal{J}\alpha, \mathcal{J}t)$ descriptions when we concern ourselves with principles. In the probability (1.11) the path $\alpha(t)$ is restricted to satisfy the initial condition $\alpha(t') = \alpha'$. If we want to remove this condition, there are two ways. First, we multiply (1.11) by the probability (1.3) of occurrence of the initial state α' and make the joint probability of a path $\alpha(t)$ in the closed interval $t' \leq t \leq t''$. Second, remembering the Markoffian character of our time series, we take the limit $t' \rightarrow -\infty$ in the expression, and consider the result only for a finite time interval. In the latter case we may select any state α' as the initial one, since no effect of the initial condition does remain after sufficiently long time. Thus we may for example suppose α' being apart from the equilibrium state α^0 infinitesimally. This means that our process is replaced with one that occurs as a spontaneous fluctuation in an aged system. For the sake of symmetry of the past and the future we write $+\infty$ in place of t'' in (1.11) and put $\alpha'' = \alpha(t'')$ to be a state infinitesimally near α^0 . Then we obtain the probability of a path $\alpha(t)$, $-\infty < t < +\infty$:

$$W[\alpha(t)] \propto \exp \left\{ -1/(2k) \cdot \int_{-\infty}^{+\infty} \mathcal{L}(\alpha, \dot{\alpha}) dt \right\}. \quad (1.12)$$

In conformity with its importance in our theory we shall call the function $\mathcal{L}(\alpha, \dot{\alpha})$ defined by (1.8) the *thermodynamical Lagrangian*.

The conditional (transition) probability of two events separated by a finite time interval is obtained by adding all contributions from paths connecting these events. This sum is represented by the *path integral*,⁽¹⁾ which is an extension of Wiener's integral⁽¹⁰⁾:

$$\mathcal{P}(\alpha', t'; \alpha'', t'') \propto \int \exp \left\{ -1/(2k) \cdot \int_{t'}^{t''} \mathcal{L}(\alpha, \dot{\alpha}) dt \right\} \mathcal{D}\alpha(t). \quad (1.13)$$

Here $\alpha(t') = \alpha'$ and $\alpha(t'') = \alpha''$ are respectively the initial and the final conditions for each path $\alpha(t)$. In the case of Gaussian process Onsager and Machlup⁽³⁾ have proved a theorem that the path integration can be replaced with a minimization procedure:

$$\mathcal{P}(\alpha', t'; \alpha'', t'') \propto \exp \left\{ -\min_{\alpha(t)} 1/(2k) \cdot \int_{t'}^{t''} \mathcal{L}(\alpha, \dot{\alpha}) dt \right\}. \quad (1.14)$$

The variation in the exponent means of course to be taken over all paths starting from α' at time t' and reaching α'' at t'' . We may confine ourselves only to the most probable path when we consider the transition probability. Namely, for a given initial and a final state there are one most probable path, which is determined by the variation principle

$$\int_{t'}^{t''} \mathcal{L}(\alpha, \dot{\alpha}) dt = \min. \quad (1.15)$$

In the preceding paragraphs we have expressed the probability in terms of the solution $\alpha(t)$ of the equations of motion, *Langevin's equation*, (1.6). Remembering the explicit form of the thermodynamical Lagrangian (1.10), we see at once the probability obtained above, (1.12), being rewritten in terms of the *fluctuating term* $\mathcal{J}(t)$. Namely, eliminating $\dot{\alpha}$ in (1.10) by making use of the equations of motion (1.6), we get

$$\mathcal{L}(\alpha, \dot{\alpha}) = 1/2 \cdot \sum_{i,j=1}^N (G^{-1})_{ij} \mathcal{J}_i(t) \mathcal{J}_j(t), \quad (1.16)$$

and hence (1.12) becomes

$$\mathcal{W}[\mathcal{J}(t)] \propto \exp \left\{ -1/4 \cdot \int_{-\infty}^{+\infty} \sum_{i,j=1}^N (D^{-1})_{ij} \mathcal{J}_i(t) \mathcal{J}_j(t) dt \right\}. \quad (1.17)$$

Here we have defined the diffusion coefficient in the α -space

$$D \equiv kG. \quad (1.18)$$

The Gaussian distribution (1.17) was the starting point of the theory of Brownian motion given by Ornstein, Uhlenbeck and others.⁽¹¹⁾ In fact, the first two moments are

$$\langle \mathcal{J}_i(t) \rangle = 0, \quad \langle \mathcal{J}_i(t) \mathcal{J}_j(t') \rangle = 2D_{ij} \delta(t' - t), \quad (1.19)$$

which agree with their expressions.⁽⁵⁾ Thus we find that our theory based on Onsager's

principle does not contain anything physically new besides those contained in the pre-War theories of Brownian motion, although our theory is extended to include all linear dissipative systems. Our formulation provides, however, *Einstein's relation* (1.18) on a general basis, and this makes the application of our theory more easily and more systematically.

§ 2. Relation to the method of Rice

In this section we shall show the equivalence of the method of the probability of path described in § 1 and the method of Fourier series (method of Rice⁽¹⁾) discussed by Wang and Uhlenbeck.⁽²⁾ First we derive the fundamental relations in the latter method by the former. Then we shall give a sketch of the proof of (1.12) by the latter method, based on the assumptions proposed by Onsager and Machlup.⁽³⁾ Our proof is not so elegant as that given by them, but we hope that it shows somewhat more clearly its relation to the method of Rice.

Onsager and Machlup have started from the assumption, that our time series is Gaussian. By this assumption, first of all, the probability of a state (1.3) has to be Gaussian, and thus we should confine ourselves to the case of quadratic entropy treated in § 3 of Part I:

$$S(\alpha) = S(\alpha^0) - 1/2 \cdot \sum_{i,j=1}^N S_{ij} (\alpha_i - \alpha_i^0) (\alpha_j - \alpha_j^0). \quad (2.1)$$

The coefficients $S_{ij} = -\partial^2 S / \partial \alpha_i \partial \alpha_j$ are the elements of a positive definite symmetric matrix S . The positive definiteness is due to the second law of thermodynamics. In this case the thermodynamical force (divided by temperature) associated with the variable α , becomes a linear combination of the deviations $\alpha_j - \alpha_j^0$ from equilibrium α^0 :

$$\partial S(\alpha) / \partial \alpha_i = - \sum_{j=1}^N S_{ij} (\alpha_j - \alpha_j^0). \quad (2.2)$$

As was done in Part I, it is convenient to introduce a matrix

$$A \equiv GS, \quad (2.3)$$

eigenvalues of which are the reciprocal of relaxation times [cf. (4.25) and (5.22)]. Then the thermodynamical Lagrangian (1.10) can be written in the form

$$\mathcal{L}^B(\alpha, \dot{\alpha}) = 1/2 \cdot \sum_{i,j=1}^N (G^{-1})_{ij} \{ \dot{\alpha}_i + \sum_{l=1}^N A_{il} (\alpha_l - \alpha_l^0) \} \{ \dot{\alpha}_j + \sum_{m=1}^N A_{jm} (\alpha_m - \alpha_m^0) \}. \quad (2.4)$$

In the method of Fourier series, we expand a function $\alpha(t)$ of time t into the Fourier series first within a sufficiently large but finite time interval $(-\theta/2, +\theta/2)$:

$$\alpha_i(t) = \alpha_i^0 + \sum_{\tau=1}^{\infty} \{ a_{i\tau} \cos(\omega_{\tau} t) + b_{i\tau} \sin(\omega_{\tau} t) \}, \quad (2.5)$$

and take the limit $\theta \rightarrow \infty$ after all required results are obtained. In the expression (2.5), ω_{τ} denotes a discrete value of angular frequency: $\omega_{\tau} = 2\pi\gamma/\theta$, $\gamma = 1, 2, \dots$. The expansion

coefficients a and b are parameters determining the shape of a path $\alpha(t)$. If we regard $\alpha(t)$ as a random function, they become a set of stochastic variables. The parametrization of paths may in general be done by making use of any complete orthonormal set of functions of t ,¹²⁾ or more generally of functions of t and i if the index i is a continuous parameter such as position coordinates, say.

First we shall derive from (1.12) the probability distribution of the parameters a and b . Returning to the parametrization by the Fourier series (2.5), we can transform the expression (2.4) into

$$\begin{aligned} \mathcal{L}(\alpha, \dot{\alpha}) = & 1/2 \cdot \sum_{i,j=1}^N (G^{-1})_{ij} \sum_{\tau,\varepsilon=1}^{\infty} \left\{ \left(\sum_{l=1}^N A_{il} a_{l\tau} + \omega_{\tau} b_{i\tau} \right) \cos(\omega_{\tau} t) \right. \\ & + \left(\sum_{l=1}^N A_{il} b_{l\tau} - \omega_{\tau} a_{i\tau} \right) \sin(\omega_{\tau} t) \left. \right\} \left\{ \left(\sum_{m=1}^N A_{jm} a_{m\varepsilon} + \omega_{\varepsilon} b_{j\varepsilon} \right) \cos(\omega_{\varepsilon} t) \right. \\ & + \left. \left(\sum_{m=1}^N A_{jm} b_{m\varepsilon} - \omega_{\varepsilon} a_{j\varepsilon} \right) \sin(\omega_{\varepsilon} t) \right\}, \end{aligned} \quad (2.6)$$

or, after integration over the time interval $(-\theta/2, +\theta/2)$,

$$\begin{aligned} & 1/(2k) \int_{-\theta/2}^{+\theta/2} \mathcal{L}(\alpha, \dot{\alpha}) dt \\ &= \theta/2 \cdot \sum_{\tau=1}^{\infty} 1/(4k) \sum_{i,j=1}^N (G^{-1})_{ij} \left\{ \left(\sum_{l=1}^N A_{il} a_{l\tau} + \omega_{\tau} b_{i\tau} \right) \left(\sum_{m=1}^N A_{jm} a_{m\tau} + \omega_{\tau} b_{j\tau} \right) \right. \\ & \quad \left. + \left(\sum_{l=1}^N A_{il} b_{l\tau} - \omega_{\tau} a_{i\tau} \right) \left(\sum_{m=1}^N A_{jm} b_{m\tau} - \omega_{\tau} a_{j\tau} \right) \right\} \\ &= (\theta/2) \sum_{\tau=1}^{\infty} 1/(4k) \sum_{i,j=1}^N (G^{-1})_{ij} \sum_{l,m=1}^N (A_{il} A_{jm} + \omega_{\tau}^2 \delta_{il} \delta_{jm}) (a_{l\tau} a_{m\tau} + b_{l\tau} b_{m\tau}) \\ &= (\theta/2) \sum_{\tau=1}^{\infty} 1/(4k) \sum_{l,m=1}^N [G^{-1} (A^2 + \omega_{\tau}^2)]_{lm} (a_{l\tau} a_{m\tau} + b_{l\tau} b_{m\tau}). \end{aligned} \quad (2.7)$$

Here we have made use of the symmetric properties of G and S . Now let us introduce the matrices

$$g_{\alpha}(\omega) \equiv (A^2 + \omega^2)^{-1} g_F(\omega), \quad g_F(\omega) \equiv 4D, \quad (2.8)$$

which are both symmetric. The symmetry of $g_{\alpha}(\omega)$ is obvious by virtue of the relation $4k g_{\alpha}^{-1}(\omega) = G^{-1} (A^2 + \omega^2) = S G S + \omega^2 G^{-1}$. From (1.12), (2.7), and (2.8) we get the probability distribution of parameters a and b , which will be obtained in the limit $\theta \rightarrow \infty$ from

$$W_{\theta}(a, b) \propto \exp \left\{ -(\theta/2) \sum_{\tau=1}^{\infty} \sum_{i,j=1}^N [g_{\alpha}^{-1}(\omega_{\tau})]_{ij} (a_{i\tau} a_{j\tau} + b_{i\tau} b_{j\tau}) \right\}. \quad (2.9)$$

This is nothing else than the starting point of the method of Rice.⁵⁾ We see from this expression that the matrices $g_{\alpha}(\omega)$ and $g_F(\omega)$ are the spectral (density) matrices of the variable $\alpha(t)$ and of the fluctuating term $\mathcal{F}(t)$, respectively. The white spectrum of $\mathcal{F}(t)$ corresponds just to the δ -function type correlation (1.19). The special dependence of $g_{\alpha}(\omega)$

on ω is of course due to the simple forms of the equations of motion (1.1) with the force (2.2).

In the preceding paragraph we have derived the probability distribution of the Fourier coefficients (2.9) starting from the probability of path (1.12) or (1.11). Now we are in a position to derive the latter from the former. We shall prove that the probability of path obtained from (2.9) can be written in the form (1.12), when we take Onsager's assumption on the average behavior of our system. Due to this assumption the spectral matrix $g_a(\omega)$ gets the special form (2.8), hence we shall not assume (2.8) for the time being. The probability of path is the limiting expression of the joint probability of $K+1$ events, with $K \rightarrow \infty$. Let us write these events as $\alpha(t_0) = \alpha^{(0)}$, $\alpha(t_1) = \alpha^{(1)}$, \dots , $\alpha(t_K) = \alpha^{(K)}$ and denote that joint probability (density) by $W(\alpha^{(0)}, t_0; \alpha^{(1)}, t_1; \dots; \alpha^{(K)}, t_K)$. The $K+1$ times, t_0, t_1, \dots, t_K , may be selected arbitrarily, but without loss of generality we can impose them the condition $-\infty < t_0 < t_1 < \dots < t_K < +\infty$. The characteristic function of this probability is the average

$$X(u^{(0)}, u^{(1)}, \dots, u^{(K)}) \equiv \langle \exp \{ i \sum_{p=0}^K \sum_{j=1}^N u_j^{(p)} \alpha_j(t_p) \} \rangle. \quad (2.10)$$

In order to calculate this average, let us consider a sufficiently large time interval $(-\theta/2, +\theta/2)$ such that $-\theta/2 < t_0 < \dots < t_K < +\theta/2$. Then we may use the Fourier expansion (2.5) in the exponent of (2.10), and calculate the average with the weight (2.9). In this calculation it is convenient to employ the relations

$$\langle \exp \{ i \sum_{j=1}^N u_j a_{j\tau} \} \rangle_0 = \langle \exp \{ i \sum_{j=1}^N u_j b_{j\tau} \} \rangle_0 = \exp \{ -1/(2\theta) \sum_{i,j=1}^N [g_a(\omega_\tau)]_{ij} u_i u_j \}. \quad (2.11)$$

Noticing that the coefficients a_τ and a_ε , b_τ and b_ε ($\varepsilon = \pm \varepsilon$), and a and b are statistically independent of each other, as is seen from (2.9), we get

$$\begin{aligned} X(u^{(0)}, \dots, u^{(K)}) &= \exp \{ i \sum_{p=0}^K \sum_{j=1}^N u_j^{(p)} \alpha_j^0 \} \lim_{\theta \rightarrow \infty} \prod_{\tau=1}^{\infty} \exp \{ -1/(2\theta) \sum_{p,q=0}^K \sum_{i,j=1}^N [g_a(\omega_\tau)]_{ij} \\ &\quad \cdot \cos(\omega_\tau [t_p - t_q]) u_i^{(p)} u_j^{(q)} \} \\ &= \exp \{ i \sum_{p=0}^K \sum_{j=1}^N u_j^{(p)} \alpha_j^0 - (1/2) \sum_{p,q=0}^K \sum_{i,j=1}^N u_i^{(p)} \varphi_{ij}(t_p - t_q) u_j^{(q)} \}, \end{aligned} \quad (2.12)$$

where we have defined the matrix

$$\varphi(t) \equiv 1/(2\pi) \int_0^\infty g_a(\omega) \cos(\omega t) d\omega. \quad (2.13)$$

It is found by observing the expression of characteristic function (2.12) that this matrix is the correlation function $\langle \{ \alpha(s) - \alpha^0 \} \cdot \{ \alpha(s+t) - \alpha^0 \} \rangle$. In order to obtain the probability of path we need the conditional probability. The characteristic function of the conditional probability is obtained by retransforming the characteristic function (2.12) with respect to the variables to be fixed and by dividing the result by the joint probability of these

fixed variables. For our purpose it suffices to fix one variable $\alpha(t_0)$ only. Let us denote this conditional probability by $\mathcal{W}(\alpha^{(0)}, t_0 | \alpha^{(1)}, t_1; \dots; \alpha^{(K)}, t_K)$. Its characteristic function is then obtained as follows:

$$\begin{aligned} & X(\alpha^{(0)}, t_0 | u^{(1)}, \dots, u^{(K)}) \\ &= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} du_1^{(0)} \dots du_N^{(0)} / [(2\pi)^N \mathcal{W}(\alpha^{(0)})] \cdot \exp \left\{ -i \sum_{j=1}^N u_j^{(0)} \alpha_j^{(0)} \right\} X(u^{(0)}, \dots, u^{(K)}) \\ &= \exp \left\{ i \sum_{p=1}^K \sum_{j=1}^N u_j^{(p)} \alpha_j^0 + i \sum_{p=1}^K \sum_{i,j=1}^N u_i^{(p)} [\varphi(t_p - t_0) \varphi^{-1}(0)]_{ij} (\alpha_j^{(0)} - \alpha_j^0) \right. \\ &\quad \left. - (1/2) \sum_{p,q=1}^K \sum_{i,j=1}^N u_i^{(p)} [\varphi(t_p - t_q) - \varphi(t_p - t_0) \varphi^{-1}(0) \varphi(t_q - t_0)]_{ij} u_j^{(q)} \right\} \\ &= \exp \left\{ i \sum_{p=1}^K \sum_{j=1}^N u_j^{(p)} \alpha_j^0 + i \sum_{p=1}^K \sum_{i,j=1}^N u_i^{(p)} \rho_{ij}(t_p - t_0) (\alpha_j^{(0)} - \alpha_j^0) \right. \\ &\quad \left. - (1/2) \sum_{p,q=1}^K \sum_{i,j=1}^N u_i^{(p)} [\{\rho(t_p - t_q) - \rho(t_p - t_0) \rho(t_q - t_0)\} \varphi(0)]_{ij} u_j^{(q)} \right\}, \end{aligned} \quad (2.14)$$

where ρ is the normalized correlation matrix:

$$\rho(t) \equiv \varphi(t) \varphi^{-1}(0). \quad (2.15)$$

The normalization condition is of course $\rho(0) = 1$.

Till now we have made use only of the assumptions that our time series is Gaussian, and that the spectral matrix $\mathcal{G}_\alpha(\omega)$ is symmetric and positive definite. At this point we should introduce Onsager's assumption that the average regression of fluctuations obeys the same law as that given by the phenomenological equations (1.1). This means, in our matrix notation,

$$\rho(t) = e^{-\Delta|t|} \quad (2.16)$$

This type of matrix has the translation operator property, or the semi-group property

$$\rho(t) \rho(t') = \rho(t + t'), \quad (tt' \geq 0) \quad (2.17)$$

and this property is sufficient to prove our process being Markoffian (Doob's theorem generalized by Wang and Uhlenbeck^{(3),(4)}). We shall show this by direct calculations. Retransforming the characteristic function (2.14) into the conditional probability $\mathcal{W}(\alpha^{(0)}, t_0 | \alpha^{(1)}, t_1; \dots; \alpha^{(K)}, t_K)$, we get a Gaussian distribution with the mean $\alpha^0 + \rho(t_p - t_0)$ $\cdot (\alpha^{(0)} - \alpha^0)$ and the variance $\{\rho(t_p - t_q) - \rho(t_p - t_0) \rho(t_q - t_0)\} \varphi(0)$. Namely there appears in its exponent a quadratic form with a super-matrix composed of elements $\varphi^{-1}(0) \{\rho(t_p - t_q) - \rho(t_p - t_0) \rho(t_q - t_0)\}^{-1}$, $p, q = 1, 2, \dots, K$. When ρ has the property (2.17), the super-matrix $\Gamma_{pq} \equiv \{\rho(t_p - t_q) - \rho(t_p - t_0) \rho(t_q - t_0)\}^{-1}$ can be determined as follows:

$$\begin{aligned} \Gamma_{KK} &= \{1 - \rho^2(t_K - t_{K-1})\}^{-1}, \\ \Gamma_{pp} &= \{1 - \rho^2(t_p - t_{p-1})\}^{-1} + \rho(t_{p+1} - t_p) \{1 - \rho^2(t_{p+1} - t_p)\}^{-1} \rho(t_{p+1} - t_p), \\ &\quad [p = 1, 2, \dots, K-1], \end{aligned}$$

$$\begin{aligned} \Gamma_{q-1,q} &= -\rho(t_q - t_{q-1}) \{1 - \rho^2(t_q - t_{q-1})\}^{-1}, \\ \Gamma_{q,q-1} &= -\{1 - \rho^2(t_q - t_{q-1})\}^{-1} \rho(t_q - t_{q-1}), \\ &[q=1, 2, \dots, K]. \end{aligned} \quad (2 \cdot 18)$$

The other matrices Γ'_{pq} are zero. In fact it is easy to show, by making use of (2·17), that

$$\sum_{p=1}^K \Gamma_{pp} \{ \rho(t_p - t_q) - \rho(t_p - t_0) \rho(t_q - t_0) \} = \delta_{pq}. \quad (2 \cdot 19)$$

Thus the quadratic form mentioned above becomes

$$\begin{aligned} (1/2) \sum_{p,q=1}^K \sum_{i,j=1}^N [\varphi^{-1}(0) \Gamma_{pq}]_{ij} \{ \alpha_i^{(p)} - \alpha_i^0 - \sum_{l=1}^N \rho_{il}(t_p - t_0) [\alpha_l^{(0)} - \alpha_l^0] \} \\ \cdot \{ \alpha_j^{(q)} - \alpha_j^0 - \sum_{m=1}^N \rho_{jm}(t_q - t_0) [\alpha_m^{(0)} - \alpha_m^0] \} \\ = \sum_{p=1}^K (1/2) \sum_{i,j=1}^N [\varphi^{-1}(0) \{1 - \rho^2(t_p - t_{p-1})\}^{-1}]_{ij} \{ \alpha_i^{(p)} - \alpha_i^0 - \sum_{l=1}^N \rho_{il}(t_p - t_{p-1}) [\alpha_l^{(p-1)} - \alpha_l^0] \} \\ \cdot \{ \alpha_j^{(p)} - \alpha_j^0 - \sum_{m=1}^N \rho_{jm}(t_p - t_{p-1}) [\alpha_m^{(p-1)} - \alpha_m^0] \}, \end{aligned} \quad (2 \cdot 20)$$

where we have employed the symmetric property of $\varphi(t)$. Since the exponent is rewritten in the sum over time points $p=1, 2, \dots, K$, the conditional probability becomes the product of functions

$$\begin{aligned} \mathcal{P}(\alpha^{(p-1)}, t_{p-1} | \alpha^{(p)}, t_p) \propto \exp \left(- (1/2) \sum_{i,j=1}^N \{ \alpha_i^{(p)} - \alpha_i^0 - \sum_{l=1}^N \rho_{il}(t_p - t_{p-1}) [\alpha_l^{(p-1)} - \alpha_l^0] \} \right. \\ \left. \cdot [\varphi^{-1}(0) \{1 - \rho^2(t_p - t_{p-1})\}^{-1}]_{ij} \{ \alpha_j^{(p)} - \alpha_j^0 - \sum_{m=1}^N \rho_{jm}(t_p - t_{p-1}) [\alpha_m^{(p-1)} - \alpha_m^0] \} \right). \end{aligned} \quad (2 \cdot 21)$$

By putting $K=1$ in the expression (2·14) and transforming back to the α -space, we see that (2·21) is nothing else than the transition probability from the time t_{p-1} to the time t_p . Thus the Markoffian character of our process is proved. In order to rewrite the conditional probability $\mathcal{P}(\alpha^{(0)}, t_0 | \alpha^{(1)}, t_1; \dots; \alpha^{(K)}, t_K)$ into the form (1·11) we need the explicit forms of $\rho(t)$ and $\varphi(0)$. $\rho(t)$ was given in (2·16). $\varphi(0)$ is determined by comparing with Boltzmann's principle (1·3), in which we should use the quadratic expression (2·1). Since we have proved the Markoffian character, the transition probability (2·21) becomes the probability $W(\alpha^{(0)})$ of a state $\alpha^{(0)}$, when we take the limit $t_{p-1} \rightarrow -\infty$. From (2·16) $\rho(\infty)=0$, and hence we get

$$\varphi^{-1}(0) = S/k. \quad (2 \cdot 22)$$

Next we divide the time interval (t', t'') into K subintervals of length Δt , and apply (2·21) to each subinterval. Since $\Delta t \rightarrow 0$ as $K \rightarrow \infty$, we can approximate the sum (2·20) by the following

$$\begin{aligned}
 & 1/(2k) \sum_{p=1}^K \sum_{i,j=1}^N [S(2\Lambda \Delta t)^{-1}]_{ij} \{ \Delta \alpha_i^{(p-1)} + \sum_{l=1}^N \Lambda_{il} (\alpha_l^{(p-1)} - \alpha_l^0) \Delta t \} \\
 & \quad \cdot \{ \Delta \alpha_j^{(p-1)} + \sum_{m=1}^N \Lambda_{jm} (\alpha_m^{(p-1)} - \alpha_m^0) \Delta t \} \\
 & = 1/(2k) \sum_{p=1}^K \Delta t \cdot (1/2) \sum_{i,j=1}^N (G^{-1})_{ij} \{ \Delta \alpha_i^{(p-1)} / \Delta t + \sum_{l=1}^N \Lambda_{il} (\alpha_l^{(p-1)} - \alpha_l^0) \} \\
 & \quad \cdot \{ \Delta \alpha_j^{(p-1)} / \Delta t + \sum_{m=1}^N \Lambda_{jm} (\alpha_m^{(p-1)} - \alpha_m^0) \}, \quad (2.23)
 \end{aligned}$$

where $\Delta \alpha^{(p-1)} \equiv \alpha^{(p)} - \alpha^{(p-1)}$. Comparing this with (2.4) we arrive at the expression (1.11). The spectral matrix $g_\alpha(\omega)$ in (2.9) is determined by the Fourier transform (2.13) from the correlation function, (2.15) with (2.22), i.e.,

$$\varphi(t) = \nu(t) \varphi(0) = e^{-\Lambda |t|} k S^{-1}, \quad (2.24)$$

and it will be found to be equal to the expression given in (2.8).

§ 3. On the analogy between $\mathcal{L}(\alpha, \dot{\alpha})$ and $S(\alpha)$

As was stated in § 1, Onsager's principle (1.12) was derived on the analogy of Boltzmann's principle (1.3), and in this analogy the thermodynamical Lagrangian $\mathcal{L}(\alpha, \dot{\alpha})$ played a similar rôle to that played by entropy $S(\alpha)$. On the other hand, the entropy selects the equilibrium state α^0 among others by virtue of the variation principle (1.2), and increases in the course of time on the average when we start from a state other than α^0 . Remembering the variation principle (1.4) or

$$\mathcal{L}(\alpha, \dot{\alpha}) = \min. \quad (3.1)$$

(with respect to $\dot{\alpha}$ leaving α constant), we may expect a similar properties of $\mathcal{L}(\alpha, \dot{\alpha})$, when we replace the equilibrium state α^0 with a special value of $\dot{\alpha}$, or more precisely with the path determined by the phenomenological relations (1.1), the *thermodynamical path*. Namely, the function $\mathcal{L}(\alpha, \dot{\alpha})$ will single out the thermodynamical path, and will increase on the average when we trace other paths. In this section we shall show that this statement, which is in a sense an extension of the second law, is correct.

First let us determine the most probable path connecting the given initial and the final points in the (α, t) -plane. The most probable path is determined by the variation principle (1.15), and therefore satisfies Euler's equations obtained from the thermodynamical Lagrangian $\mathcal{L}(\alpha, \dot{\alpha})$. These equations are of the second order. It will be more convenient to separate them into two groups of equations of the first order by introducing a set of new variables. This separation can be done, by passing through equations like the canonical equations in mechanics. Let us define the "momentum" π corresponding to the variable α by

$$\pi_i \equiv \partial \mathcal{L}(\alpha, \dot{\alpha}) / \partial \dot{\alpha}_i, \quad (3.2)$$

where the thermodynamical Lagrangian $\mathcal{L}(\alpha, \dot{\alpha})$ should be regarded as a function of α and $\dot{\alpha}$. Remembering the explicit form of $\mathcal{L}(\alpha, \dot{\alpha})$, (2.4), and the symmetric property

of G , we get

$$\pi_i = \sum_{j=1}^N (G^{-1})_{ij} \{ \dot{\alpha}_j + \sum_{m=1}^N A_{jm} (\alpha_m - \alpha_m^0) \}. \quad (3.3)$$

Next we introduce the "Hamiltonian" after the general procedure in mechanics:

$$\mathcal{H}(\alpha, \pi) \equiv \sum_{i=1}^N \pi_i \dot{\alpha}_i - \mathcal{L}(\alpha, \dot{\alpha}), \quad (3.4)$$

which becomes*

$$\begin{aligned} \mathcal{H}(\alpha, \pi) &= (1/2) \sum_{i,j=1}^N (G^{-1})_{ij} \{ \dot{\alpha}_i - \sum_{l=1}^N A_{il} (\alpha_l - \alpha_l^0) \} \{ \dot{\alpha}_j + \sum_{m=1}^N A_{jm} (\alpha_m - \alpha_m^0) \} \\ &= (1/2) \sum_{i,j=1}^N \pi_i G_{ij} \{ \pi_j - 2 \sum_{l=1}^N S_{jl} (\alpha_l - \alpha_l^0) \}. \end{aligned} \quad (3.5)$$

Then the "canonical" equations of motion are given by

$$\begin{aligned} \dot{\alpha}_i &= \partial \mathcal{H} / \partial \pi_i = \sum_{j=1}^N G_{ij} \{ \pi_j - \sum_{l=1}^N S_{jl} (\alpha_l - \alpha_l^0) \}, \\ \dot{\pi}_i &= -\partial \mathcal{H} / \partial \alpha_i = \sum_{j=1}^N \pi_j A_{ji}. \end{aligned} \quad (3.6)$$

Now let us introduce a set of new variables

$$\begin{aligned} \xi_i &\equiv \dot{\alpha}_i + \sum_{l=1}^N A_{il} (\alpha_l - \alpha_l^0) = \sum_{j=1}^N G_{ij} \pi_j, \\ \eta_i &\equiv \dot{\alpha}_i - \sum_{l=1}^N A_{il} (\alpha_l - \alpha_l^0) = \sum_{j=1}^N G_{ij} \{ \pi_j - 2 \sum_{l=1}^N S_{jl} (\alpha_l - \alpha_l^0) \}. \end{aligned} \quad (3.7)$$

Remembering the definition (2.3) and the symmetric property of G and S , we can determine the equations of motion for these ξ and η by direct calculations:

$$\begin{aligned} \dot{\xi}_i &= \sum_{j=1}^N G_{ij} \dot{\pi}_j = \sum_{j,l=1}^N G_{ij} \pi_l A_{lj} = \sum_{l,j=1}^N \pi_l (GS)_{lj} G_{ji} \\ &= \sum_{l,j,k=1}^N \pi_l G_{lk} S_{kj} G_{ji} = \sum_{j,k=1}^N \xi_k G_{ij} S_{jk} = \sum_{k=1}^N A_{ik} \xi_k, \\ \therefore \quad \dot{\xi}_i &= \sum_{j=1}^N A_{ij} \xi_j; \end{aligned} \quad (3.8)$$

$$\begin{aligned} \dot{\eta}_i &= \sum_{j=1}^N G_{ij} \{ \dot{\pi}_j - 2 \sum_{l=1}^N S_{jl} \dot{\alpha}_l \} = \sum_{j=1}^N A_{ij} \{ \xi_j - 2 \dot{\alpha}_j \} \\ &= \sum_{j=1}^N A_{ij} [\xi_j - 2 \{ \xi_j - \sum_{l=1}^N A_{jl} (\alpha_l - \alpha_l^0) \}] \\ &= - \sum_{j=1}^N A_{ij} \{ \xi_j - 2 \sum_{l=1}^N A_{jl} (\alpha_l - \alpha_l^0) \}, \end{aligned}$$

* On the analogy of Schrödinger's equation in quantum mechanics, Fokker-Planck's equation [(2.8) of Part I] can be derived by the "quantization" $\pi \rightarrow -2k\partial/\partial\alpha$. This formulation was given by N. Saito and M. Namiki, Bussei-ron Kenkyu, No. 78 (1954) 95.

$$\therefore \dot{\eta}_i = - \sum_{j=1}^N A_{ij} \eta_j. \quad (3.9)$$

Thus our equations are separated into the equations of the first order. Their solutions are

$$\xi_i(t) = \sum_{j=1}^N [e^{+\Delta t}]_{ij} \xi_j(0), \quad \eta_i(t) = \sum_{j=1}^N [e^{-\Delta t}]_{ij} \eta_j(0). \quad (3.10)$$

On the other hand, according to the definition (3.7), the variable $\alpha(t)$ is given by the linear combination of $\xi(t)$ and $\eta(t)$:

$$\alpha_i(t) - \alpha_i^0 = (1/2) \sum_{j=1}^N (A^{-1})_{ij} \{ \xi_j(t) - \eta_j(t) \}. \quad (3.11)$$

The most probable path is determined by (3.11) and (3.10), and hence it is the superposition of the curves exponentially increasing and of those decreasing in the course of time. The integration constants $\xi(0)$ and $\eta(0)$ should be determined by the initial and the final conditions, $\alpha(t') = \alpha'$ and $\alpha(t'') = \alpha''$. If $\xi(0)$ is zero, the components exponentially increasing, $\xi(t)$, vanish and the path (3.11) becomes the thermodynamical one. This can also be seen from the first equation of (3.7).

Now the proof of our statement on the properties of $\mathcal{L}(\alpha, \dot{\alpha})$ is easy. If we introduce the new variables ξ into the expression of $\mathcal{L}(\alpha, \dot{\alpha})$, (2.4), we obtain*

$$\mathcal{L}(\alpha, \dot{\alpha}) = (1/2) \sum_{i,j=1}^N \xi_i (G^{-1})_{ij} \dot{\xi}_j \geq 0. \quad (3.12)$$

The signs \geq are due to the positive definiteness of the matrix G , and the equality sign holds when and only when $\dot{\xi} = 0$, i.e., for the thermodynamical paths. Thus we find that the value of thermodynamical Lagrangian becomes minimum for the thermodynamical path. Next let us insert the explicit form of $\xi(t)$ from (3.9) into (3.12). Then we see at once that the value of thermodynamical Lagrangian always increases with the lapse of time, except for the case $\dot{\xi} = 0$. That is to say, the value of $\mathcal{L}(\alpha, \dot{\alpha})$ is not decreasing for the most probable path. We can also show this by direct calculations. Differentiating (3.12) with respect to time, and remembering the equations of motion (3.8), we get

$$d\mathcal{L}(\alpha, \dot{\alpha})/dt = \sum_{i,j=1}^N \xi_i (G^{-1})_{ij} \dot{\xi}_j = \sum_{i,j,k=1}^N \xi_i (G^{-1})_{ij} A_{jk} \xi_k = \sum_{i,k=1}^N \xi_i S_{ik} \xi_k \geq 0, \quad (3.13)$$

where the signs \geq are due to the positive definiteness of the matrix S , which was in turn the consequence of the second law. The quantity ξ can always be defined by (3.7), and the expression (3.12) remains valid for an arbitrary path $\alpha(t)$. In this case (3.8) does not hold, and then the value of thermodynamical Lagrangian increases as well as decreases. On the other hand the average path coincides with the most probable path because of the

* A similar theorem has been proved by R. T. Cox, Rev. Mod. Phys. 22 (1950) 238. In his minimal principle, a set of quantities f_i , which are in our notation formally corresponding to $-\pi_i$, is introduced because of an external action. In our theory, however, the quantity π or ξ arises from the action of internal degrees of freedom, i.e., the fluctuating force $\mathcal{F}(t)$ [see Langevin's equation (1.6)].

Gaussian character of our process. Thus we have proved that *on the average the value of $\mathcal{L}(\alpha, \dot{\alpha})$ does increase except for the thermodynamical path, for which the value of $\mathcal{L}(\alpha, \dot{\alpha})$ remains constant, i.e., at the value zero.*

When our system has only one relaxation time :

$$A_{ij} = \tau^{-1} \delta_{ij}, \quad (3.14)$$

we can examine more precisely the above-mentioned analogy between the function $\mathcal{L}(\alpha, \dot{\alpha})$ and the entropy $S(\alpha)$. In this case the most probable path (3.10) describes a trajectory, whose projection on each (ξ_i, η_j) -plane is a hyperbola

$$\xi_i(t) \eta_j(t) = \text{const.} \quad (3.15)$$

The representative points of the system trace these hyperbolas from their branches along the asymptotes $\eta_j = 0$ to those along the other asymptotes $\xi_i = 0$, for the values of η_j are decreasing and those of ξ_i are increasing with the lapse of time. On the other hand the contours of the thermodynamical Lagrangian (3.12) in the $(\bar{\epsilon}, \eta)$ plane are straight lines parallel to the η -axes, and the valley of the $\mathcal{L}(\alpha, \dot{\alpha})$ -surface just lies on the η -axes. Thus our representative point climbs the side-wall of the valley in the course of time. In the present case we can determine the rate of increase of the thermodynamical Lagrangian. Remembering the definition of A , (2.3), we get, from (3.13),

$$d\mathcal{L}(\alpha, \dot{\alpha})/dt = (1/\tau) \sum_{i,j=1}^N \xi_i (G^{-1})_{ij} \xi_j = 2/\tau \cdot \mathcal{L}(\alpha, \dot{\alpha}). \quad (3.16)$$

Namely the value of $\mathcal{L}(\alpha, \dot{\alpha})$ increases exponentially with the time constant $\tau/2$. This equation can also be used to simplify Onsager-Machlup's expression (1.14) of the transition probability. The simplified expression will show the analogy of \mathcal{L} and S more clearly, though it is only a special form of the general analogy. In the expression (1.14) the time integral should be carried out by making use of the most probable path, and therefore the value of $\mathcal{L}(\alpha, \dot{\alpha})$ thus determined should satisfy the equation (3.16). Substituting the value of $\mathcal{L}(\alpha, \dot{\alpha})$ for $\mathcal{L}(\alpha, \dot{\alpha})$ from the equation (3.16), we can carry out the time integration, and obtain

$$\Psi(\alpha', t'; \alpha'', t'') \propto \exp \left\{ -\tau/(4k) (\mathcal{L}[\alpha(t''), \dot{\alpha}(t'')] - \mathcal{L}[\alpha(t'), \dot{\alpha}(t')]) \right\}. \quad (3.17)$$

At a glance we see the analogy to Boltzmann's principle (1.3). In Onsager's principle there appear the transition probability and the difference of the thermodynamical Lagrangian, while in Boltzmann's principle the probability of a state $W(\alpha)$ and the difference of the entropy $S(\alpha) - S(\alpha'')$. There is, however, a little difference. In the exponent of (3.17), two values of the thermodynamical Lagrangian can not be given independently, provided that the time interval $t'' - t'$ is fixed. And so it is difficult to regard the thermodynamical Lagrangian as the "state function", while the entropy is strictly a state function. This situation arises of course from the after-effect of the initial state on the final one, and it can not be avoided. Nevertheless it rather shows a close connection between the two principles. As is well known the probability $W(\alpha)$ is obtained from the transition probability in the limit of infinite time interval, i.e., the expression (3.17) should reduce

to the expression (1.3) in the limit $t' \rightarrow \infty$. This can easily be shown by making use of (3.12), (3.16), (3.11), and (2.1).

We have neglected the effect of inertia of the system in all the preceding paragraphs. When this effect can not be neglected, we get the equations of motion of damped oscillators in place of (3.8), by a similar procedure to the above mentioned. The equations corresponding to (3.9) are also obtained by the time reversal. Thus our representative point oscillates first along the ξ -axis, and finally along the η -valley. The amplitude of oscillation decreases first, passes a minimum value, and finally increases again. The plane of oscillation rotates gradually from the direction of the ξ -axis into that of the η -axis.

§ 4. Fluctuations in the distribution function

As was stated in § 1, the principal aim of the present paper is the application of our statistical theory to such variables that are more microscopic than the thermodynamical variables. Among such variables the most important one is obviously the molecular or electronic distribution function, which is considered in the kinetic theory. In the next section we shall treat the fluctuation of the electronic distribution in metals. There we shall consider the electronic distribution function. The fluctuations of these distribution functions are of course the result of the fluctuations in the motion of each molecule or electron. We shall show that it is much simpler to treat the fluctuation of the distribution function than to consider the motion of each particle colliding with others, say.

In this section we shall concern ourselves with a set of harmonic oscillators immersed into a heat reservoir in order to illustrate the method, as a preliminary to the next section. Such a system has been treated in the theory of the shape of collision-broadened spectral line. In this theory radiating molecules or atoms are replaced with a classical harmonic oscillators, and non-radiating molecules ("foreign-gas") constitute the heat reservoir. If the number of radiating molecules is much smaller than that of non-radiating molecules, the former molecules may be regarded as practically independent of each other. Thus essentially, we are concerning with an ideal gas, whose state is described completely by the (one-particle) distribution function. In the subsection a) we shall start by assuming that each harmonic oscillator makes the Brownian motion, and then evaluate the polarization of the gas. This type of model has been proposed by Gross.¹¹⁾ In the subsection b), conversely, we shall first treat the fluctuation of the molecular distribution in the gas and next reduce it to the Brownian motion of the average molecule. Thus we shall see the equivalence of the two methods: one is treating the distribution, and the other the motion of each molecule. In the subsection c) we shall examine the possibility of generalization of our method of fluctuating distribution function to the models based on the various strong-collision assumptions. In this section, for the sake of simplicity, we shall confine ourselves to the one-dimensional case.

a) Brownian Motion of the Lorentz Oscillator

First let us consider one harmonic oscillator with charge $-e$, mass m , and characteristic (angular) frequency ω_0 , immersed in a heat reservoir of temperature T . If x and p

denote its displacement and the corresponding momentum, respectively, Langevin's equations are given by

$$\dot{x} = p/m, \quad \dot{p} = -m\omega_0^2 x - eE(t) - p/\tau + \mathcal{F}(t). \quad (4.1)$$

$\tau = m/\zeta$ is the relaxation time of the momentum, and ζ the friction constant. The term $-p/\tau$ in the last equation represents the dissipative term associated with the fluctuating term $\mathcal{F}(t)$. As was discussed in § 4 of Part I, we have to introduce the antisymmetric part of the kinetic coefficients G in order to write (4.1) into the form (1.6). Then we can apply our general theory to the total isolated system, which is composed of the harmonic oscillator, the heat reservoir, and the work source generating the electric field $E(t)$. We shall neglect all the changes of the states of the reservoir and the source. Thus we can determine the diffusion constant $D_p = \zeta kT$ and get from (1.17) the probability distribution of $\mathcal{F}(t)$:

$$W[\mathcal{F}(t)] \propto \exp \left\{ -1/(4D_p) \int_{-\infty}^{+\infty} \mathcal{F}^2(t) dt \right\}. \quad (4.2)$$

The statistical property of the motion of the harmonic oscillator is completely determined by the stochastic differential equations (4.1) and this probability distribution (4.2).

Let us consider only the under-damped case: $\omega_0 > 1/2\tau$. Then the formal solution of (4.1) is given by

$$\begin{aligned} x(t) = & x(t_0) e^{-(t-t_0)/2\tau} \left\{ \cos(\omega_1[t-t_0]) + \frac{\sin(\omega_1[t-t_0])}{2\tau\omega_1} \right\} \\ & + \frac{p(t_0)}{m} e^{-(t-t_0)/2\tau} \sin(\omega_1[t-t_0]) \\ & + \frac{1}{m\omega_1} \int_{t_0}^t \{ -eE(s) + \mathcal{F}(s) \} e^{-(t-s)/2\tau} \sin(\omega_1[t-s]) ds, \\ p(t) = & -x(t_0) \frac{m\omega_0^2}{\omega_1} e^{-(t-t_0)/2\tau} \sin(\omega_1[t-t_0]) \\ & + p(t_0) e^{-(t-t_0)/2\tau} \left\{ \cos(\omega_1[t-t_0]) - \frac{\sin(\omega_1[t-t_0])}{2\tau\omega_1} \right\} \\ & + \int_{t_0}^t \{ -eE(s) + \mathcal{F}(s) \} e^{-(t-s)/2\tau} \left\{ \cos(\omega_1[t-s]) \right. \\ & \left. - \frac{\sin(\omega_1[t-s])}{2\tau\omega_1} \right\} ds, \end{aligned} \quad (4.3)$$

where we have defined $\omega_1 = \sqrt{\omega_0^2 - (1/2\tau)^2}$. $x(t_0)$ and $p(t_0)$ are the given initial values of x and p respectively. In order to discuss the polarization of the gas it is enough to

consider only an aged system. When the system is aged ($t_0 \rightarrow -\infty$), the formal solution (4.3) can be written in the form:

$$\begin{aligned} x(t) &= \int_{-\infty}^t \varphi_x(t-s) \{-eE(s) + \mathcal{J}(s)\} ds, \\ p(t) &= \int_{-\infty}^t \varphi_p(t-s) \{-eE(s) + \mathcal{J}(s)\} ds, \end{aligned} \quad (4.4)$$

where $\varphi_x(t)$ and $\varphi_p(t)$ denote the decay functions of x and p , respectively:

$$\begin{aligned} \varphi_x(t) &\equiv \frac{1}{m\omega_1} e^{-t/(2\tau)} \sin(\omega_1 t), \\ \varphi_p(t) &\equiv e^{-t/(2\tau)} \left\{ \cos(\omega_1 t) - \frac{\sin(\omega_1 t)}{2\tau\omega_1} \right\}, \end{aligned} \quad (t \geq 0) \quad (4.5)$$

The probability distribution $W_\infty(x, p, t)$ of x and p at time t will in general be determined by the usual procedure¹⁾ from the formal solution (4.3), or (4.4), and the probability distribution (4.2). In the present case we can give it at once, since we know that, when the electric field is absent, the required distribution reduces to Maxwell-Boltzmann's distribution

$$W_0(x, p) \equiv \frac{\omega_0}{2\pi kT} \exp \left\{ -\frac{1}{kT} \left(\frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 \right) \right\}. \quad (4.6)$$

If we insert the formal solution (4.4) with $E=0$ into this distribution, we find the probability for the time integrals of $\mathcal{J}(t)$ appearing on the right-hand side of (4.4). On the other hand the fluctuating term $\mathcal{J}(t)$ is assumed to be independent of x and p , as is seen from (4.2), and therefore the probability obtained above will remain valid when the electric field is introduced. Thus we see that the probability distribution of $x(t)$ and $p(t)$ have the same form as (4.6) provided that $x(t)$ and $p(t)$ are shifted by the terms including the electric field on the right-hand side of (4.4):

$$\begin{aligned} W_\infty(x, p, t) &= \frac{\omega_0}{2\pi kT} \exp \left\{ -\frac{m\omega_0^2}{2kT} \left| x + e \int_{-\infty}^t \varphi_x(t-s) E(s) ds \right|^2 \right. \\ &\quad \left. - \frac{1}{2mkT} \left| p + e \int_{-\infty}^t \varphi_p(t-s) E(s) ds \right|^2 \right\}. \end{aligned} \quad (4.7)$$

We find that the stochastic variables x and p become independent of each other when the system is aged. It should be noticed here that the distribution (4.7) does not in general coincide with the "equilibrium" distribution under the given electric field $E(t)$

$$W_e(x, p, t) \equiv \frac{\omega_0}{2\pi kT} \exp \left[-\frac{1}{kT} \left\{ \frac{p^2}{2m} + \frac{m\omega_0^2}{2} \left(x + \frac{eE(t)}{m\omega_0^2} \right)^2 \right\} \right]. \quad (4.8)$$

The coincidence occurs when and only when the electric field $E(t)$ is constant.*

In the preceding paragraphs we have considered only one "radiating molecule." Now let us suppose that there are N radiating molecules per unit volume, but the interaction between them may be neglected. The state of the gas composed of these molecules is described by the coordinates x_1, x_2, \dots, x_N and the momenta p_1, p_2, \dots, p_N . Since the molecules are independent of each other, the probability of a state $(x_1, \dots, x_N; p_1, \dots, p_N)$ is given by the product of (4.7). In the following we shall, however, confine ourselves to the coordinates (x_1, \dots, x_N) . When the system is aged, (x_1, \dots, x_N) become independent of (p_1, \dots, p_N) , and their probability is given by

$$W_N(x_1, x_2, \dots, x_N; t) = \prod_{i=1}^N W_\infty(x_i, t) \\ \equiv \prod_{i=1}^N \sqrt{\frac{m\omega_0^2}{2\pi kT}} \exp\left\{-\frac{m\omega_0^2}{2kT} \left|x_i + e \int_{-\infty}^t \varphi_x(t-s)E(s)ds\right|^2\right\}. \quad (4.9)$$

In the theory of the shape of spectral line we call the polarization of the gas in question. This quantity is defined as follows:

$$P(t) \equiv -e \sum_{i=1}^N x_i(t), \quad (4.10)$$

and its probability distribution is obtained by Markoff's method⁽⁴⁾:

$$W(P, t) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \delta(P + e \sum_{i=1}^N x_i) W_N(x_1, \dots, x_N; t) dx_1 \dots dx_N \\ = 1/(2\pi) \int_{-\infty}^{+\infty} e^{iPu} \left[\int_{-\infty}^{+\infty} e^{iuex} W_\infty(x, t) dx \right]^N du \\ = \sqrt{\frac{m\omega_0^2}{2\pi N e^2 kT}} \exp\left\{-\frac{m\omega_0^2}{2N e^2 kT} \left|P - Ne^2 \int_{-\infty}^t \varphi_x(t-s)E(s)ds\right|^2\right\}. \quad (4.11)$$

From this we can easily determine the probability distribution of the electric displacement $D(t) \equiv E(t) + 4\pi P(t)$. Its average is given by

$$\langle D(t) \rangle = E(t) + \int_{-\infty}^t \varphi_D(t-s)E(s)ds, \quad (4.12)$$

* The necessary and sufficient condition for the coincidence of (4.7) and (4.8) are

$$\int_{-\infty}^t E(s) \varphi_x(t-s)ds = E(t)/m(\omega_0^2), \quad \int_{-\infty}^t E(s) \varphi_p(t-s)ds = 0,$$

or, if we insert the explicit expressions of the decay functions from (4.5),

$$\int_{-\infty}^t E(s) e^{-(t-s)/2\tau} \cos(\omega_1[t-s])ds = E(t)/(2\tau\omega_0^2) = 1/(2\tau\omega_1) \int_{-\infty}^t E(s) e^{-(t-s)/2\tau} \sin(\omega_1[t-s])ds.$$

Differentiating the second equation with respect to t and making use of the first, we obtain at once $dE(t)/dt = 0$. It is obvious that a constant $E(t)$ satisfies the two conditions mentioned above.

where $\varphi_D(t)$ denotes the decay function of the electric displacement and is found to be

$$\varphi_D(t) = 4\pi N e^2 \varphi_x(t) = (4\pi N e^2 / m \omega_1) e^{-t/(2\tau)} \sin(\omega_1 t), \quad (t \geq 0). \quad (4.13)$$

As far as we assume the Brownian motion for each harmonic oscillator, the decay function $\varphi_D(t)$ should have this form. The complex dielectric constant $\varepsilon(\omega)$ corresponding to this decay function is¹⁵⁾

$$\begin{aligned} \varepsilon(\omega) - 1 &= \int_0^\infty \varphi_D(t) e^{i\omega t} dt \\ &= \frac{2\pi N e^2}{im\omega_1} \left\{ \frac{1}{(1/2\tau) - i(\omega + \omega_1)} - \frac{1}{(1/2\tau) - i(\omega - \omega_1)} \right\} \\ &= \frac{4\pi N e^2}{m} \left(\omega_0^2 - \omega^2 + i \frac{\omega}{\tau} \right)^{-1}. \end{aligned} \quad (4.14)$$

This complex dielectric constant agrees completely with the one obtained by Gross¹³⁾ based on his new collision model. The reason for this agreement will be given in the subsection c).

From the probability distribution of P , (4.11), the variance of the polarization is found to be

$$\langle \{P(t) - \langle P(t) \rangle\}^2 \rangle = (N e^2 / m \omega_0^2) kT. \quad (4.15)$$

The correlation function $\langle \{P(t) - \langle P(t) \rangle\} \{P(t') - \langle P(t') \rangle\} \rangle$ can not be derived from (4.11). In order to obtain it, we should return to the formal solution (4.3) and construct the joint probability of $P(t)$ and $P(t')$. A method based on the path integral will be given in the appendix A. Here we shall have recourse to the general theorem on thermal fluctuations given by Takahasi,¹⁶⁾ which states that the correlation function is given by kT times a function representing the average response of the polarization when the unit electric field is suddenly removed. Thus we obtain

$$\begin{aligned} &\langle \{P(t) - \langle P(t) \rangle\} \{P(t+s) - \langle P(t+s) \rangle\} \rangle \\ &= 1/(4\pi) \int_{-\infty}^0 \varphi_D(s-s') ds' \cdot kT \\ &= \frac{N e^2 kT}{m \omega_0^2} e^{-s/(2\tau)} \left\{ \cos(\omega_1 s) + \frac{\sin(\omega_1 s)}{2\tau \omega_1} \right\}, \quad (s \geq 0). \end{aligned} \quad (4.16)$$

b) Fluctuations of the Distribution Function of the Lorentz Oscillator

In this subsection we shall show that the results obtained in the preceding subsection a), especially the correlation function (4.16), can be derived more easily by the method of fluctuating distribution function.

We have derived the probability distribution (4.7) from Langevin's equations (4.1). It can be obtained also from the corresponding Fokker-Planck's equation

$$\left[\frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - \{ m\omega_0^2 x + eE(t) \} \frac{\partial}{\partial p} \right] W = \frac{\partial}{\partial p} \left\{ D_p \frac{\partial W}{\partial p} + \frac{p}{\tau} W \right\}. \quad (4.17)$$

The latter method will in general be easier than the former, since in the most practical cases we need only approximate solutions which are valid up to linear terms in the electric field $E(t)$. When N oscillators may be regarded as independent of each other, the number of molecules with displacement x and momentum p found in the range dx and dp is on the average equal to $NW_\infty(x, p, t)dx dp$. Thus the distribution function $f(x, p, t)$ satisfies on the average the same equation (4.17). On the other hand the state of the gas as a whole is described by the coordinates and momenta $(x_1, \dots, x_N; p_1, \dots, p_N)$, whose probability distribution is given by the product $\prod_{i=1}^N W_\infty(x_i, p_i, t)$. Since the distribution function $f(x, p, t)$ is by nature equivalent to the set of variables $(x_1, \dots, x_N; p_1, \dots, p_N)$, it should be a random variable, whose probability distribution will be derived from $\prod_{i=1}^N W_\infty(x_i, p_i, t)$. The arguments x and p are now not the random variables, in contrast to the case of Langevin's equation. They correspond to the suffix i in the case of the variable α_i of our general theory developed in §§ 1-3. If the distribution function $f(x, p, t)$ corresponds to the variable α_i , the equation (4.17), with f in place of W , describing the average behaviour of the gas will correspond to the general equation (1.1), and then our statistical theory will be applied. The distribution function $f(x, p, t)$ is, however, not the thermodynamical variable. It describes the state of the gas much more precisely than the latter. Nevertheless, at least when the gas is ideal, it has "thermodynamical" properties. First the entropy of the gas represented in terms of the distribution function is additive or extensive:

$$S[f] = -k \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp f(x, p, t) \{ \log f(x, p, t) - 1 \}. \quad (4.18)$$

Here we have assumed Boltzmann's statistics. Second the equilibrium value of the distribution function, i.e., Maxwell-Boltzmann's distribution when there is no electric field, say, is determined by following the same procedure as that in thermodynamics, i.e., the maximization of entropy, (1.2). Therefore we may treat the distribution function in a way similar to that for the thermodynamical variables.* This treatment will lead us to an extension of Boltzmann-Planck's method in statistical mechanics to the case of irreversible processes.

According to our general theory, when the average behaviour of the gas is described by the equation (4.17), in which we replace W with f , the corresponding stochastic differential equation will be given by adding a fluctuating term \mathcal{J} :

* Many authors, e.g., J. G. Kirkwood, J. Chem. Phys. **14** (1946) 180, have tried to treat the motion of each molecule in a liquid, say, by the theory of Brownian motion. If these theories are correct, our treatment will be supported also in the case of translational motion by the argument similar to that developed in the subsections a) and b).

$$\left[\frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - \{m\omega_0^2 x + eE(t)\} \frac{\partial}{\partial p} \right] f(x, p, t) \\ = \frac{\partial}{\partial p} \left\{ D_p \frac{\partial f(x, p, t)}{\partial p} + \frac{p}{\tau} f(x, p, t) \right\} + \mathcal{J}_t(x, p). \quad (4.19)$$

This is Langevin's equation describing the state of the gas, and corresponds to the general equation (1.6). In the present case we have to generalize the equation (1.6) in the following two points. First, as was stated in § 1, the kinetic coefficient matrix G should include its antisymmetric part $G^{(a)}$ besides the symmetric part $G^{(s)}$.⁽²⁾ Second, since the distribution function may make a reversible change under the action of forces $-m\omega_0^2 x + eE(t)$, we must add a new term $V(\alpha)$ to the time derivative $d\alpha/dt$, or interpret it as composed of two terms $\partial\alpha/\partial t + V(\alpha)$.^{(1), (7)} The meaning of this new term will be apparent when we see its explicit form, for instance (5.17) or (B.5) in the appendix B. Thus we write in place of (1.6)

$$\partial\alpha_i/\partial t + V_i(\alpha) = \sum_{j=1}^N [G^{(s)} + G^{(a)}]_{ij} \cdot \partial S(\alpha) / \partial \alpha_j + \mathcal{J}_i(t). \quad (4.20)$$

As was stated in Part I, the statistical property of the fluctuating term $\mathcal{J}(t)$ is determined by the diffusion coefficient matrix D , which is in turn given by the symmetric part $G^{(s)}$:

$$D = kG^{(s)}. \quad (4.21)$$

To determine the matrix $G^{(s)}$ or D , the explicit expression of the generalized force $\partial S/\partial \alpha$ is necessary. The entropy $S(\alpha)$ appearing in this force is the total entropy of an isolated system, which in the present case is composed of the set of oscillators (radiating molecules), their heat reservoir (non-radiating molecules), and the work source generating the electric field $E(t)$. The existence of the heat reservoir and the source does not, however, contribute the second derivatives of the entropy: their effect appears only in the displacement of the "equilibrium" position. Thus we may use the entropy of the set of oscillators to determine the matrix elements $S_{ij} \equiv -[\partial^2 S / \partial \alpha_i \partial \alpha_j]_{\alpha=\alpha_0}$ in the expression (2.1) or (2.2). From (4.18) we obtain

$$S(x, p; x', p') \equiv \left[\frac{-\partial^2 S[f]}{\partial f(x, p, t) \partial f(x', p', t)} \right]_{f=f^0} = \frac{k\delta(x-x')\delta(p-p')}{f^0(x, p, t)}, \quad (4.22)$$

where $\delta/\delta f$ denotes the functional derivative, and f^0 a certain equilibrium distribution function. The explicit form of this distribution function f^0 is not necessary, because for the following purpose we may replace it with Maxwell-Boltzmann's distribution function

$$f_0(x, p) \equiv \frac{N\omega_0}{2\pi kT} \exp \left\{ -\frac{1}{kT} \left(\frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 \right) \right\}, \quad (4.23)$$

which corresponds to (4.6). When we approximate the force $\partial S/\partial \alpha$ by the linear one (2.2), in the equation (4.20) the matrix G appears in the combination $A = GS$, (2.3). On the other hand, as was stated in Part I, the matrix $G^{(s)}$ represents the dissipative terms, which in the present case is the first term on the right-hand side of (4.19). The

operator $-\partial/\partial p \cdot \{D_p \cdot \partial/\partial p + p/\tau\}$ thus corresponds to the dissipative part of the matrix A ,

$$A^{(d)} \equiv G^{(s)} S. \quad (4.24)$$

In the last paragraph of this subsection we shall show that this operator will effectively be replaced with $1/\tau$. Thus we find

$$A^{(d)}(x, p; x', p') = \tau^{-1} \delta(x-x') \delta(p-p') \quad (4.25)$$

and hence

$$D(x, p; x', p') = \tau^{-1} f^0(x, p, t) \delta(x-x') \delta(p-p'). \quad (4.26)$$

According to (1.17), the probability of our fluctuating term $\mathcal{J}_t(x, p)$ is now given by

$$W[\mathcal{J}_t(x, p)] \propto \exp \left\{ -\frac{\tau}{4} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dp \frac{\mathcal{J}_t^2(x, p)}{f^0(x, p, t)} \right\}. \quad (4.27)$$

The statistical property of our system is completely determined by the stochastic equation (4.19) and the probability distribution (4.27).

Now let us seek for the solution. In order to determine the polarization we need only a solution that is correct up to linear terms in the electric field $E(t)$. Such a solution is obtained by assuming the required distribution function $f(x, p, t)$ to be a product $f_0(x, p)h(x, p, t)$, expanding the unknown function $h(x, p, t)$ in a power series of x and p , or more precisely in terms of some orthonormal set of polynomials, and by retaining only its linear terms (Fröhlich-Gross method^{(15), (17)}). Since xf_0 and pf_0 are proportional to $\partial f_0/\partial x$ and $\partial f_0/\partial p$ respectively, it is more convenient to write our trial function as follows:

$$f(x, p, t) = f_0(x, p) - a(t) \cdot \partial f_0/\partial x - b(t) \cdot \partial f_0/\partial p, \quad (4.28)$$

where the coefficients $a(t)$ and $b(t)$ are of the order of electric field $E(t)$. Since in our approximation we have neglected higher terms, this trial function is equivalent to the shifting of the arguments x and p in Maxwell-Boltzmann's distribution function $f_0(x, p)$, (4.23), by amounts $a(t)$ and $b(t)$, respectively,

$$f(x, p, t) = f_0(x - a(t), p - b(t)). \quad (4.29)$$

In this sense we shall call our method *shift approximation*. From (4.29) we know at once the meaning of the expansion coefficients $a(t)$ and $b(t)$:

$$a(t) = (1/N) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f(x, p, t) dx dp, \quad b(t) = (1/N) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p f(x, p, t) dx dp. \quad (4.30)$$

Namely, they are respectively the displacement and the momentum of the average or representative oscillator of the gas. The normalization condition of the distribution function is of course

$$N = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, p, t) dx dp. \quad (4.31)$$

Since the distribution function was a random variable, and since the function $f_0(x, p)$ is not fluctuating, the expansion coefficients $a(t)$ and $b(t)$ must be random variables. In our shift approximation the state of our system is described only by them, and therefore we shall be able to get their stochastic equations. These equations will be obtained from the original equation (4.19) by the "transformation" of the variable $f(x, p, t)$ into the coefficients $a(t)$, $b(t)$, etc., (4.28). Corresponding to this transformation the fluctuating term $\mathcal{F}_t(x, p)$ in (4.19) will be transformed into a set of variables $\mathcal{A}(t)$, $\mathcal{B}(t)$, etc.:

$$\mathcal{F}_t(x, p) = \mathcal{E}(t) f_0(x, p) - \mathcal{A}(t) \cdot \partial f_0 / \partial x - \mathcal{B}(t) \cdot \partial f_0 / \partial p, \quad (4.32)$$

where we have neglected higher terms. The insertion of the approximate expressions (4.28) and (4.32) into the equation (4.19) will give the stochastic equations of $a(t)$ and $b(t)$. In fact we get

$$\text{the left-hand side of (4.19)} = -(\dot{a} - b/m) \partial f_0 / \partial x - \{\dot{b} + m\omega_0^2 a + eE(t)\} \partial f_0 / \partial p, \quad (4.33)$$

$$\text{the dissipative term on the right-hand side of (4.19)} = b/\tau \cdot \partial f_0 / \partial p,$$

and hence, comparing the coefficients of x and p , or of $\partial f_0 / \partial x$ and $\partial f_0 / \partial p$,

$$\begin{aligned} 0 &= \mathcal{E}(t), \quad \dot{a} - b/m = \mathcal{A}(t), \\ \dot{b} + m\omega_0^2 a + eE(t) &= -b/\tau + \mathcal{B}(t). \end{aligned} \quad (4.34)$$

We see that the first term in (4.32) vanishes. The second term should also vanish:

$$\mathcal{A}(t) = 0. \quad (4.35)$$

The reason is as follows. As was emphasized in Part I, the fluctuating term must be coupled with the corresponding dissipative term. However, the term $\mathcal{A}(t)$ in the second equation of (4.34) does not have its partner, so that it should vanish. Thus we have returned to Langevin's equations (4.1), although here we concern ourselves to the average oscillator with displacement $a(t)$ and momentum $b(t)$. The effect of average (4.30) will appear in the magnitude of fluctuations. The probability distribution of the new fluctuating term $\mathcal{B}(t)$ is determined from the expression (4.27). For the sake of later convenience, we write it retaining the term $\mathcal{A}(t)$. Noticing that the expression (4.32) can be written in the form

$$\mathcal{F}_t(x, p) = \frac{1}{kT} \left\{ \frac{p}{m} \mathcal{B}(t) + m\omega_0^2 x \mathcal{C}(t) \right\} f_0(x, p), \quad (4.36)$$

we can carry out the integrations with respect to x and p in the exponent of (4.27):

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathcal{F}_t^2(x, p)}{f_0(x, p, t)} dx dp \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f_0(x, p)}{f_0(x, p, t)} \left(\frac{1}{kT} \right)^2 \left\{ \frac{p}{m} \mathcal{B}(t) + m\omega_0^2 x \mathcal{C}(t) \right\}^2 f_0(x, p) dx dp \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{kT} \right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{ \frac{2}{m} \mathcal{B}^2(t) \frac{p^2}{2m} + 2m\omega_0^2 \mathcal{A}^2(t) \frac{m\omega_0^2}{2} x^2 \right\} f_0(x, p) dx dp \\
&= \frac{2N}{kT} \left\{ \frac{\mathcal{B}^2(t)}{2m} + \frac{m\omega_0^2}{2} \mathcal{A}^2(t) \right\}, \quad (4.37)
\end{aligned}$$

where we have approximated f^n by f_0 , because $(f^n - f_0)/f_0$ will be of the order $eE(t)/kT$ as will be seen in (4.53) in the next subsection, say. Inserting (4.37) into (4.27) we obtain

$$W[\mathcal{A}(t), \mathcal{B}(t)] \propto \exp \left[-\frac{N\tau}{2kT} \int_{-\infty}^{+\infty} \left\{ \frac{\mathcal{B}^2(t)}{2m} + \frac{m\omega_0^2}{2} \mathcal{A}^2(t) \right\} dt \right]. \quad (4.38)$$

If we put $\mathcal{A}(t) = 0$ as was done in (4.35), this probability distribution gets a form similar to the probability distribution (4.2) of the previous fluctuating term $\mathcal{F}(t)$. We should remember here the definitions $D_p \equiv \zeta kT$ and $\tau \equiv m/\zeta$, which give $\tau/(mkT) = 1/(\zeta kT) = 1/D_p$. The only difference is the factor N in the exponent of (4.38). This means, as was expected previously, that the magnitude of fluctuations of the averaged oscillator is \sqrt{N} times smaller.

In the shift approximation the statistical property of our system is completely characterized by Langevin's equations (4.34) and the probability distribution (4.38), with (4.35). The formal solution of (4.34) is given by (4.3) or (4.4) with $\mathcal{B}(t)$ in place of $\mathcal{F}(t)$. Here we take (4.4) as before. Then from (4.28) we get the formal solution of (4.19) in the shift approximation as follows:

$$f(x, p, t) = f_\infty(x, p, t) - \int_{-\infty}^t \left\{ \varphi_x(t-s) \frac{\partial f_0}{\partial x} + \varphi_p(t-s) \frac{\partial f_0}{\partial p} \right\} \mathcal{B}(s) ds, \quad (4.39)$$

where the first term $f_\infty(x, p, t)$ is defined by making use of (4.7) as

$$f_\infty(x, p, t) \equiv N W_\infty(x, p, t). \quad (4.40)$$

[Remember the relation (4.29).] The probability distribution (4.38) shows that the term $\mathcal{B}(t)$ has the Gaussian distribution with mean and variance

$$\langle \mathcal{B}(t) \rangle = 0, \quad \langle \mathcal{B}(t) \mathcal{B}(t') \rangle = (2mkT/\tau) \delta(t-t'), \quad (4.41)$$

respectively, and we know that its linear combination (4.39) will also have a Gaussian distribution. The mean of the distribution function (4.39) is thus found to be $f_\infty(x, p, t)$. For the comparison of the present method with that of the preceding subsection we need, however, not the variance of the distribution function itself. The polarization of the gas composed of oscillators is given by

$$P(t) = -e \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x f(x, p, t) dx dp = -Ne a(t) \quad (4.42)$$

in correspondence to the definition (4.10) in the previous subsection. Thus we need the

mean and the variance of $a(t)$. These can be evaluated by making use of the formal solution (4.4), or it will be more convenient to compute (4.42) by using (4.39) directly :

$$P(t) = N\epsilon^2 \int_{-\infty}^t \varphi_x(t-s) E(s) ds - N\epsilon^2 \int_0^{\infty} \varphi_x(u) \mathcal{B}(t-u) du. \quad (4.43)$$

In the second term on the right-hand side we have changed the integration variable from s to $u \equiv t-s$. According to (4.41) the first term represents the average behaviour of our system when it is aged ;

$$\langle P(t) \rangle = \int_{-\infty}^t N\epsilon^2 \varphi_x(t-s) E(s) ds. \quad (4.44)$$

The second term gives the deviation from this average due to fluctuations. The correlation function of the polarization can be determined easily :

$$\begin{aligned} & \langle \{P(t) - \langle P(t) \rangle\} \{P(t') - \langle P(t') \rangle\} \rangle \\ &= N^2 \epsilon^2 \int_0^{\infty} du \int_0^{\infty} du' \varphi_x(u) \varphi_x(u') \langle \mathcal{B}(t-u) \mathcal{B}(t'-u') \rangle \\ &= \frac{2N\epsilon^2 kT}{m\omega_1^2 \tau} \int_0^{\infty} dv \int_{-2v}^{+2v} dv' e^{-v/\tau} \sin\left(\omega_1 \left[v - \frac{v'}{2}\right]\right) \sin\left(\omega_1 \left[v + \frac{v'}{2}\right]\right) \delta(t-t'-v) \\ &= \frac{N\epsilon^2 kT}{m\omega_1^2 \tau} \int_{|t'-t|/2}^{\infty} dv e^{-v/\tau} \{\cos(\omega_1[t'-t]) - \cos(2\omega_1 v)\} \\ &= \frac{N\epsilon^2 kT}{m\omega_0^2} e^{-|t'-t|/2\tau} \left\{ \cos(\omega_1|t'-t|) + \frac{\sin(\omega_1|t'-t|)}{2\tau\omega_1} \right\}. \end{aligned} \quad (4.45)$$

Here we have used the explicit form of the decay function $\varphi_x(t)$ given in (4.5). It will be seen that this correlation function is completely in accord with the result (4.16) in the previous subsection.

Before proceeding further, we have to go back to the proof of the statement, which led us to the expression (4.25), that the diffusion term in (4.19) may be effectively replaced by $-f(x, p, t)/\tau$ plus some term. This is evident from the second equation of (4.33). According to (4.28), $(b/\tau) \partial f_0 / \partial p$ equals $-1/\tau$ times the distribution function f minus some extra term. The meaning of this extra term will be given in the next subsection (Gross' strong collision assumption). In order to determine the matrix $A^{(0)}$, it was sufficient to compare the term linear in $f(x, p, t)$, so that the coefficient $-1/\tau$ determined above gives in fact the expression (4.25).

c) Weak versus Strong Collisions

In the preceding subsection we have shown that the method of fluctuating distribution function is at least within the range of the shift approximation equivalent to the method of the Brownian motion of each particle. The latter method is, as is well-known, based

on the weak collision assumption, while the former method can be extended to include the cases based on the various strong collision assumptions. Since our method of introducing the probability distribution of the fluctuating term is based on the theory of Brownian motion, i.e., on a kind of weak collision assumption, at first sight we might expect that this extension would contain contradictions. In the case of Gaussian process, however, there is no essential difference between the two kinds of assumptions, since by virtue of the central limit theorem the difference in effects of each collision does not remain in the resulting Gaussian probability distribution, although the average behaviour of the system may of course change from assumption to assumption. In this subsection we shall consider the three strong collision assumptions given by Lorentz, by Van Vleck and Weisskopf,¹⁸⁾ and by Gross,¹⁹⁾ and show that Gross' assumption gives the same results as obtained in the preceding subsection.

In all the strong collision assumptions, we suppose that the intermolecular collisions occur instantaneously, i.e., the duration of collision ($\sim 10^{-12}$ sec¹⁹⁾) be much shorter than the average time τ_c between collisions, and that immediately after each collision the distribution function $f(x, p, t)$ be drawn back to a certain equilibrium distribution $f^i(x, p, t)$. From the first condition the fluctuating term $\mathcal{J}(t)$ should have a δ -function like correlation function, such as given in our statistical theory, (1.19). The variety of strong collision assumptions arises from the second condition. Lorentz assumed

$$f^0(x, p, t) = f_0(x, p), \quad (4.46)$$

while Van Vleck and Weisskopf have proposed

$$f^0(x, p, t) = f_e(x, p, t), \quad (4.47)$$

where $f_e(x, p, t)$ denotes the equilibrium distribution function corresponding to the instantaneous value of the electric field $E(t)$, i.e.,

$$f_e(x, p, t) \equiv NW_e(x, p, t), \quad (4.48)$$

$W_e(x, p, t)$ being defined in (4.8). Gross' new assumption is

$$f^0(x, p, t) = f_{l.e.}(x, p, t), \quad (4.49)$$

where $f_{l.e.}(x, p, t)$ means the local equilibrium distribution with the true number density determined by the true distribution function $f(x, p, t)$:

$$\begin{aligned} f_{l.e.}(x, p, t) &\equiv N(x, t) / \sqrt{2\pi mkT} \cdot \exp(-p^2/2mkT), \\ N(x, t) &\equiv \int_{-\infty}^{+\infty} f(x, p, t) dp. \end{aligned} \quad (4.50)$$

When we make use of the strong collision assumptions, we replace Fokker-Planck's equation (4.17) in the previous subsection with Lorentz-Boltzmann's equation

$$\left[\frac{\partial}{\partial t} + \frac{p}{m} \frac{\partial}{\partial x} - \{m\omega_0^2 x + eE(t)\} \frac{\partial}{\partial p} \right] f = -\frac{f - f^0}{\tau_c} + \mathcal{J}_t(x, p). \quad (4.51)$$

Here we have added the fluctuating term $\mathcal{J}_t(x, p)$ as was done in (4.19). From the form of the dissipative term on the right-hand side we find at once the dissipative part of the matrix A to be given by

$$A^{(D)}(x, p; x', p') = \tau_c^{-1} \delta(x - x') \delta(p - p') \quad (4.52)$$

as in (4.25) [see the appendix B]. Thus the probability distribution of the fluctuating term $\mathcal{J}_t(x, p)$ will have the same form as (4.27), provided that we replace τ with τ_c .

Next let us derive Langevin's equations for the averaged oscillator by making use of the shift approximation (4.28). Noticing the approximate expressions

$$\begin{aligned} f_e(x, p, t) &= f_0(x, p) + (eE(t)/m\omega_0^2) \partial f_0 / \partial x, \\ f_{l.e.}(x, p, t) &= f_0(x, p) - a(t) \cdot \partial f_0 / \partial x, \end{aligned} \quad (4.53)$$

we can proceed in the same way as in the subsection b). First, it will be found that the extra term stated in the last paragraph of the preceding subsection is $f_{l.e.}(x, p, t)$, for $(b/\tau) \partial f_0 / \partial p$ in the second equation of (4.33) is equal to $(-1/\tau) \{f - f_0 + a(t) \partial f_0 / \partial x\} = -\{f - f_{l.e.}\} / \tau$. Thus if we put $\tau = \tau_c$, the stochastic differential equation (4.51) based on Gross' strong collision assumption agrees completely with the previous equation (4.19) based on the weak collision assumption within the range of the shift approximation. Therefore we need not discuss the case of Gross' assumption. In the case of Lorentz', and of Van Vleck-Weisskopf's assumptions the resulting Langevin's equations are as follows:

$$\left. \begin{aligned} \dot{a} - b/m &= -a/\tau_c + \mathcal{O}(t), \\ \dot{b} + m\omega_0^2 a + eE(t) &= -b/\tau_c + \mathcal{B}(t); \end{aligned} \right\} \quad (\text{Lorentz}) \quad (4.54)$$

$$\left. \begin{aligned} \dot{a} - b/m &= -1/\tau_c \cdot \{a - (-eE(t)/m\omega_0^2)\} + \mathcal{O}(t), \\ \dot{b} + m\omega_0^2 a + eE(t) &= -b/\tau_c + \mathcal{B}(t). \end{aligned} \right\} \quad (\text{Van Vleck-Weisskopf}) \quad (4.55)$$

The equations for momentum $b(t)$ are the same in all cases, including the case of Gross' assumption. The difference appears in the equations of displacement $a(t)$. In Lorentz' theory there appears a "frictional" term, which is proportional to the displacement $a(t)$ from the origin, i.e., from the equilibrium position under Maxwell-Boltzmann's distribution $f_0(x, p)$, (4.23). Similarly in Van Vleck-Weisskopf's theory the "frictional" term is proportional to the deviation from the equilibrium position $-eE(t)/(m\omega_0^2)$ under the distribution $f_e(x, p, t)$, (4.48). According to the strong collision assumptions, these terms are interpreted as due to jumps of the average oscillator occurring at the instant of collision from the position $a(t)$ immediately before collision to the respective equilibrium positions. Certainly, we can derive the averaged form of the equations (4.54) and (4.55) by a technique similar to that used by Kerppl and Schwinger.¹⁹⁾ These jumps seem, however, difficult to be understood within the classical theory. Apart from this point, we have found that our method of fluctuating distribution function is open to such strong collision assumptions. This makes it possible to apply our statistical theory to stochastic problems on free electrons in metals, which we shall consider in the next section.

§ 5. Thermal noise in metals

In this section we shall try to apply our statistical theory to fluctuations of the electronic distribution in metals. When the electronic distribution fluctuates, the electronic current and the electronic energy flow will do so too. The fluctuation in the electronic current will then induce a fluctuating voltage between the ends of a conductor, which can be measured experimentally as "noise." This is the famous *thermal* or *Johnson noise*, and its thermodynamical theory was given by Nyquist.⁽²⁰⁾ The fluctuation of the electronic distribution in metals arises from the "Brownian motion" of electrons in the conduction band. The kinetic theories based on Lorentz' free electron model were proposed and improved by many authors, especially by Brillouin,⁽²¹⁾ Bernamont,⁽²²⁾ Bakker-Heller,⁽²³⁾ and by Spence.⁽²⁴⁾ Brillouin's method is based on the idea that we first construct formally the mean square of the total electronic current from the expression of the current in terms of the distribution function f and then insert the well-known formula in statistics, $\langle |f - \langle f \rangle|^2 \rangle = \langle f \rangle$, which is valid at equilibrium. Bernamont introduced the probability distribution of the time between collision (actually the distribution of the free path of an electron) and derived the correlation function of current, but he gave only Nyquist's formula at low frequency. Bakker and Heller considered the various statistics, especially the Fermi statistics, and replace the formula $\langle |f - \langle f \rangle|^2 \rangle = \langle f \rangle$ with the one obtained by statistical mechanics, $\langle |f - \langle f \rangle|^2 \rangle = -kT \partial \langle f \rangle / \partial \epsilon$, where ϵ is the energy per electron. They calculated the correlation function in a way similar to Bernamont's method, but got the results valid over all frequency range by making use of Wiener-Khinchin's theorem, which states essentially the relation (2.13). The key point in their theory may be said to be the determination of the fluctuations of the electronic distribution function $f(\mathbf{x}, \mathbf{p}, t)$ in the following form

$$d\langle |f(\mathbf{x}, \mathbf{p}) - \langle f(\mathbf{x}, \mathbf{p}) \rangle|^2 \rangle = -kT \frac{\partial \langle f(\mathbf{x}, \mathbf{p}) \rangle}{\partial \epsilon} \frac{e^{-s\tau_c}}{\tau_c} ds d\mathbf{p}. \quad (5.1)$$

The quantity on the left-hand side means the contribution to the variance $\langle |f - \langle f \rangle|^2 \rangle$ from the electrons, whose momentum \mathbf{p} and time between collision (free time) s fall into the ranges $d\mathbf{p} = dp_x dp_y dp_z$ and ds , respectively. Bernamont, and Bakker-Heller made use of the conductivity at frequency zero given by Lorentz' theory of electrons, while Spence employed the conductivity valid at an arbitrary frequency, which is obtained by solving Lorentz-Boltzmann's equation

$$\left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} - e\mathbf{E}(t) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f = -\frac{f - f^0}{\tau_c} \quad (5.2)$$

in the case of alternating field. $f^0(\mathbf{x}, \mathbf{p})$ denotes the Fermi distribution. The result given by Spence is in accord with that of Bakker-Heller. His method was, however, different. He obtained directly the power spectrum of the electric current by the method of Fourier series.

All these theories are based on the assumption of instantaneous collisions. In the quantum mechanical theory of transport phenomena in metals, the equation (5.2) is derived from Bloch's equation.⁽²⁵⁾ When we derive Bloch's equation by the time-dependent perturbation

theory, we have to assume a kind of instantaneous collision.²⁶⁾ In the following we shall therefore proceed on this assumption. In the quantum mechanical expression, the momentum \mathbf{p} is the wave vector \mathbf{k} multiplied by \hbar , and the velocity \mathbf{p}/m the group velocity $\hbar^{-1}\partial\epsilon/\partial\mathbf{k}$. For simplicity let us suppose the energy per electron ϵ to be spherical:

$$\epsilon(\mathbf{p}) = \mathbf{p}^2/(2m), \quad (5.3)$$

m being the effective mass, and further the mean free time τ_c to be a function of the energy ϵ and the temperature T . Then we may use the classical equation (5.2) also in quantum mechanics. Besides these assumptions, when we want to apply our statistical theory, we should make the assumption of Gaussian process. In the case of noise problems this assumption is usually done. Thus we shall start from the stochastic differential equation corresponding to (5.2):

$$\left\{ \frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} - e\mathbf{E}(t) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f = -\frac{f - f^0}{\tau_c} + \mathcal{J}_t(\mathbf{x}, \mathbf{p}). \quad (5.4)$$

In the previous theories of Lorentz and others the determination of the conductivity and that of the fluctuation of current were done separately, i.e., the former was done under the existence of electric field in a non-equilibrium condition and the latter at equilibrium with no field. In our theory based on the stochastic equation (5.4), these will be done simultaneously when the current exists under the action of electric field. Brillouin considered also this case, but he produced the average current by shifting the Fermi sphere without deformation and derived a term proportional to the square of the average current. As is well-known, this term, i.e., the so-called "current noise," arises from different mechanisms,²⁷⁾ and such a term should vanish when we consider only the Brownian motion of electrons in the conduction bands. Our theory will not give such a term.

According to our general theory the statistical property of the fluctuating term $\mathcal{J}_t(\mathbf{x}, \mathbf{p})$ in (5.4) is completely determined by the diffusion matrix D , (4.21), and this matrix is connected with the matrix $G^{(s)}$ or $A^{(d)}$, (4.24), which will appear in the stochastic differential equation (5.4). Thus let us first determine the matrix S , the second derivative of the entropy with minus sign, or the generalized force (2.2), the first derivative of the entropy. This entropy is the total entropy of our system composed of the free electron gas, the crystal lattice, the heat reservoirs, and the work source. The heat reservoirs are used to fix or change local temperatures of the lattice. Let us assume each local temperature to be changed sufficiently slowly, so that the thermodynamical considerations may be applied to determine the generalized force above mentioned. Similarly the electric field generated by the work source is supposed to change slowly. Under these conditions the local equilibrium distribution

$$f^0(\mathbf{x}, \mathbf{p}, t) \equiv \left\{ \exp \frac{\epsilon - \mu(\mathbf{x}, t)}{kT(\mathbf{x}, t)} + 1 \right\}^{-1} \quad (5.5)$$

will be a good approximation to the electronic distribution function $f(\mathbf{x}, \mathbf{p}, t)$. As was done in the previous section, let us neglect the fluctuations of temperature and of electric field, and regard them as given functions of position vector \mathbf{x} and time t . Consequently,

our state variables are only the distribution function, and we need only the generalized forces corresponding to it. This force is of course not the derivative of the local electronic entropy :

$$S(\mathbf{x}) \equiv -k \int \{f \log f + (1-f) \log (1-f)\} 2d\mathbf{p}/h^3. \quad (5.6)$$

If we change the electronic distribution $f(\mathbf{x}, \mathbf{p}, t)$, certain amount of heat will be supplied from the heat reservoir and work will be done by the work source automatically. Thus the generalized force will be given by the derivative of a kind of free energy, or of the minimum work, which is necessary to take the electron gas out of the local equilibrium f^0 into the given state f . Since local electron gases are quasi-independent of each other, we may determine the force for each local gas with fixed volume $d\mathbf{x} = dx dy dz$ and a fixed position \mathbf{x} in the crystal. In this case, however, we have to bear in mind the redistribution of electrons between local gases, which occurs in response to the value of chemical potential $\mu(\mathbf{x}, t)$ at each position. Since the value of chemical potential is determined by temperature, we may regard it as given. Thus if we consider one local gas in order to determine the generalized force, an electron source characterized by the given value of chemical potential $\mu(\mathbf{x}, t)$ should be attached to that local gas. Taking these reservoirs and sources into account, the change of local minimum work is given by

$$\delta W_{\min.}(\mathbf{x}) = \delta U(\mathbf{x}) - T(\mathbf{x}) \delta S(\mathbf{x}) - \mu(\mathbf{x}) \delta N(\mathbf{x}), \quad (5.7)$$

where $U(\mathbf{x})$ and $N(\mathbf{x})$ denote the internal energy and the number of electrons per unit of volume, respectively :

$$U(\mathbf{x}) \equiv \int \varepsilon f(\mathbf{x}, \mathbf{p}, t) 2d\mathbf{p}/h^3, \quad N(\mathbf{x}) \equiv \int f(\mathbf{x}, \mathbf{p}, t) 2d\mathbf{p}/h^3. \quad (5.8)$$

The generalized force associated with the state variable $f(\mathbf{x}, \mathbf{p}, t)$ is given by

$$A(\mathbf{x}, \mathbf{p}, t) \equiv - \frac{\delta W_{\min.}(\mathbf{x})}{\delta f(\mathbf{x}, \mathbf{p}, t)} = \mu(\mathbf{x}) - \varepsilon - kT(\mathbf{x}) \log \frac{f(\mathbf{x}, \mathbf{p}, t)}{1-f(\mathbf{x}, \mathbf{p}, t)}. \quad (5.9)$$

The Fermi distribution (5.5) is obtained just by the equilibrium condition $A(\mathbf{x}, \mathbf{p}, t) = 0$. In the case of Gaussian process, we restrict ourselves to the linear approximation of the force with respect to the deviation $f(\mathbf{x}, \mathbf{p}, t) - f^0(\mathbf{x}, \mathbf{p}, t)$. To get a linear term it is convenient to introduce a function $w(\mathbf{x}, \mathbf{p}, t)$, which measures the deviation in energy unit :

$$f(\mathbf{x}, \mathbf{p}, t) \equiv \left\{ \exp \frac{\varepsilon + w - \mu}{kT} + 1 \right\}^{-1}. \quad (5.10)$$

Inserting this relation (5.10) into the expression of force (5.9), we find that this function w is nothing else than the force A itself. On the other hand we know that this function w appears just in the expression of the local rate of entropy production⁽³⁾

$$(dS/dt)_{\text{irr.}}(\mathbf{x}) = 1/T(\mathbf{x}) \cdot \int w(\mathbf{x}, \mathbf{p}, t) (df/dt)_{\text{irr.}} 2d\mathbf{p}/h^3, \quad (5.11)$$

where $(df/dt)_{\text{irr.}}$ denotes the rate of change of f due to collisions with lattice. This shows

the correctness of our force A , since it appears in the rate of entropy production as the product with the dissipative part of the generalized "velocity" (see Part I, § 4). By expanding (5.10) with respect to w/kT and retaining only the zero-th and the first powers, we get

$$f(\mathbf{x}, \mathbf{p}, t) = f^0 + (\partial f^0 / \partial \varepsilon) w(\mathbf{x}, \mathbf{p}, t) \quad (5.12)$$

and hence

$$A(\mathbf{x}, \mathbf{p}, t) = w(\mathbf{x}, \mathbf{p}, t) = \frac{f(\mathbf{x}, \mathbf{p}, t) - f^0(\mathbf{x}, \mathbf{p}, t)}{\partial f^0 / \partial \varepsilon}. \quad (5.13)$$

The generalized force (2.2) is obtained by dividing this $A(\mathbf{x}, \mathbf{p}, t)$ by the local temperature $T(\mathbf{x})^{11}$:

$$\frac{\partial S[f]}{\partial f(\mathbf{x}, \mathbf{p}, t)} = \frac{A(\mathbf{x}, \mathbf{p}, t)}{T(\mathbf{x}, t)} = \frac{f - f^0}{T \partial f^0 / \partial \varepsilon}, \quad (5.14)$$

where $S[f]$ denotes the entropy of our total system including all the reservoirs and sources. The matrix S in the expression (2.2) is in the present case given by

$$S(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') \equiv \frac{-\partial^2 S[f]}{\partial f(\mathbf{x}, \mathbf{p}, t) \partial f(\mathbf{x}', \mathbf{p}', t)} = -\frac{\partial(\mathbf{x} - \mathbf{x}') \partial(\mathbf{p} - \mathbf{p}')}{T(\mathbf{x}) \partial f^0 / \partial \varepsilon}. \quad (5.15)$$

This is the generalization of (4.22) to quantum statistics.

Next let us determine the matrix D , or $G^{(s)}$, or $A^{(d)}$. To do so we should compare our stochastic differential equation (5.4) with the one in the general theory (4.20), or

$$\begin{aligned} \frac{\partial f(\mathbf{x}, \mathbf{p})}{\partial t} + V(\mathbf{x}, \mathbf{p}) &= \iint \{G^{(s)} + G^{(d)}\}(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') \frac{\partial S[f]}{\partial f(\mathbf{x}', \mathbf{p}', t)} \frac{2d\mathbf{x}' d\mathbf{p}'}{h^3} \\ &+ \mathcal{J}_t(\mathbf{x}, \mathbf{p}) \end{aligned} \quad (5.16)$$

in the present notations. Inserting $\partial S[f] / \partial f$ from (5.14), we get at once by inspection

$$\begin{aligned} V(\mathbf{x}, \mathbf{p}) &= \left\{ \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} - e\mathbf{E}(t) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f^0 \\ &= \frac{\partial f^0}{\partial \varepsilon} \frac{\mathbf{p}}{m} \cdot \left[\left\{ -e\mathbf{E}(t) - T \frac{\partial}{\partial \mathbf{x}} \left(\frac{\mu}{T} \right) \right\} + \varepsilon \left\{ -\frac{\partial \log T}{\partial \mathbf{x}} \right\} \right], \end{aligned} \quad (5.17)$$

$$G^{(d)}(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') = -T \frac{\partial f^0}{\partial \varepsilon} \left\{ \frac{\mathbf{p}}{m} \cdot \frac{\partial}{\partial \mathbf{x}} - e\mathbf{E}(t) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{p} - \mathbf{p}'), \quad (5.18)$$

$$G^{(s)}(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') = -\frac{T}{\tau_c} \frac{\partial f^0}{\partial \varepsilon} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{p} - \mathbf{p}'). \quad (5.19)$$

When we derive (5.19), we have remembered the fact that the symmetric part $G^{(s)}$ should be coupled with the dissipative term on the right-hand side of (5.4). Then the other terms are settled. From the expression (5.17) we know that $V(\mathbf{x}, \mathbf{p})$ may be

interpreted as the rate of change of the local equilibrium f^0 due to the work done by the generalized forces

$$X_I(\mathbf{x}, t) \equiv -e\mathbf{E}(t) - T \frac{\partial}{\partial \mathbf{x}} \left(\frac{\mu}{T} \right), \quad X_Q(\mathbf{x}, t) \equiv -\frac{\partial \log T}{\partial \mathbf{x}}, \quad (5.20)$$

which drive the electronic current and the electronic energy flow, respectively. These forces (5.20) are those treated in the thermodynamics of irreversible processes. In our theory there appear two kinds of generalized forces: the previous force A associated with the distribution function, (5.13), and the present forces X , (5.20). The former arises from the thermal disturbing effect of the lattice, and makes the distribution function f approach to the local equilibrium distribution f^0 . The latter forces are generated by the external reservoirs and sources, and tend to alter the local equilibrium f^0 . When these two effects are balanced, a local steady state will be realized. Although we have neglected the fluctuation of forces X , the force A is regarded as fluctuating, so that this balance can occur only on the average. This will be seen in the next paragraph. Remembering (4.21), we get from (5.19) the matrix D :

$$D(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') = \frac{kT(\mathbf{x})}{\tau_e(\varepsilon)} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{p} - \mathbf{p}'). \quad (5.21)$$

The matrix $A^{(0)}$, (4.24), is at once given as follows:

$$A^{(0)}(\mathbf{x}, \mathbf{p}; \mathbf{x}', \mathbf{p}') = \tau_e^{-1}(\varepsilon) \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{p} - \mathbf{p}'), \quad (5.22)$$

by making use of the expression (5.15). These (5.21) and (5.22) are the generalization of the previous results (4.26) and (4.25), respectively. According to (1.17) and (1.19), the probability distribution of the fluctuating term $\mathcal{F}_t(\mathbf{x}, \mathbf{p})$ and its first two moments are thus determined as follows:

$$W[\mathcal{F}_t(\mathbf{x}, \mathbf{p})] \propto \exp \left\{ -\frac{1}{4} \int_{-\infty}^{+\infty} dt \iint \frac{\tau_e(\varepsilon)}{-kT(\mathbf{x}) \partial f^0 / \partial \varepsilon} \frac{2d\mathbf{p} d\mathbf{x}}{h^3} \right\}; \quad (5.23)$$

$$\langle \mathcal{F}_t(\mathbf{x}, \mathbf{p}) \rangle = 0,$$

$$\langle \mathcal{F}_t(\mathbf{x}, \mathbf{p}) \mathcal{F}_{t'}(\mathbf{x}', \mathbf{p}') \rangle = \frac{2kT}{\tau_e} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \delta(\mathbf{x} - \mathbf{x}') \delta(\mathbf{p} - \mathbf{p}') \delta(t - t'). \quad (5.24)$$

Now let us solve our stochastic differential equation (5.4) under the assumption that terms of higher order than the second with respect to the force X and terms containing the space and time derivatives of X may all be neglected. Namely we seek for a solution linear in the force X . In this approximation $w(\mathbf{x}, \mathbf{p}, t)$ introduced previously in (5.10) may also be assumed as linear in the forces X . Thus we may use the expansion (5.12). We shall call this type of solution *linear response solution*. As was done in the preceding section, in correspondence to the expansion (5.12), we approximate the fluctuating term $\mathcal{F}_t(\mathbf{x}, \mathbf{p})$ in the form

$$\mathcal{F}_t(\mathbf{x}, \mathbf{p}) = \partial f^0 / \partial \varepsilon \cdot \mathcal{U}_t(\mathbf{x}, \mathbf{p}), \quad (5.25)$$

higher terms being neglected. Inserting the expressions (5.12) and (5.25) into the stochastic equation (5.4), and comparing the coefficients of $\partial f^0/\partial \varepsilon$, we obtain

$$\partial w/\partial t + w/\tau_e = -\mathbf{p}/m \cdot \{X_I(\mathbf{x}, t) + \varepsilon X_Q(\mathbf{x}, t)\} + \mathcal{W}_t(\mathbf{x}, \mathbf{p}). \quad (5.26)$$

When we derive this equation, the terms $\partial w/\partial \mathbf{x}$ and $\partial^2 f^0/\partial \mathbf{x} \partial \varepsilon$ have been dropped because they contain the spatial derivatives of X , the term $\partial w/\partial \mathbf{p}$ because it appears as the product with the electric field $\mathbf{E}(t)$, and the term $w \partial^2 f^0/\partial \mathbf{p} \partial \varepsilon$ because it contains the second order derivative $\partial^2 f^0/\partial \varepsilon^2$, which was neglected in the expansion (5.12). According to (5.23), the fluctuating term $\mathcal{J}_t(\mathbf{x}, \mathbf{p})$ and therefore $\mathcal{W}_t(\mathbf{x}, \mathbf{p})$ do not depend on $w(\mathbf{x}, \mathbf{p}, t)$. Thus the right-hand side of the equation (5.26) may be for the time being regarded as a given function \mathbf{x} , \mathbf{p} , and t . Then we can write down at once the formal solution of (5.26). If we impose the initial condition at a time t_0 , we get

$$w(\mathbf{x}, \mathbf{p}, t) = w(\mathbf{x}, \mathbf{p}, t_0) e^{-(t-t_0)/\tau_e} - \mathbf{p}/m \cdot \int_{t_0}^t e^{-(t-s)/\tau_e} \{X_I(\mathbf{x}, s) + \varepsilon X_Q(\mathbf{x}, s)\} ds + \int_{t_0}^t e^{-(t-s)/\tau_e} \mathcal{W}_s(\mathbf{x}, \mathbf{p}) ds. \quad (5.27)$$

When our system is aged ($t_0 \rightarrow -\infty$), the distribution function is thus determined as follows:

$$f(\mathbf{x}, \mathbf{p}, t) = f^\infty(\mathbf{x}, \mathbf{p}, t) + \int_{-\infty}^t e^{-(t-s)/\tau_e(\varepsilon)} \mathcal{J}_s(\mathbf{x}, \mathbf{p}) ds, \quad (5.28)$$

where we have defined a new distribution function

$$f^\infty \equiv f^0 - (\partial f^0/\partial \varepsilon) \mathbf{p}/m \cdot \int_{-\infty}^t e^{-(t-s)/\tau_e} \{X_I + \varepsilon X_Q\} ds. \quad (5.29)$$

This distribution function f^∞ reduces to the one usually called as Lorentz' solution, if the generalized forces X are independent of time. It also corresponds to the solution obtained by Spenke, when the forces X are proportional to $e^{i\omega t}$. f^∞ contains the effect of work done by the generalized forces X , which was not included in f^0 . The distribution function f^0 is the object or target, toward which the distribution function f approaches after each collision, and the function f^∞ is the distribution which is really arrived at, on the average, against the action of the generalized forces X .

Remembering the first equation of (5.24), we see that the distribution function f^∞ describes the average behaviour of our system:

$$\langle f(\mathbf{x}, \mathbf{p}, t) \rangle = f^\infty(\mathbf{x}, \mathbf{p}, t). \quad (5.30)$$

Similarly by making use of the second equation of (5.24) we can evaluate the correlation function:

$$\langle \{f(\mathbf{x}, \mathbf{p}, t) - \langle f(\mathbf{x}, \mathbf{p}, t) \rangle\} \{f(\mathbf{x}', \mathbf{p}', t') - \langle f(\mathbf{x}', \mathbf{p}', t') \rangle\} \rangle$$

$$\begin{aligned}
&= \int_0^\infty du \int_0^\infty du' e^{-(u+u')/\tau_0} \langle \mathcal{J}_{t-u}(\mathbf{x}, \mathbf{p}) \mathcal{J}_{t'-u'}(\mathbf{x}', \mathbf{p}') \rangle \\
&= (2kT/\tau_0) \{ -\partial f^0/\partial \varepsilon \} \delta(\mathbf{x}-\mathbf{x}') \delta(\mathbf{p}-\mathbf{p}') \int_0^\infty du \int_0^\infty du' \delta(t-t'-u+u') e^{-(u+u')/\tau_0} \\
&= (2kT/\tau_0) \{ -\partial f^0/\partial \varepsilon \} \delta(\mathbf{x}-\mathbf{x}') \delta(\mathbf{p}-\mathbf{p}') \int_0^\infty dv \int_{-2\tau_0}^{2\tau_0} dv' \delta(t-t'-v) e^{-2v/\tau_0} \\
&= (2kT/\tau_0) \{ -\partial f^0/\partial \varepsilon \} \delta(\mathbf{x}-\mathbf{x}') \delta(\mathbf{p}-\mathbf{p}') \int_{|t'-t|}^\infty dv e^{-2v/\tau_0}, \\
&\therefore \langle \{ f(\mathbf{x}, \mathbf{p}, t) - \langle f(\mathbf{x}, \mathbf{p}, t) \rangle \} \{ f(\mathbf{x}', \mathbf{p}', t') - \langle f(\mathbf{x}', \mathbf{p}', t') \rangle \} \rangle \\
&= kT(\mathbf{x}) \{ -\partial f^0/\partial \varepsilon \} \delta(\mathbf{x}-\mathbf{x}') \delta(\mathbf{p}-\mathbf{p}') e^{-|t'-t|/\tau_0(\varepsilon)}. \tag{5.31}
\end{aligned}$$

This correlation function does correspond just to the expression (5.1) in the previous theories. In our expression there appears the distribution function f^∞ as the average instead of f^0 , while in the expression (5.1) the average $\langle f \rangle$ is the Maxwellian or Fermi distribution f^0 , since Bernamont, Bakker and Heller considered fluctuations at complete equilibrium with no generalized forces X . On the right-hand side of (5.31) we have got, however, $\partial f^0/\partial \varepsilon$ where we expect the derivative of f^∞ . This is due to the situation that in our statistical theory the fluctuating term $\mathcal{J}_t(\mathbf{x}, \mathbf{p})$ does not depend upon the state variable f and continues to have local equilibrium fluctuations even when the average $\langle f \rangle$ is displaced from f^0 to f^∞ by the action of the generalized forces X . The factor $-kT \partial f^0/\partial \varepsilon \cdot \delta(\mathbf{x}-\mathbf{x}') \delta(\mathbf{p}-\mathbf{p}')$ in (5.31) arises from the matrix kS^{-1} [cf. (5.15)], i.e., the correlation function for $t'=t$, and shows that fluctuations will occur at the Fermi surface, as it should be. The remaining factor $\exp(-|t'-t|/\tau_0)$ can be understood by the same argument as given by Bernamont. This is essentially concerned with the definition of mean free path given by Lorentz. We shall give it here in terms of the mean free time τ_0 . Let us suppose N electrons with momentum \mathbf{p} at time t . Among them dN will on the average suffer collisions with lattice within a small time interval $(t, t+dt)$. The mean free time τ_0 is defined by the relation $dN(t) = N(t) dt/\tau_0$, or in other words $N(t+s) = N(t) \cdot \exp(-s/\tau_0)$, ($s \geq 0$). $N(t+s)$ gives the average number of electrons that do not collide till the time $t+s$. The average is taken under the condition that the value of $N(t)$ be fixed. Thus, multiplying by $N(t)$ and averaging over all possible values of $N(t)$, we obtain the correlation function $\langle N(t) N(t+s) \rangle = \langle N^2 \rangle \exp(-s/\tau_0)$. This is essentially the expression (5.31).

Next we shall consider the fluctuations of electric current and of heat flow. The electron flow \mathbf{I} and the electronic energy flow \mathbf{Q} are defined by

$$\mathbf{I}(\mathbf{x}, t) \equiv \int \frac{\mathbf{p}}{m} f(\mathbf{x}, \mathbf{p}, t) \frac{2d\mathbf{p}}{h^3}, \quad \mathbf{Q} \equiv \int \varepsilon \frac{\mathbf{p}}{m} f \frac{2d\mathbf{p}}{h^3}. \tag{5.32}$$

According to (5.28) the distribution function $f(\mathbf{x}, \mathbf{p}, t)$ is a linear combination of the

fluctuating term $\mathcal{J}_i(\mathbf{x}, \mathbf{p})$, and the latter obeys the Gaussian distribution law (5.23). Therefore the probability distribution of $f(\mathbf{x}, \mathbf{p}, t)$ is Gaussian, and its first two moments are given by (5.30) and (5.31). From (5.32), the flows \mathbf{I} and \mathbf{Q} are again linear combinations of $f(\mathbf{x}, \mathbf{p}, t)$, whence they will have a Gaussian distribution. Their means and variances will be determined either by making use of (5.30) and (5.31), or directly by using (5.24). For the sake of later convenience we shall take the second method. Inserting the expression of the distribution function (5.28) into (5.32), we can separate the systematic and the fluctuating parts of the electron flow \mathbf{I} and of the energy flow \mathbf{Q} as follows :

$$\begin{pmatrix} I_i(\mathbf{x}, t) \\ Q_i(\mathbf{x}, t) \end{pmatrix} = \int_{-\infty}^t ds \begin{pmatrix} \bar{K}_1(t-s), \bar{K}_2(t-s) \\ \bar{K}_2(t-s), \bar{K}_3(t-s) \end{pmatrix} \begin{pmatrix} X_{ji}(\mathbf{x}, s) \\ X_{oi}(\mathbf{x}, s) \end{pmatrix} + \begin{pmatrix} \mathcal{I}_{ii}(\mathbf{x}) \\ \mathcal{Q}_{ii}(\mathbf{x}) \end{pmatrix},$$

$$[i=x, y, z]. \quad (5.33)$$

Here the fluctuating parts \mathcal{I}_i and \mathcal{Q}_i have been defined by

$$\begin{aligned} \mathcal{I}_{ii}(\mathbf{x}) &\equiv \int_{-\infty}^t ds \int e^{-(t-s)/\tau_0} \frac{p_i}{m} \mathcal{J}_s(\mathbf{x}, \mathbf{p}) \frac{2d\mathbf{p}}{h^3}, \\ \mathcal{Q}_{ii}(\mathbf{x}) &\equiv \int_{-\infty}^t ds \int e^{-(t-s)/\tau_0} \varepsilon \frac{p_i}{m} \mathcal{J}_s(\mathbf{x}, \mathbf{p}) \frac{2d\mathbf{p}}{h^3}, \end{aligned} \quad (5.34)$$

and the decay functions \bar{K}_γ by

$$K_\gamma(t) \equiv \frac{2}{3m} \int e^{-|t|/\tau_0} \varepsilon^\gamma \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3}, \quad [\gamma=1, 2, 3]. \quad (5.35)$$

According to the first equation of (5.24) we know that the first term of (5.33) describes the average response and the second term the deviation from it :

$$\langle \mathcal{I}_{ii}(\mathbf{x}) \rangle = 0, \quad \langle \mathcal{Q}_{ii}(\mathbf{x}) \rangle = 0. \quad (5.36)$$

The correlation functions of the fluctuating terms can be evaluated at once by making use of the second equation of (5.24) :

$$\begin{aligned} &\left(\begin{aligned} &\langle \mathcal{I}_{ii}(\mathbf{x}) \mathcal{I}_{ij}(\mathbf{x}') \rangle, \quad \langle \mathcal{I}_{ii}(\mathbf{x}) \mathcal{Q}_{ij}(\mathbf{x}') \rangle \\ &\langle \mathcal{Q}_{ii}(\mathbf{x}) \mathcal{I}_{ij}(\mathbf{x}') \rangle, \quad \langle \mathcal{Q}_{ii}(\mathbf{x}) \mathcal{Q}_{ij}(\mathbf{x}') \rangle \end{aligned} \right) \\ &= kT(\mathbf{x}) \begin{pmatrix} \bar{K}_1(t'-t), & \bar{K}_2(t'-t) \\ \bar{K}_2(t'-t), & \bar{K}_3(t'-t) \end{pmatrix} \partial_{ij} \partial(\mathbf{x}-\mathbf{x}'), \\ &[i, j=x, y, z]. \end{aligned} \quad (5.37)$$

It will be found that the correlation functions are completely determined by the decay functions \bar{K} . This is true, even when the generalized forces X are constant and the kinetic coefficients

$$K_T \equiv \int_{-\infty}^{\infty} \bar{K}_T(u) du = \frac{2}{3m} \int \tau_c(\varepsilon) \varepsilon^\tau \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3} \quad (5.38)$$

appear in the first term of (5.33). On the right-hand side of (5.37) the time integral of the decay functions does not exist, in contrast to the case of polarization (4.16). The reason is as follows. In the case of polarization the quantity conjugate to the electric field was the electric polarization, but in the present case the quantity conjugate to the voltage is the electric charge and not the electric current.¹⁰⁾ The electric current is given by the time derivative of the electric charge. From (5.33) the decay function for the electric charge will be a time integral of \bar{K}_1 . The correlation function of charge is then given by a double integral of K_1 , according to the general theorem of Takahasi. Since the correlation function of electric current is given by the second derivative of the correlation function of charge, the former will be \bar{K}_1 .

Usually we analyse the fluctuating flows \mathcal{J}_i and \mathcal{Q}_i into the Fourier components:

$$\begin{aligned} \mathfrak{J}_{\omega i}(\mathbf{x}) &\equiv \frac{\sqrt{2}}{2\pi} \int_{-\infty}^{+\infty} \mathcal{J}_{it}(\mathbf{x}) e^{-i\omega t} dt, \\ \mathfrak{Q}_{\omega i}(\mathbf{x}) &\equiv \frac{\sqrt{2}}{2\pi} \int_{-\infty}^{+\infty} \mathcal{Q}_{it}(\mathbf{x}) e^{-i\omega t} dt. \end{aligned} \quad [i=x, y, z] \quad (5.39)$$

$\sqrt{2}$ is necessary in order to make the absolute values equal to the root-mean-square values. Since these components are linear combinations of \mathcal{J}_i or \mathcal{Q}_i , their probability distribution is Gaussian, and their means and correlation functions are

$$\langle \mathfrak{J}_{\omega i}(\mathbf{x}) \rangle = 0, \quad \langle \mathfrak{Q}_{\omega i}(\mathbf{x}) \rangle = 0; \quad (5.40)$$

$$\begin{aligned} &\left(\begin{aligned} &\langle \mathfrak{J}_{\omega i}^*(\mathbf{x}) \mathfrak{J}_{\omega' j}(\mathbf{x}') \rangle, \quad \langle \mathfrak{J}_{\omega i}^*(\mathbf{x}) \mathfrak{Q}_{\omega' j}(\mathbf{x}') \rangle \\ &\langle \mathfrak{Q}_{\omega i}^*(\mathbf{x}) \mathfrak{J}_{\omega' j}(\mathbf{x}') \rangle, \quad \langle \mathfrak{Q}_{\omega i}^*(\mathbf{x}) \mathfrak{Q}_{\omega' j}(\mathbf{x}') \rangle \end{aligned} \right) \\ &= \frac{2}{\pi} kT(\mathbf{x}) \begin{pmatrix} g_1(\omega) & g_2(\omega) \\ g_2(\omega) & g_3(\omega) \end{pmatrix} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(\omega - \omega'). \end{aligned} \quad (5.41)$$

Here the asterisk denotes the complex conjugate. The coefficients are the Fourier transforms of the decay functions K :

$$g_T(\omega) \equiv \int_{-\infty}^{\infty} \bar{K}_T(t) \cos(\omega t) dt = \frac{2}{3m} \int \frac{\tau_c(\varepsilon) \varepsilon^\tau}{1 + \omega^2 \tau_c^2(\varepsilon)} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3}, \quad (5.42)$$

and are the real part of the complex quantities:

$$g_T(\omega) \equiv \int_{-\infty}^{\infty} \bar{K}_T(t) e^{-i\omega t} dt = \frac{2}{3m} \int \frac{\tau_c(\varepsilon) \varepsilon^\tau}{1 + i\omega \tau_c(\varepsilon)} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3}. \quad (5.43)$$

The electric current is defined by $-e\mathbf{I}$, and therefore the conductivity is given by

$$\sigma(\omega) = e^2 g_1(\omega) = -\frac{2e^2}{3m} \int \frac{\tau_e(\varepsilon) \varepsilon^\tau}{1 + \omega^2 \tau_e^2(\varepsilon)} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3}. \quad (5.44)$$

From (5.41) the correlation function of electric current becomes

$$\langle e_{\mathfrak{N}i}^* (\mathbf{x}) e_{\mathfrak{N}j} (\mathbf{x}') \rangle = 2/\pi \cdot kT(\mathbf{x}) \sigma(\omega) \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') \delta(\omega - \omega'). \quad (5.45)$$

This is the localized expression of the well-known Nyquist formula. In order to obtain the usual expression we should integrate (5.45) over the volume of a conductor. For simplicity let us assume that the conductor is of the form of a cylinder with length L and cross section Q , and further that it is isotropic and uniform or homogeneous. Then its conductance is given by

$$G(\omega) = Q\sigma(\omega)/L. \quad (5.46)$$

Let us take the x -axis, say, in the direction of length. The total fluctuating current flowing in this direction and having frequencies in a small band $(\omega, \omega + \Delta\omega)$ is

$$J(\omega) \equiv \int_{\omega}^{\omega + \Delta\omega} d\omega' \int \mathfrak{N}_{\omega'x}(\mathbf{x}) d\mathbf{x} / L, \quad (5.47)$$

where the volume integration should be taken over the conductor. Since the conductor is uniform, the current density $\mathfrak{N}(\mathbf{x})$ is constant in the direction of x -axis, and therefore this volume integration is effectively the integration over the cross section. Remembering (5.45), we get

$$\begin{aligned} \langle |J(\omega)|^2 \rangle &= L^{-2} \iint d\mathbf{x} d\mathbf{x}' \iint d\omega d\omega' \langle \mathfrak{N}_{\omega x}^*(\mathbf{x}) \mathfrak{N}_{\omega' x}(\mathbf{x}') \rangle \\ &= 2/\pi \cdot L^{-2} \int d\mathbf{x} kT(\mathbf{x}) \int_{\omega}^{\omega + \Delta\omega} \sigma(\omega) d\omega \\ &= (2/\pi) kT(Q/L) \sigma(\omega) \Delta\omega, \\ \therefore \langle |J(\omega)|^2 \rangle &= (2/\pi) \cdot kTG(\omega) \Delta\omega. \end{aligned} \quad (5.48)$$

Here we have assumed that the temperature T be constant over the conductor, and that the conductivity $\sigma(\omega)$ be approximately constant over the frequency band $(\omega, \omega + \Delta\omega)$. The expression (5.48) is in agreement with that given by Spenke, though he gave the expression for the unit frequency interval $\Delta\omega = 2\pi$. Bakker and Heller employed the conductance at frequency zero, $1/R_0$, and introduced a factor

$$\int \frac{\tau_e \varepsilon}{1 + \omega^2 \tau_e^2} \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3} / \int \tau_e \varepsilon \left\{ -\frac{\partial f^0}{\partial \varepsilon} \right\} \frac{2d\mathbf{p}}{h^3} = \frac{G(\omega)}{G(0)}, \quad (5.49)$$

as was pointed out by Spenke. The quantity $g_1(\omega)$ introduced in (5.43) is the complex admittance derived by Spenke, if it is multiplied by Qe^2/L .

§ 6. General remarks

In this section we shall give a few remarks on the nature of our statistical theory

developed in the preceding sections of this paper and in Part I.

In our theory the system under consideration is assumed to be described by the two classes of state variables. The variables belonging to the first class are those describing the macroscopic modes of motion ("collective motion") of the system, and the variables of the second class describes the microscopic modes of motion ("internal motion"). We, however, pay attention only to the former variables, and the effect of the latter variables on them is incorporated into our theory only as the origin or a source of their fluctuations. Namely, the degrees of freedom of the latter variables are regarded as constructing a kind of "heat reservoir." This will be seen most apparently in our treatment in § 4 of Part I. In this case the momentum of a colloidal particle was the only macroscopic variable of the system, and its statistical distribution (when the system is aged) was Maxwellian. This distribution was nothing else than that given by Boltzmann's principle (1.3), which can be written in general in the form given by Einstein²⁹⁾

$$W(\alpha) \propto \exp \{ -W_{\min}(\alpha)/kT \}, \quad (6.1)$$

when there is only one heat reservoir of temperature T . $W_{\min}(\alpha)$ means the minimum work [see § 4 of Part I]. If we call the macroscopic degrees of freedom "the system" and the microscopic ones "the heat reservoir," the probability distribution (6.1) may be said as "canonical distribution." The canonical distribution in statistical mechanics is valid for microscopic variables, but as is well-known,³⁰⁾ it also valid for macroscopic variables, provided that the energy is replaced with the minimum work or free energy determined by Gibbs-Helmholtz' relation from that energy. And the canonical distribution thus obtained is the one given by (6.1). If the distribution (6.1) or (1.3) is "canonical" in this sense, its generalization, the path probability (1.11) or (1.12) will be the "canonical distribution of path." From this view-point the kinetic coefficients $G^{(i)}$ are found only to be "coupling constants" between "the system" and "the reservoir." The energy is exchanged between the macroscopic and the microscopic degrees of freedom at a rate determined by these "coupling constants."

In § 4 c) we have referred to the strong collision assumptions. The strong collision theories do not immediately lead us to the Gaussian process, since they assume Poisson's distribution for the number of collisions occurring in a sufficiently long time interval $(-\theta/2, +\theta/2)$.¹⁾ Only in the limit of infinitely large number of collisions per unit of time they lead us to the Gaussian process.²⁾ In this sense the theories based on the strong collision assumptions describes more precisely the phenomenon than our theory based on the weak collision assumption. Thus we see a possibility of extending our statistical theory, especially the stochastic differential equation, to include the case of a finite number of collisions per unit of time. In this extended theory, the dissipative and the fluctuating terms of (4.1), say, will be combined into the following form

$$\sum_k \Delta_k p \cdot \delta(t - t_k), \quad (6.2)$$

where the random variables t_k and $\Delta_k p$ denote the instant of the k -th collision and the value of momentum change by this collision, respectively. The δ -function means that the

instantaneous collisions are assumed. This extended theory will be closely related to the theory of shot effect.⁴⁾

In § 5 we have derived the correlation function of the distribution function, (5.31), which was a refined form of the previous results (5.1). Spenke introduced the factor $1-f^0$ in the expression $-kT\partial f^0/\partial \varepsilon = f^0(1-f^0)$ on the right-hand side of (5.1) by reason of the quantum exclusion-effect for the collision. According to his argument the momentum in this factor $1-f^0$ should have in reality the value after collision, while that in the factor f^0 is the one before collision. The concepts of "before" and "after" collisions are strange for thermodynamics, and they can not at once be introduced into our statistical theory. There exists, however, the possibility of introducing these concepts.*

Also in § 5 we have got Nyquist's formula (5.48). In this expression, however, the classical factor kT appears instead of the quantal average energy of harmonic oscillator

$$E(\omega, T) \equiv \frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT) - 1}. \quad (6.3)$$

Namely, the dynamical quantum effect has escaped out of our theory, although the Fermi statistics was incorporated. This of course arises from our starting equation of motion (5.2), or in other words from the assumption of instantaneous collisions. Our statistical theory is essentially semi-classical, and the detailed description of the collision process is of course beyond the realm of our theory.

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Appendix A. Path-integral evaluated by the method of characteristic function for the case of Brownian motion of the harmonic oscillator

In § 4 a) we have had recourse to the general theorem of Takahasi, when we determined the correlation function of the polarization. If we want to calculate it directly, the method of path integral is convenient. The path integral (1.13) may of course be carried out by Onsager-Machlup's theorem (1.14). We shall give here, however, another method based on the use of characteristic function.

The transition probability corresponding to Langevin's equation (4.1) is, as is well-known, given by

* The averaged equation (5.2) becomes then the usual Lorentz-Boltzmann equation containing the cross-section of collision, from which the simplified equation (5.2) is derived. We can derive an equation similar to Fokker-Planck's equation in this case too. In this generalized Fokker-Planck equation the functional derivatives $\delta/\delta f$ with " f before collision" and $\delta/\delta f'$ with " f' after collision" appear in the combination $\delta/\delta f - \delta/\delta f'$. These equations have been obtained by H. Satô, *Bussei-ron Kenkyu*, No. 67 (1953) 91, using the second order perturbation theory of quantum mechanics.

$$\Psi(x, p; \Delta t; x + \Delta x, p + \Delta p) = \frac{\delta(\Delta x - p/m \cdot \Delta t)}{\sqrt{4\pi D_p \Delta t}} \exp \left[-\frac{|\Delta p + \{p(\tau + m\omega_0^2 x + eE(t))\} \Delta t|^2}{4D_p \Delta t} \right]. \quad (\text{A} \cdot 1)$$

This can also be derived from (1.17). In this case the δ -function on the right hand side will be obtained by introducing a virtual friction constant γ in the equation of x and by taking the limit $\gamma \rightarrow 0$ in the final expression of the transition probability. The path integral (1.13) is the convolution of the transition probability. It will be convenient to introduce the characteristic function of the transition probability. For this purpose we rewrite (A.1) into the following form

$$\begin{aligned} &\Psi(x(s), p(s), s; x(s+ds), p(s+ds), s+ds) \\ &= \frac{\delta[x(s+ds) - x(s) - p(s)ds/m]}{\sqrt{4\pi D_p ds}} \\ &\times \exp \left[-\frac{|p(s+ds) - p(s)(1 - ds/\tau) + \{m\omega_0^2 x(s) + eE(s)\} ds|^2}{4D_p ds} \right]. \quad (\text{A} \cdot 2) \end{aligned}$$

Then its characteristic function becomes

$$\begin{aligned} &X(x(s), p(s), s; \lambda(s+ds), \mu(s+ds), s+ds) \\ &\equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(x(s), p(s), s; x', p', s+ds) dx' dp' \exp \{i\lambda(s+ds)x' + i\mu(s+ds)p'\} \\ &= \exp \{-D_p \mu^2(s+ds)ds - i\mu(s+ds)eE(s)ds\} \cdot \exp[i\{\lambda(s+ds) \\ &\quad - m\omega_0^2 \mu(s+ds)ds\}x(s) + i\{\lambda(s+ds)/m \cdot ds + \mu(s+ds)(1 - ds/\tau)\}p(s)]. \quad (\text{A} \cdot 3) \end{aligned}$$

The characteristic function of the probability of a state (x, p) at time $s+ds$ is determined from that at time s by the relation

$$\begin{aligned} &X(\lambda(s+ds), \mu(s+ds); s+ds) \\ &= \exp \{-D_p \mu^2(s+ds)ds - i\mu(s+ds)eE(s)ds\} X(\lambda(s), \mu(s); s), \quad (\text{A} \cdot 4) \end{aligned}$$

where $\lambda(s)$ and $\mu(s)$ on the right-hand side are related to $\lambda(s+ds)$ and $\mu(s+ds)$ by

$$\begin{aligned} \lambda(s) &= \lambda(s+ds) - m\omega_0^2 \mu(s+ds)ds, \\ \mu(s) &= \mu(s+ds)(1 - ds/\tau) + \lambda(s+ds)/m \cdot ds. \end{aligned} \quad (\text{A} \cdot 5)$$

The repeated use of (A.4) gives a formula

$$\begin{aligned} &X(\lambda(t), \mu(t); t) \\ &= \exp \left\{ -D_p \int_{t_0}^t \mu^2(s)ds - ie \int_{t_0}^t E(s)\mu(s)ds \right\} X(\lambda(t_0), \mu(t_0); t_0), \quad (\text{A} \cdot 6) \end{aligned}$$

which determines the probability at time t in terms of the probability at time t_0 . Especial-

ly, if we put $X(\lambda(t_0), \mu(t_0); t_0) = \exp \{i\lambda(t_0)x_0 + i\mu(t_0)p_0\}$, we obtain characteristic function of the transition probability $\mathcal{P}(x_0, p_0, t_0; x, p, t)$:

$$X(x_0, p_0, t_0 | \lambda(t), \mu(t), t) = \exp \left\{ -D_p \int_{t_0}^t \mu^2(s) ds \right\} \cdot \exp \left\{ i\lambda(t_0)x_0 + i\mu(t_0)p_0 - ie \int_{t_0}^t E(s) \mu(s) ds \right\}. \quad (\text{A} \cdot 7)$$

Here the function $\lambda(s)$ and $\mu(s)$, including $\lambda(t_0)$ and $\mu(t_0)$, are determined by the relations (A.5) or the differential equations

$$\dot{\lambda} = m\omega_0^2 \mu, \quad \dot{\mu} = -\frac{\mu}{\tau} - \frac{\lambda}{m}. \quad (\text{A} \cdot 8)$$

Since the characteristic function (A.7) should be regarded as a function of the final values $\lambda(t)$ and $\mu(t)$, we need the solution of (A.8) in terms of them:

$$\begin{pmatrix} \lambda(s) \\ \mu(s) \end{pmatrix} = e^{-(t-s)/2\tau} \begin{pmatrix} \cos(\omega_1[t-s]) + \frac{\sin(\omega_1[t-s])}{2\tau\omega_1}, & -\frac{m\omega_0^2}{\omega_1} \sin(\omega_1[t-s]) \\ \frac{1}{m\omega_1} \sin(\omega_1[t-s]), & \cos(\omega_1[t-s]) - \frac{\sin(\omega_1[t-s])}{2\tau\omega_1} \end{pmatrix} \begin{pmatrix} \lambda(t) \\ \mu(t) \end{pmatrix}. \quad (\text{A} \cdot 9)$$

First let us determine the characteristic function of the probability of a state, which can be obtained by taking the limit $t_0 \rightarrow -\infty$ in the expression (A.7). Since from (A.9) we obtain $\lambda(-\infty) = 0$ and $\mu(-\infty) = 0$, it becomes

$$\begin{aligned} X(\lambda(t), \mu(t), t) &= \exp \left\{ -D_p \int_{-\infty}^t \mu^2(s) ds - ie \int_{-\infty}^t E(s) \mu(s) ds \right\} \\ &= \exp \left\{ -\frac{\lambda^2(t)}{2} \frac{kT}{m\omega_0^2} - \frac{\mu^2(t)}{2} mkT - i\lambda(t) \int_{-\infty}^t \varphi_x(t-s) eE(s) ds \right. \\ &\quad \left. - i\mu(t) \int_{-\infty}^t \varphi_p(t-s) eE(s) ds \right\}. \end{aligned} \quad (\text{A} \cdot 10)$$

This gives certainly the probability distribution (4.7) in the text. Next we shall consider the joint probability of two states at times t' and t'' . Its characteristic function is of course given by the average

$$\begin{aligned} X(\lambda', \mu', t'; \lambda'', \mu'', t'') &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx' dp' W_{\infty}(x', p', t') \exp \{i\lambda'x' + i\mu'p'\} X(x', p', t' | \lambda'', \mu'', t'') \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx' dp' W_{\infty}(x', p', t') \exp [i\{\lambda' + \lambda(t')\}x' + i\{\mu' + \mu(t')\}p'] \end{aligned}$$

$$\begin{aligned}
& \cdot \exp \left\{ -D_p \int_{t'}^{t''} \mu^2(s) ds - ie \int_{t'}^{t''} E(s) \mu(s) ds \right\} \\
& = X(\lambda' + \lambda(t'), \mu' + \mu(t') ; t') \exp \left\{ -D_p \int_{t'}^{t''} \mu^2(s) ds - ie \int_{t'}^{t''} E(s) \mu(s) ds \right\} \\
& = \exp \left[-\frac{kT}{2m\omega_0^2} \{ \lambda' + \lambda(t') \}^2 - \frac{mkT}{2} \{ \mu' + \mu(t') \}^2 - D_p \int_{t'}^{t''} \mu^2(s) ds \right. \\
& \quad \left. - i \{ \lambda' + \lambda(t') \} \int_{-\infty}^t \varphi_x(t-s) eE(s) ds - i \{ \mu' + \mu(t') \} \int_{-\infty}^t \varphi_p(t-s) eE(s) ds \right. \\
& \quad \left. - i \int_{t'}^{t''} eE(s) \mu(s) ds \right], \tag{A.11}
\end{aligned}$$

where $\lambda(s)$, $\mu(s)$, $\lambda(t')$, and $\mu(t')$ should be determined by the relation (A.9) with the final conditions $\lambda'' = \lambda(t'')$ and $\mu'' = \mu(t'')$, and thus they are functions of λ'' and μ'' . The auto- and cross-correlations constructed with $x(t')$, $p(t')$, $x(t'')$, and $p(t'')$ are, as is well-known, given by the coefficients of $\lambda'\lambda''$, $\lambda'\mu''$, and so on. Remembering (A.9) we can write down at once

$$\begin{aligned}
& \left(\langle \{x(t') - \langle x(t') \rangle\} \{x(t'') - \langle x(t'') \rangle\} \rangle, \langle \{x(t') - \langle x(t') \rangle\} \{p(t'') - \langle p(t'') \rangle\} \rangle \right) \\
& \left(\langle \{p(t') - \langle p(t') \rangle\} \{x(t'') - \langle x(t'') \rangle\} \rangle, \langle \{p(t') - \langle p(t') \rangle\} \{p(t'') - \langle p(t'') \rangle\} \rangle \right) \\
& = e^{-(t''-t')/\tau} \left(\begin{aligned} & \frac{kT}{m\omega_0^2} \left\{ \cos(\omega_1[t''-t']) + \frac{\sin(\omega_1[t''-t'])}{2\tau\omega_1} \right\}, \quad -\frac{kT}{\omega_1} \sin(\omega_1[t''-t']) \\ & -\frac{kT}{\omega_1} \sin(\omega_1[t''-t']), \quad mkT \left\{ \cos(\omega_1[t''-t']) - \frac{\sin(\omega_1[t''-t'])}{2\tau\omega_1} \right\} \end{aligned} \right). \tag{A.12}
\end{aligned}$$

These correlation functions have already been given by Wang and Uhlenbeck.²⁹ It will be found that the electric field does not affect these correlation functions, although the means $\langle x(t') \rangle$, etc., are shifted by it. Returning to the definition of the polarization (4.10), we can at once derive the relation (4.16) from (A.12) by taking into account the independence of N oscillators.

Appendix B. The fluctuation of the distribution function in the theory of Van Vleck-Weisskopf and Fröhlich

In the microwave region Van Vleck-Weisskopf-Fröhlich's theory is commonly accepted to analyse experimental data, though there are theoretical objections.^{14), 33)} So it will be convenient to summarize here the relations based on their assumption, which we have not given in the text, § 4 c).

First we have to construct a force, which makes the distribution $f(x, p, t)$ approach to the "equilibrium" distribution $f_e(x, p, t)$, (4.48). This distribution function f_e is the

equilibrium distribution under the electric field $E(t)$, provided that the value of $E(t)$ is frozen to that at the instant of collision. We should therefore, in thermodynamical consideration, introduce a work source attached to the set of oscillators, which supplies automatically the work $E(t) \delta P(t)$ with this frozen value of $E(t)$ when the polarization $P(t)$ of our system is changed. Besides this work source, the set of oscillators has the heat reservoir of temperature T composed of non-radiating molecules. Thus the change of minimum work is given by

$$\delta W_{\min.} = \delta U - T \delta S - E(t) \delta P, \quad (\text{B} \cdot 1)$$

where U denotes the internal energy of the set of oscillators, per unit of volume :

$$U \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 \right\} f(x, p, t) dx dp. \quad (\text{B} \cdot 2)$$

The entropy S and the polarization P are defined by (4.18) and (4.42) in the text, respectively. Inserting these expressions into (B.1) we get

$$\delta W_{\min.} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 + eE(t)x + kT \log f \right\} \delta f dx dp. \quad (\text{B} \cdot 3)$$

In the present case all the $\delta f(x, p, t)$ can not be changed independently, since there is the normalization condition (4.31). Taking this into account we can derive the generalized force associated with the state variable $f(x, p, t)$ in the same way as was done in § 5 :

$$A(x, p, t) = w(x, p, t) = -kT/f_e \cdot (f - f_e), \quad (\text{B} \cdot 4)$$

where $w(x, p, t)$ is a function measuring the deviation of f from f_e in unit of energy. By making use of this force we can determine the quantities V , $G^{(c)}$, and $G^{(u)}$ in the equation (4.20) or (5.16) :

$$\begin{aligned} V(x, p, t) &= \left[\frac{p}{m} \frac{\partial}{\partial x} - \{m\omega_0^2 x + eE(t)\} \frac{\partial}{\partial p} \right] f_e(x, p, t), \\ kG^{(c)}(x, p; x', p') &= f_e(x', p', t) \left[\frac{p}{m} \frac{\partial}{\partial x} - \{m\omega_0^2 x + eE(t)\} \frac{\partial}{\partial p} \right] \\ &\quad \cdot \delta(x - x') \delta(p - p'), \\ kG^{(u)}(x, p; x', p') &= D(x, p; x', p') = \frac{f_e(x, p, t)}{\tau_c} \delta(x - x') \delta(p - p'). \end{aligned} \quad (\text{B} \cdot 5)$$

Remembering (4.47), the last expression (B.5) of the diffusion matrix agrees with the previous one (4.26), and thus, as was described in the text, we shall arrive at the stochastic differential equations (4.55).

When our system is aged, the formal solution of (4.55) is given by

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \int_{-\infty}^t ds \begin{pmatrix} \varphi_{aa}(t-s), & \varphi_{ab}(t-s) \\ \varphi_{ba}(t-s), & \varphi_{bb}(t-s) \end{pmatrix} \begin{pmatrix} -eE(s)/(m\omega_0^2 \tau_c) + \mathcal{A}(s) \\ -eE(s) + \mathcal{B}(s) \end{pmatrix}, \quad (\text{B} \cdot 6)$$

where we have introduced the following decay functions

$$\left. \begin{aligned} \varphi_{aa}(t) &\equiv e^{-t/\tau_c} \cos(\omega_0 t), & \varphi_{ab}(t) &\equiv \frac{1}{m\omega_0} e^{-t/\tau_c} \sin(\omega_0 t), \\ \varphi_{ba}(t) &\equiv -m\omega_0 e^{-t/\tau_c} \sin(\omega_0 t), & \varphi_{bb}(t) &\equiv e^{-t/\tau_c} \cos(\omega_0 t), \end{aligned} \right\} (t \geq 0). \quad (\text{B} \cdot 7)$$

Since the probability distribution of the fluctuating terms $\mathcal{A}(t)$ and $\mathcal{B}(t)$ is given by (4.38), we obtain

$$\begin{aligned} \langle \mathcal{A}(t) \rangle &= 0, & \langle \mathcal{A}(t) \mathcal{A}(t') \rangle &= (2kT/N\tau_c m\omega_0^2) \delta(t' - t), \\ \langle \mathcal{A}(t) \mathcal{B}(t') \rangle &= 0 \end{aligned} \quad (\text{B} \cdot 8)$$

besides (4.41) in the text. By making use of the formal solution (B.6), the distribution function (4.28) can be written in the form

$$\begin{aligned} \frac{f(x, p, t) - \langle f(x, p, t) \rangle}{f_0(x, p, t)} &= \frac{m\omega_0^2}{kT} \int_{-\infty}^t \left\{ x\varphi_{aa}(t-s) + \frac{p}{(m\omega_0)^2} \varphi_{ba}(t-s) \right\} \mathcal{A}(s) ds \\ &+ \frac{1}{mkT} \int_{-\infty}^t \left\{ (m\omega_0)^2 x\varphi_{ab}(t-s) + p\varphi_{bb}(t-s) \right\} \mathcal{B}(s) ds, \end{aligned} \quad (\text{B} \cdot 9)$$

where the averaged distribution is

$$\begin{aligned} \langle f(x, p, t) \rangle &= \frac{N\omega_0}{2\pi kT} \exp \left\{ -\frac{m\omega_0^2}{2kT} \left| x + e \int_{-\infty}^t \varphi_a(t-s) E(s) ds \right|^2 \right. \\ &\quad \left. - \frac{1}{2mkT} \left| p + e \int_{-\infty}^t \varphi_b(t-s) E(s) ds \right|^2 \right\}; \end{aligned} \quad (\text{B} \cdot 10)$$

$$\left. \begin{aligned} \varphi_a(t) &\equiv \frac{1}{m\omega_0} e^{-t/\tau_c} \left\{ \sin(\omega_0 t) + \frac{\cos(\omega_0 t)}{\omega_0 \tau_c} \right\}, \\ \varphi_b(t) &\equiv e^{-t/\tau_c} \left\{ \cos(\omega_0 t) - \frac{\sin(\omega_0 t)}{\omega_0 \tau_c} \right\}, \end{aligned} \right\} (t \geq 0). \quad (\text{B} \cdot 11)$$

These decay functions $\varphi_a(t)$ and $\varphi_b(t)$ correspond to those given in (4.5) in the case of Brownian motion. Remembering (B.8) and (4.41), we can determine the correlation function of $f(x, p, t)$, (B.9). The result is as follows:

$$\begin{aligned} &\langle \{f(x, p, t) - \langle f(x, p, t) \rangle\} \{f(x', p', t') - \langle f(x', p', t') \rangle\} \rangle \\ &= f_0(x, p) f_0(x', p') \frac{m\omega_0^2}{NkT} e^{-|t'-t|/\tau_c} \left[\left\{ xx' + \frac{pp'}{(m\omega_0)^2} \right\} \cos(\omega_0[t'-t]) \right. \\ &\quad \left. + \frac{x'p - xp'}{m\omega_0} \sin(\omega_0[t'-t]) \right]. \end{aligned} \quad (\text{B} \cdot 12)$$

From (B.10), we see at once that the decay function of electric displacement $D(t) \equiv E(t) + 4\pi P(t)$ has the form proposed by Fröhlich¹⁵⁾:

$$\varphi_D(t) = 4\pi N e^2 \varphi_a(t) = \frac{4\pi N e^2}{m\omega_0} e^{-t/\tau_0} \left\{ \sin(\omega_0 t) + \frac{\cos(\omega_0 t)}{\omega_0 \tau_0} \right\}, \quad (t \geq 0). \quad (\text{B} \cdot 13)$$

The correlation of the polarization can be determined from (B·12) :

$$\langle \{P(t) - \langle P(t) \rangle\} \{P(t') - \langle P(t') \rangle\} \rangle = kT \frac{N e^2}{m\omega_0^2} e^{-|t'-t|/\tau_0} \cos(\omega_0[t'-t]). \quad (\text{B} \cdot 14)$$

This of course satisfies the requirement of Takahasi's theorem.

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Letters to the Editor

Some Relations between the Bound State Problem and Scattering Theory

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It was shown by Sawada¹⁾ a few years ago that there exists a linear relation between the self-energy of the nucleon due to the pion field and the eigen phase shifts of pion-nucleon scattering. However, this is not correct in some cases, for instance, in the static neutral scalar theory, where the pion is never scattered by the nucleon, but still there appears a self-energy of the nucleon. Quite recently, A. Reifman, B. S. DeWitt, and R. Newton²⁾ have discussed the relation between the bound state problem and scattering theory in the ordinary non-relativistic quantum mechanics, according to which the energy shift of the system should be given in terms of the diagonal element of the reaction matrix in the limit of infinite volume. Thus it seems to give a definite justification for the boundary condition which Brueckner and others³⁾ took in their works in connection with nuclear models.

The purpose of this paper is to show that Sawada's relation holds exactly in the non-relativistic case insofar as the range of the potential is negligibly small compared

with the radius R of the sphere in which the system is enclosed. The energy shift ΔE goes to zero as $1/R$, not as $1/R^3$, as R tends to infinity, if we consider this problem at some fixed energy value independent of R , as was assumed by the above authors.²⁾ Therefore, the level spacing and the energy shift is always at least of the same order of magnitude contrary to their expectation, so that their proof seems to be wrong.* Turning to field theory, we would like to give the expression for self-energy of the nucleon in the static model in terms of the renormalized coupling constant and the pion-nucleon scattering phase shifts, using Miyazawa's⁴⁾ paper on the anomalous magnetic moment of the nucleon. This expression is obviously correct even in the neutral scalar theory.

We consider, first, a one-particle system in a central field $V(r)$ which is enclosed in a large sphere with radius R and impose the boundary condition that the wave function ψ^l should be zero on the surface of this sphere. As is well known, it is convenient in this case to adopt the angular momentum representation and to consider some state belonging to a definite angular momentum l . Then, writing

$$\psi^l = R_l(r) Y_{lm}(\theta, \varphi) \quad (1)$$

$R_l(r)$ has the asymptotic form for large r

$$R_l(r) \sim \frac{1}{k_0 r} \sin(k_0 r - \frac{1}{2} l \pi) \quad (2)$$

$$R_l(r) \sim \frac{1}{kr} \sin(kr - \frac{1}{2} l \pi + \delta_l) \quad (3)$$

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* This will be reexamined in a forthcoming paper by the author and R. G. Newton.

where the suffix o specifies the case of free particle and δ_l denotes the eigen phase shift which is to be determined quite independently of R . Now, from the above mentioned boundary condition, the characteristic values of k_0 and k (absolute values of wave vectors) are given by

$$k_{0n} \sim \frac{(\frac{1}{2}l\pi + n)\pi}{R} \quad \text{and} \\ k_n \sim \frac{(\frac{1}{2}l\pi + n)\pi - \delta_l}{R} \quad (n=0, 1, \dots) \quad (4)$$

where k_n goes over into k_{0n} as $V(r)$ tends to zero. Since the energy level is given by $\hbar^2 k^2/2m$, the energy-shift ΔE_l corresponding to some fixed value of k and l turns out to be, in first order in $1/R$, equal to

$$\Delta E_l(k) = (\hbar^2 k/m) \Delta k_l \\ = -(\hbar^2 k/m) \delta_l/R. \quad (5)$$

In the case of meson theory, the total Hamiltonian H is assumed to be of the following form, in the static limit of the one nucleon problem:

$$H = \sum_s \omega_s a_s^* a_s + \sum_s (G_s a_s + G_s^* a_s^*), \\ \omega_s = \sqrt{\mu^2 + k^2} \quad (6)$$

where a_s and a_s^* denote the annihilation and creation operations, respectively, G_s includes only nucleon variables, and s specifies the meson state. Using the relations obtained by Miyazawa, we get easily the self-energy ΔE of the nucleon, i. e., the expectation of H with respect to the physical nucleon state as follows

$$\Delta E = \langle |H| \rangle \\ = \sum_{s, \gamma_s} \langle |G_s| n \rangle \frac{\omega_s}{(E_n + \omega_s)^2} \langle n | G_s^* | \rangle \\ - 2 \sum_{s, \gamma_s} \langle |G_s| n \rangle \frac{1}{E_n + \omega_s} \langle n | G_s^* | \rangle \quad (7)$$

where $|n\rangle$ denotes the incoming-wave

solutions which are assumed to form a complete, orthonormal set. The term $n=0$, corresponding to physical nucleon states, in eq. (7) is the same as the second order self-energy with the renormalized coupling constant, and the other terms are written in terms of the transition matrix elements

$$T_s(n) = \langle n | G_s | \rangle \quad \text{for} \quad s \rightarrow n \quad (8)$$

defined by Chew and Low⁵⁾. In the neutral scalar theory the T 's are all equal to zero; therefore the selfenergy is given by the second order perturbation theory as it should be. In the case of pseudoscalar p -wave theory, the explicit formula of (7) turns out to be

$$\Delta E = \Delta E^{(2)} + \Delta E' \\ \Delta E^{(2)} = -\frac{f_\pi^3}{4\pi} \frac{3}{\pi\mu^2} \int dk \frac{k^4 |v(k)|^2}{\omega_k^2} \\ \Delta E' = -\frac{1}{\pi^2} \iint dk dl \\ \times \frac{|v(k)|^2 k^4 (2\omega_l - \omega_k)}{(\omega_l + \omega_k)^2 l^2 \omega_l \omega_k} \{4 \sin^2 \delta_{33}(l) \\ + 4 \sin^2 \delta_{31}(l) + \sin^2 \delta_{11}(l)\} \quad (9)$$

where $v(k)$ is the cut-off function, $\delta_{ij}(k)$ the eigen phase shift for the state of total isotopic spin $i/2$ and total angular momentum $j/2$, and the multiple production pions is neglected.

Detailed discussions, especially in connection with the work of Brueckner and others³⁾, will be given at another opportunity by the author and R. G. Newton.

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Note on the Nuclear Level of Spin 2⁺

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The nuclear level scheme of even-even nuclei has been studied by many authors.¹⁾⁻⁴⁾

Recently the gamma-gamma angular correlation measurements make it possible not only to assign the spin value to the excited level but also to determine the radiation characters of cascade gamma ray,

Table

Element	Level Energy	Level Order	γ_1	γ_2	γ_3	E_2/E_1	γ_2/γ_3	δ	Element studied	Reference
²⁰ Ca ₂₄ ⁴⁴	2.54	2	1.16	1.38	2.54	2.19	~0	—	Sc ⁴⁴	(a)
²⁶ Fe ₃₀ ⁵⁶	2.59	3	0.845	1.81	~2.7	3.06	~300	+0.23	Mn ⁵⁶	(b)
	2.88	4	0.845	2.06	~3.0	3.40	~100	-0.23	Mn ⁵⁶	(b)
³⁰ Zn ₃₆ ⁶⁶	2.76	2	1.044	1.72	2.76	2.64	~0.1	—	Ga ⁶⁶	(c)
³² Ge ₄₀ ⁷²	1.46	3	0.84	0.63	1.46	1.74	8.0	—	Ga ⁷²	(d)
	1.73	4	0.84	0.90	—	2.06	—	—	Ga ⁷²	(d)
³⁴ Se ₄₂ ⁷⁶	1.21	2	0.567	0.64	1.21	2.14	~0.4, -1~-0.5	—	As ⁷⁶	(e)
⁴² Mo ₅₂ ⁹⁴	3.27	3	0.87	2.40	3.27	3.76	~0	—	Tc ⁹⁴	(f)
⁴⁶ Pd ₆₀ ¹⁰⁶	1.55	3	0.513	1.04	1.55	3.02	~9	-0.18	Rh ¹⁰⁶	(g)
⁴⁸ Cd ₇₀ ¹¹⁴	1.263	2	0.550	0.713	1.263	2.30	~16	-0.106	In ¹¹⁴	(h)
⁵⁰ Sn ₆₆ ¹¹⁶	2.09	2	1.27	0.80	2.09	1.65	~1	~+3	In ¹¹⁶	(i)
⁵² Te ₇₀ ¹²²	1.24	2	0.568	0.67	1.24	2.18	4.5	+3	Sb ¹²²	(j)
⁵² Te ₇₄ ¹²⁶	1.42	2	0.55	0.74	1.42	2.18	10	—	I ¹²⁶	(k)
⁵⁴ Xe ₇₂ ¹²⁶	0.86	2	0.38	0.48	0.86	2.26	5	—	I ¹²⁶	(l)
⁷⁴ W ₁₀₈ ¹⁸²	1.222	3	0.101	1.122	1.222	12.1	~1	Q+D	Ta ¹⁸²	(m)
⁷⁶ Pt ₁₁₄ ¹⁹²	0.612	2	0.316	0.296	0.612	1.95	4.3	±4.4	Ir ¹⁹²	(n)
⁷⁶ Pt ₁₁₆ ¹⁹⁴	0.620	2	0.327	0.293	0.620	1.92	large	±10	Ir ¹⁹⁴	(o)
⁷⁸ Pt ₁₁₈ ¹⁹⁶	0.688	2	0.358	0.32	0.688	1.92	100>	-4.35~-4.0	Au ¹⁹⁶	(p)
⁸⁰ Hg ₁₁₈ ¹⁹⁸	1.087	2	0.411	0.676	1.087	2.64	3.5	-0.85	Au ¹⁹⁸	(q)
⁸⁰ Hg ₁₂₀ ²⁰⁰	0.947	2	0.368	0.579	—	2.57	large	very small	Tl ²⁰⁰	(r)

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thanks to the advanced scintillation counter techniques. In case of even-even nuclei it happens to reveal that there are many nuclei which have excited levels of spin 2^+ other than the well-known first 2^+ excited level. We have collected in table a list of all nuclei known up to present for which the spin value 2^+ seemed to be established. In Table γ_1 and γ_2 are the gamma rays decaying from the first excited state to the ground state and from the 2^+ level to the first excited state respectively and γ_3 is the cross over gamma ray. The branching ratio of γ_2 to γ_3 is denoted by γ_2/γ_3 . δ is defined as the ratio of the reduced matrix elements of quadrupole radiation to dipole radiation.⁵⁾ E_2/E_1 is the ratio of the energy of 2^+ level in question to that of the first 2^+ level. Since the data are not sufficient and some of the results of angular correlation measurement do not seem quite dependable, it may be dangerous to draw a definite conclusion from this table. But the author believes it is worth noting the following several points.

1) The proton number of the nuclei collected in the table occurs in the vicinity of the magic number. As for the neutron number such a fact does not seem found so far as the present table is concerned.

2) When the branching ratio γ_2/γ_3 is large, δ is always either small or large. Moreover γ_2 is emitted mostly by a magnetic dipole radiation if the proton number is just below the magic number.

Several isotopes have a surprising similarity not only as to the energy scheme but also as to the branching ratio and the δ value. Pt^{192} , Pt^{194} and Pt^{196} present a good example.

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On the Renormalization Cut-off

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It has been pointed out that the renormalization theory can be internally consistent only when the observed value g_{ob} of the coupling constant lies in the normal zone¹⁾ $N(g_r)$ (g_r : the renormalized coupling constant). Then, it is natural to ask how the renormalization theory fails in experiments, when g_{ob} is out of $N(g_r)$.

Since all of the S -matrix elements are described with the renormalized quantities, it is interesting to find essential differences between the renormalized Green's function with g_{ob} within $N(g_r)$ and the one with g_{ob} out of $N(g_r)$. In order to obtain the renormalized quantities G_c , $\Gamma_{\mu c}$, etc., we first cut off the effects of the high internal momenta ($k > \Lambda$). Then, we make the limiting process $\Lambda \rightarrow \infty$, and replace g_r by g_{ob} and renormalized mass m by the observed

mass m_{ob} , after the renormalization, to obtain $G_c(g_{ob})$ and $I'_{\nu c}(g_{ob})$.

The limiting process $\Lambda \rightarrow \infty$ can be avoided, when use is made of a cut-off method which with finite Λ leads to the ordinary renormalized Green's function being independent of Λ . We shall show that such a cut-off, called *the renormalization cut-off* (or simply RCO), really exists in cases of interactions of the first kind.²⁾

As a first step for deriving RCO, we shall cut off the high internal momenta for Green's function of the scalar field (or the electromagnetic field)* as follows;

$$[G_{\Lambda}(-k^2)]^{-1} = k^2 + \kappa^2 + \sum'(-k^2) + (k^2 + m^2) \frac{\sum'(\cdot F) - \sum'(m^2)}{\cdot F^2 - m^2} \quad (I)$$

Here G_{Λ} is the unrenormalized Green's function, κ is the mechanical mass and $\sum'(-k^2)$ is the mass-operator. The last term in (I) has been determined so as to satisfy the three conditions; i) the renormalized Green's function should not depend on the cut-off except through the renormalized coupling g_{Λ} (which should be replaced by g_{ob}), and only the Z factor depends on Λ , ii) the renormalized mass should remain unchanged through the cut-off, and iii) $G_{\Lambda}(-k^2)$ should be the free propagator at $-k^2 = \cdot F^2$, that is, $[G_{\Lambda}(\cdot F)]^{-1} = -\cdot F^2 + m^2$. It can be shown that this cut-off is just identical with Ward's one used Gell-Mann and Low³⁾.

However, this cut-off method has a defect that the Z factor $Z(\cdot F, g_{\Lambda})$ becomes complex.

$$[Z(\cdot F, g_{\Lambda})]^{-1} = \frac{\partial}{\partial k^2} [G_{\Lambda}(-k^2)]^{-1} \Big|_{-k^2 = m^2} \quad (1)$$

* It is easy to extend the similar discussion to cases of spinor fields.

This is because $\sum'(-k^2)$ is, in general, a complex quantity which is due to the interference thresholds¹⁾ being branching points. The imaginary part $Z^I(\cdot F, g_{ob})$ of $Z(\cdot F, g_{ob})$ is the contribution from the displaced poles¹⁾. Thus, it should be finite even for $\Lambda \rightarrow \infty$.

As the renormalization factor should be a real quantity, we modify eq. (I) as follows,

$$[G_{\Lambda}(-k^2)]^{-1} = k^2 + \kappa^2 + \sum'(-k^2) + (k^2 + m^2) \frac{Re[\sum'(\cdot F)] - \sum'(m^2)}{\cdot F^2 - m^2} \quad (II)$$

This is our RCO. We can derive the following results:

$$G_{\Lambda}(-k^2, g_{ob}) = G^0(-k^2) \frac{Z^R(\cdot F, g_{ob})}{Z(-k^2, g_{ob})} = G_c(-k^2, g_{ob}) Z^R(\cdot F, g_{ob}). \quad (2)$$

Here $G^0(-k^2)$ is the free propagator, $Z^R(\cdot F, g_{\Lambda})$ is the renormalization factor defined by Eq. (1) and (II). The $Z(-k^2, g_{ob})$ is the function obtained by replacing $\cdot F^2$ by $-k^2$ and g_{Λ} by g_{ob} in $Z(\cdot F, g_{\Lambda})$. It can be proved that $Z^R(\cdot F, g_{ob})$ is just the real part of $Z(\cdot F, g_{ob})$. For (II), the conditions (i), (ii) are still valid, while the condition (iii) is violated only by the correction due to $Z^I(\cdot F, g_{ob})/Z^R(\cdot F, g_{ob})$. Eq. (2) reduces to the result of the ordinary renormalization theory as $\cdot F^2 \rightarrow \infty$, but even if we keep Λ finite, to replace g_{Λ} by g_{ob} leads to the same renormalized Green's function.

The renormalization cut-off for the vertex Γ_{μ} can be introduced in a way similar to the case of Green's function. Then we can show that, in the quantum electrodynamics, the Ward identity holds even for the finite Λ .

It is remarkable feature of RCO that

the renormalized propagator $G_c(-k^2)$ with g_{ob} (i. e., $g_r \rightarrow g_{ob}$) can be obtained by calculating $Z(\Lambda^2, g_\Lambda)$ with various values of the cut-off.

Eq. (2) shows that the propagator $G_\Lambda(-k^2)$ becomes that of the free field at $-k^2 = \Lambda^2 \gg m^2$ except an imaginary correction. This means that if we enter the inside of the proper field we should observe its source field.

Since the order of divergences of $Z(g_{ob})$ is logarithmic, we may approximately calculate $Z^R(\Lambda^2, g_{ob})$ with $\Lambda^2 \gg m^2$ by using the straight cut-off instead of using the RCO; the error induced in this case is made up of terms of $O(m^2/\Lambda^2) \log(\Lambda^2/m^2)$.

When g_{ob} is out of $N(g_r)$ and the renormalization factor is negative, there

should exist a high energy region where $Z^R(-k^2, g_{ob})$ is negative. Eq. (2) shows that the sign of Z^R has various observable effects; e. g., it affects the scattering phase shift. To analyse the high energy phenomena, it seems convenient to introduce the "effective coupling". Detailed accounts of the RCO method and its application to the high energy phenomena will be presently published in *Nuovo Cimento*.

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ERRATA

Note on the Decay Interactions Hyperons and Heavy Mesons

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P. 181 left, 9th line, for $\Delta I_3 = \pm 1/2$ read $\Delta I = \pm 1/2$ and $\Delta I_3 = \pm 1/2$

P. 181 right ii) Case of odd spin and odd parity, "Remarks" in Table,

for $\tau(\theta^\circ) = \tau(\theta^+)$ read $\tau(\theta^\circ) = 4\tau(\theta^+)$

P. 181 right 5th line from the bottom, for $\tau(\theta^\circ) = \tau(\theta^+)$ read $\tau(\theta^\circ) = 4\tau(\theta^+)$

P. 181 right footnote** for $T(\theta^\circ)$ read $\tau(\theta^\circ)$

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CONTENTS

	Page
Shobu KANEKO and Taro KIHARA; Second Virial Coefficient of Helium from the Square-well Potential.....	471
Tokuo SUITA and Noriaki ITOH; Dielectric Breakdown and Pulse Conduction of Silver Halide Crystals.....	474
Yasutada UEMURA; On the Coupled Dislocations along a Grain Boundary	479
Kazuo KIMOTO; An Electron Diffraction Study on the Plastic Deformation of Aluminium Single Crystals.....	485
Yoshikazu ISHIKAWA and Shozo SAWADA; The Study on Substances having the Ilmenite Structure I. Physical Properties of Synthesized FeTiO_3 and NiTiO_3 Ceramics.....	496
F. E. FUJITA, D. WATANABE, M. YAMAMOTO and S. OGAWA; Study of Plastic Deformation in Aluminium Crystals by Electron Diffraction and Electron Microscopy	502
Saiyu MARUYAMA and Hiroshi KIHO; Intersections of $\{301\}$, $\{101\}$ Twin Bands in Tin	516
Tadami TAOKA and Shoichiro AOYAGI; Electron Microscopic Study on the Structure of Mosaic Boundaries in Ni-Mn Single Crystal	522
Goro HONJO, Norihisa KITAMURA, Kohji SHIMAOKA and Kazuhiro MIHAMA; Low Temperature Specimen Method for Electron Diffraction and Electron Microscopy	527
Tadami TAOKA; Magnetic After-Effect in Ni_3Mn Alloy	537
Ryogo KUBO and Yukio OBATA; Note on the Paramagnetic Susceptibility and the Gyromagnetic Ratio in Metals	547
Sōshin CHIKAZUMI; Study of Magnetic Annealing on Ni_3Fe Single Crystal	551
Hiroomi FUJIKAWA; The Forces Acting on Two Equal Circular Cylinders placed in a Uniform Stream at Low Values of Reynolds Number.....	558
Mitutosi KAWAGUTI; On the Viscous Shear Flow around a Circular Cylinder II. Oseen's Approximation.....	570
Ryuma KAWAMURA and Haruo SAITO; Reflection of Shock Waves—I Pseudo-Stationary Case	584
Sunao OGIH; On the Coarsening of Non-Sag Tungsten Lamp Filament Wires	593
Itsuro KIMURA and Seichiro KUMAGAI; Spark Ignition of Flowing Gases.....	599

SHORT NOTES

Satoshi TANIGUCHI and Mikio YAMAMOTO; On the Uniaxial Anisotropy Induced by Magnetic Annealing in Ferrites.....	604
Chiyo YAMANAKA, Hidetsugu SAKAI and Tokuo SUITA; Electron Bombardment Conductivity in Silver Bromide Single Crystal II	605
Takahiko KAMIGAICHI, Tadami KIHARA, Hideo TAZAKI and Eiji HIRAHARA; Electrical Conductivity of Iron Sulfides Single Crystals at the Temperature Range of α -transformation.....	606
Miyuki MURAKAMI and Eiji HIRAHARA; On the Lithium Resonance in Lithium Sulfate Monohydrate $\text{Li}_2\text{SO}_4 \cdot \text{H}_2\text{O}$	607
Shigeharu KOSHINO; On the Low Temperature Conduction Phenomena in Semiconductors	608
Shigehiro KOBAYASHI; Fermi Part γ_0 of the TF Function for Free Positive Ion	609
Hiroshi TUBOTA, Hiromichi SUZUKI and On MATUMURA; The Temperature Dependence of the Optical Absorption of Zinc Selenide	610
Hidekazu HASEMOTO; Note on Rayleigh's Problem for a Circular Cylinder with Uniform Suction and Related Unsteady Flow Problem.....	611
Teturo INUI, Susumu HARASAWA and Yukio OBATA; Note on the Electron Spin Resonance of the V_1 -center.....	612

Errata

Akira SAKURAI; Decrement of Blast Wave.....	613
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The Atmospheric Effects on the Intensity of High Energy μ -Mesons

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In order to reconsider the present model of the cosmic ray in middle energy region, calculations are made for the atmospheric effects on the oblique intensity of high energy μ -mesons under the following assumptions:

- 1) The μ -mesons are originated only through the decay of π -mesons.
- 2) The source function of these π -mesons decreases exponentially with an atmospheric depth.

Zenithal dependence and the absolute intensity of the μ -mesons at sea level deduced theoretically are compared with a narrow angle counter telescope experiment by Sekido, Yoshida and Kamiya. Satisfactory accordance seems to suggest the consistency of the present assumptions. Correlation coefficients of the temperature effect are also obtained on the oblique μ -mesons. It is shown that the negative temperature effect can be mainly attributed to the variation of the height of μ -meson production, but is partly dependent on the variation of the density distribution of the air. The correlation coefficients of the temperature effects on the vertical μ -meson are also compared with the experimental results of Duperier.

§ 1. Introduction

It is well known that the intensity of cosmic rays is influenced by the temperature of the atmosphere. This temperature effect is known to be related to the unstable components in the cosmic rays. Besides the so-called 'negative' temperature effect, Duperier¹⁾ showed the existence of the 'positive' correlation between the sea level intensity of μ -mesons and the temperature in the upper atmosphere from the analysis of the observed intensity variation. The negative temperature effect comes from the variation of the height of the μ -meson production and the positive one is considered to be caused by the variation of air density of the atmospheric layer where π -mesons decay into μ -mesons.

Several reports have been published on the analysis of the intensity of μ -mesons¹⁾⁻⁹⁾ with regard to the positive temperature effect. Barrett et al.⁶⁾ showed that the value of the coefficient of this effect was in agreement with that predicted theoretically for the intensity of very high energy μ -mesons underground, provided that all of them originated from the decay of π -mesons. But appreciable discrepancies are seen among the results obtained by other authors for the μ -meson intensity of lower energy at sea level.

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Since December 1951, a narrow counter telescope with a thick iron absorber has been operated by Sekido et al.¹⁰⁾ to investigate the intensity distribution of cosmic rays over the celestial sphere. To make free from the complicated geomagnetic deflections acting on the primary particles, the observed particles are selected as to be energetic by keeping the counter telescope nearly horizontal (with a zenith distance 80°). The median energy of μ -mesons observed under this condition is, as will be shown in the section 3, about 40 Bev which lies between that of particles observed by Barrett et al. and by other authors with ordinary counter telescopes at sea level. Therefore, the continuous intensity measurement by this telescope is expected to offer a useful data for the analysis of the temperature effect. In this experiment, the zenithal dependence of the μ -meson intensity was also measured with three different thickness of the absorber in the telescope.

In virtue of this experiment, it becomes possible to test the present model of the cosmic ray in the intermediary energy. With this intention, the authors have estimated the μ -meson intensity theoretically as a function of an atmospheric temperature, energy and zenith angle, and compared the flux and the zenithal dependence of the μ -meson intensity with the observed results. The temperature effects of the nearly horizontal cosmic ray particles are also calculated. Our analysis shows satisfactory accordance of the theory with experiments, except of the positive temperature effect.

§ 2. Intensity as a function of depth and zenith angle

Most of the particles observed by a nearly horizontal cosmic ray telescope should be high energy μ -mesons originated from the decays of π -mesons in the upper atmosphere. On the production of these π -mesons in the atmosphere, we make simply such an assumption that its source density decreases exponentially with atmospheric depth x along with the line of motion. Then, we can obtain the following equation for the variation of $\pi(E', x)$ which is the π -meson intensity of energy E' at x ,

$$d\pi(E')/dx = F(E') \exp(-x/\lambda_p) / \lambda_p - \pi(E') (1/\lambda_\pi + m_\pi c/E' \rho \tau_\pi). \quad (1)$$

The first term is the rate of the production of π -mesons, and λ_p and $F(E')$ are the absorption mean free path of π -meson producers and the differential energy spectrum of the produced π -mesons, respectively. In the following, we can safely take $F(E')$ of the form $I_0 \cdot (E')^{-2.8}$ where I_0 is a constant. The second term indicates the processes which correlate with the decrease of π -meson intensity; nuclear interaction and $\pi \rightarrow \mu$ decay. λ_π , m_π and τ_π are the absorption mean free path in air, the rest mass and the proper life time of the π -meson, respectively. ρ , the density of air is a function of x , which can be expressed as $\rho = x \cos \theta / H$ where θ is the zenith distance. H is RT_1/Mg (M =molecular weight of air, g =acceleration of gravity, R =gas constant) and T_1 , the atmospheric temperature, is regarded to be independent of x in the upper atmosphere. Therefore, in the following discussion, H can be treated as a constant with a value 6.46×10^5 cm.

Based on this approximation, the solution of the equation (1) is easily obtained, namely

$$\pi(E', x, \theta) = F(E')/\lambda_p \cdot e^{-x/\lambda_\pi} x^{-(B_\pi/E \cos \theta)} \int_0^x e^{-x'/\lambda'} x'^{B_\pi/E \cos \theta} dx'$$

$$= F(E') e^{-x/\lambda_\pi} x/\lambda_p \cdot \left\{ \frac{1}{1+B_\pi/E \cos \theta} - \frac{x/\lambda'}{2+B_\pi/E \cos \theta} + \frac{1}{2!} \frac{(x/\lambda')^2}{3+B_\pi/E \cos \theta} \dots \right\}, \quad (2)$$

where $1/\lambda' = 1/\lambda_p - 1/\lambda_\pi$, $B_\pi = m_\pi c^2 r H / c \tau_\pi$, $E' = E/r$ and $r = m_\mu / m_\pi$, m_μ being the rest mass of the μ -meson. Now we shall take account of the μ -meson component which arises from the source function $\pi(E', x, \theta)$. The probability $P(x)$ of the surviving μ -meson of energy E to travel from x to x_0 is obtained from the equation,

$$dP_\mu(x')/dx' = -P_\mu(x') \cdot m_\mu c / E(x') \rho(x') \tau_\mu, \quad (3)$$

where $P_\mu(x')$ is the survival probability of μ -mesons from x to x' , m_μ is the rest mass and τ_μ is the proper life time of the μ -meson. The solution of the equation (3) leads to the following

$$P(x_0, x, E, \theta) = \left\{ \frac{x}{x_0} \cdot \frac{E - \beta(x_0 - x)}{E} \right\}^{B_\mu / (E + \beta x) \cos \theta}, \quad (4)$$

where $B_\mu = m_\mu c^2 H' / \tau_\mu c$ and $H' = RT_2 / Mg$. β means the ionization loss of μ -mesons through 1 gram/cm² of air. T_2 is the mean temperature of the lower atmosphere. The energy of the μ -meson originating from the decay of the π -meson with energy E' has a small range between E' and $r^2 E'$. But it is a good approximation to put it as a unique energy rE' . Then the increase dN_μ of the number of μ -mesons by $\pi \rightarrow \mu$ decay within the atmospheric depth dx is as follows

$$dN_\mu(x) = (B_\pi / E x \cos \theta) \pi(E/r, x, \theta) dx. \quad (5)$$

By connecting the equations (2), (4) and (5), we can obtain the number of μ -mesons at the atmospheric depth x_0 as follows,

$$\mu(E, \theta) = \int_0^{x_0} [F(E/r)/\lambda_p] e^{-x/\lambda_\pi} \cdot \frac{B_\pi}{E \cos \theta} \cdot \left\{ \frac{x}{x_0} \cdot \frac{E - \beta(x_0 - x)}{E} \right\}^{B_\mu / (E + \beta x) \cos \theta}$$

$$\cdot \left\{ \sum_{n=0}^{\infty} (-1)^n (x/\lambda')^n \frac{1}{n!} \frac{1}{n+1+B_\pi/E \cos \theta} \right\} dx. \quad (6)$$

This integration can be performed analytically with following two approximations. First, x in the term $[x/x_0 \cdot \{E - \beta(x_0 - x)\} / E]^{B_\mu / (E + \beta x) \cos \theta}$ may be neglected when $E \gg \beta x$, and replaced by x_1 when E is not so larger than βx , where x_1 is the mean depth of the π -meson production. Hereafter we shall put x_1 equal to λ_π . Secondly, the upper limit x_0 of this integration is replaced by infinity, because the integrand involves the term e^{-x/λ_π} which rapidly falls off with x . Through these procedures, we finally reach the following equation,

$$\mu(E, \theta) = [F(E/r)] \frac{\lambda_\pi}{\lambda_p} \cdot \frac{B_\pi}{E \cos \theta} \cdot \left[\frac{\lambda_\pi \{E - \beta(x_0 - x_1)\}}{E x_0} \right]^{B_\mu / (E + \beta x_1) \cos \theta}$$

$$\times \Gamma \left(1 + \frac{B_\mu}{(E + \beta x_1) \cos \theta} \right) \left\{ \frac{1}{1+B_\pi/E \cos \theta} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{\lambda_\pi}{\lambda'} \right)^n \frac{1}{n!} \prod_{i=1}^n \left[i + \frac{B_\mu}{(E + \beta x_1) \cos \theta} \right] \right\}. \quad (7)$$

The intensity of μ -mesons at x_0 is given by integrating the equation (7) with energy E ,

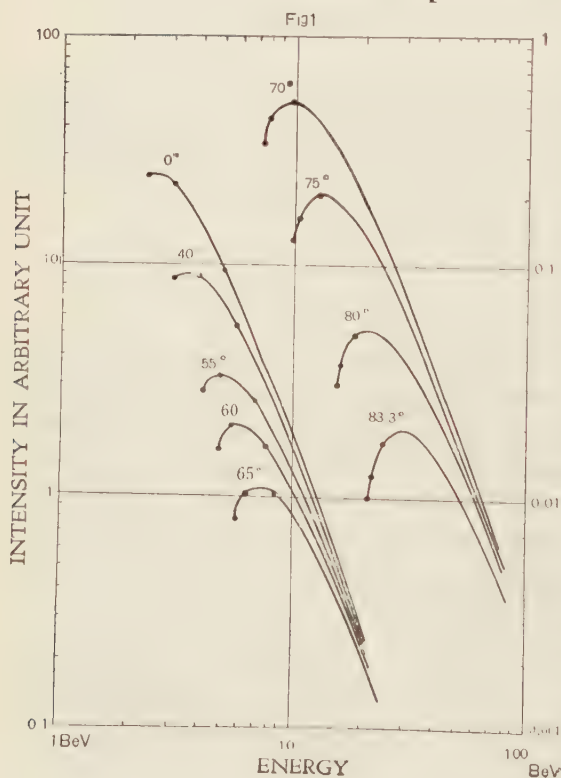
$$I(\theta) = \int_{E_0}^{\infty} \mu(E, \theta) dE. \quad (8)$$

E_0 means the minimum energy at the point of the origination of the μ -meson which is detectable with the telescope. We can approximate that $E_0 = E_{\min} + \beta(x_0 - x_1)$, where E_{\min} is the minimum energy to penetrate the absorber in the telescope. The integration of the equation (8) is performed numerically.

The numerical values of various constants in the equation (7) are as follows,

- 1) $\lambda_p = 120 \text{ g/cm}^2$, $\lambda_\pi = 60 \text{ g/cm}^2$.
- 2) $B_\pi = (m_\pi c^2 r / c \tau_\pi) \cdot RT_1 / Mg = 90 \text{ Bev.}$, $B_\mu = m_\mu c^2 RT_2 / c \tau_\mu Mg = 1.25 \text{ Bev.}$, where $T_1 = 222 \text{ K}$ and $T_2 = 256 \text{ K}$, respectively.
- 3) β : β is the energy loss of μ -mesons per 1 g/cm^2 of air, which is a gradually increasing function $\beta(E)$ of the μ -meson energy E in the high energy region. For the relationship $\beta(E)$ we used the one which has been given by Halpern and Hall⁽¹¹⁾, and to calculate $\mu(E, \theta)$ by the equation (7) we adopted the constant value of β which is the mean between $\beta(E)$ and $\beta(E_2)$, where E_2 , the energy of μ -mesons at sea level, is a function of E .

§ 3. Zenithal dependence of μ -meson intensity



The zenithal dependence of the μ -meson intensity at sea level has been obtained from the equation (7). The calculations were done at the nine zenith angles: 0° , 40° , 55° , 60° , 65° , 70° , 75° , 80° and 83.3° . It should be noted that in the case of nearly horizontal direction (83.3°) the relation $\rho = x \cos \theta / H$ no more holds because of the roundness of the earth surface. In order to include this effect in a simple approximation, θ^* was used instead of θ in the term $B_\pi / E \cos \theta$ in the equation (7), where θ^* is slightly smaller than θ .

Fig. 1 Zenith angle dependence of the differential energy distribution of cosmic ray μ -mesons at the originating point. Three points attached to each curve are minimum energies of particles to be able to penetrate the absorber of 211 g/cm^2 Fe, 523 g/cm^2 Fe, 1588 g/cm^2 Fe at sea level.

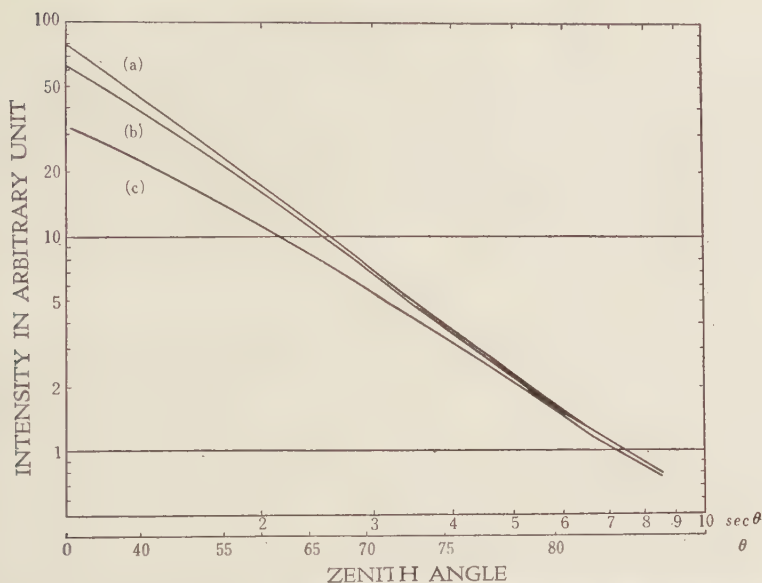


Fig. 2 Theoretical curves for the zenith angle distribution of cosmic rays capable of penetrating a thick iron absorber. Three curves (a), (b), (c), correspond to the cases of 211g/cm² Fe, 523g/cm² Fe, and 1588g/cm² Fe, respectively.

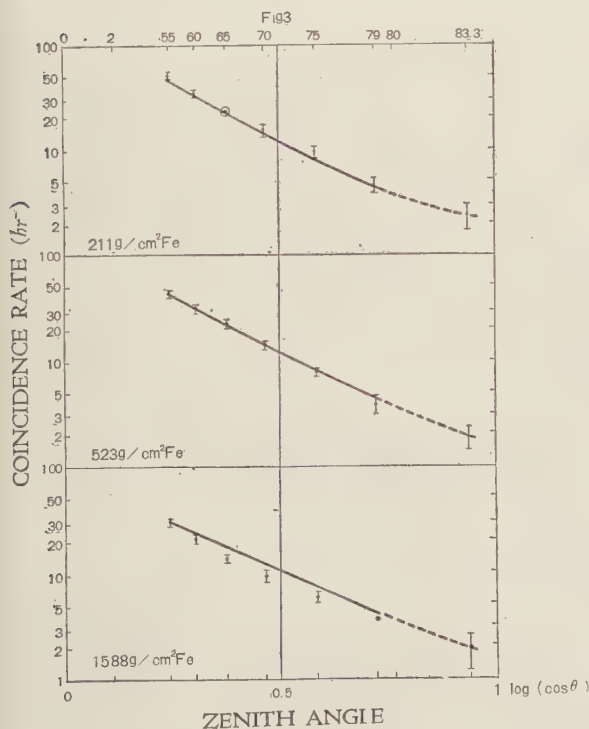


Fig. 3 Observed zenith angle distribution of cosmic ray intensity by a counter telescope with an iron absorber is compared with theoretical curves in the three cases of different absorber thicknesses.

○: normalization point

The results are shown in Fig. 1 and Fig. 2. Fig. 1 gives the energy spectra of the μ -mesons at the points of origination. Fig. 2 shows the zenithal dependence of the μ -meson intensity at sea level for the different thickness of the absorber in the telescope. These curves in Fig. 2, however, do not directly correspond to the experimental data, since the telescope has a finite solid angle. In Fig. 3 the solid lines represent the theoretical curves with the above correction, which should be compared with the experimental values. Normalization between them is given by only one point. Accordance between the theoretical and the experimental values is satisfactory, as shown in this figure.

§ 4. Atmospheric effect

In this section, we shall study the atmospheric effect on the oblique cosmic ray intensity at sea level. Particularly, calculations are done for a particular condition corresponding to the observation by Sekido et al. whose equipment consists of the counter telescope with an absorber of $1,588\text{ g/cm}^2$ Fe, the zenith distance being kept at 80° .

The variation of the μ -meson intensity due to the change of atmospheric condition is usually expressed as,

$$\delta I/I = A_p \delta T + A_H \delta H + A_p \delta p, \quad (9)$$

where δT is the variation of the temperature of the atmospheric layer where π -mesons are produced, δH is the variation of the height of the μ -meson production and δp is that of the ground pressure. As pointed out by Olbert, however, this expression is incomplete because it does not include the possible variation of the intensity due to the temperature distribution of the lower atmosphere. This variation is more closely related to the ionization loss of the μ -meson and it brings the change of the air density along the path of the particles. Accordingly, this variation results possibly in the change of the survival probability of μ -mesons, even when there is no change in the production height H of μ -mesons. Therefore, the following equation

$$\delta I/I = A_1 \delta T_1 + A_2 \delta T_2 + A_3 \delta T_3 + A_p \delta p \quad (10)$$

is better than the equation (9) for the estimation of the atmospheric effect on the intensity of μ -mesons. In this equation, T_1 is the temperature of the π -meson region, which is about 20 mb for our particular condition. The atmosphere traversed by μ -mesons is divided into two parts, δT_2 and δT_3 being the temperature changes of the upper and lower part, respectively. For the determination of the boundary height between these two regions, the yearly change of the atmospheric temperatures at different pressure levels have been examined with regard to the data obtained by two meteorological observatories in Japan. As a result, an appreciable discrepancy is seen between the temperature change of the upper and lower part and the boundary height of them lies near 300 milibar. Therefore, in the following discussion, we shall adopt 300 milibar as a boundary height between the atmospheric layers relating to the second and third terms in the equation (10).

To obtain the theoretical values of the regression coefficients attached to each term of of equation (10), the equation (7) was modified as follows:

$$\begin{aligned} \mu(E, T_1, T_2, T_3, x_0) &= K_\pi T_1 F(E/r) \frac{\lambda_\pi}{\lambda_p} \left\{ \frac{\lambda_\pi}{x_{300}} \cdot \frac{E - \beta(x_{300} - x_1)}{E} \right\}^{K_\mu T_2} \\ &\times F(1 + K_\mu T_2) \left\{ \frac{1}{1 + K_\pi T_1} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{\lambda_\pi}{\lambda'} \right)^n \frac{1}{n!} \frac{H^n(i + K_\mu T_2)}{n + 1 + K_\pi T_1} \right\} \\ &\times \left(\frac{x_{300}}{x_0} \cdot \frac{E - \beta(x_0 - x_1)}{E - \beta(x_{300} - x_1)} \right)^{K_\mu T_3}, \end{aligned} \quad (11)$$

where $K_\pi = m_\pi c^2 R / c \tau_\pi E \cos 80^\circ$ Mg and $K_\mu = m_\mu c^2 R / c \tau_\mu (E + \beta x_1) \cos 80^\circ$ Mg. T_1 , T_2 and T_3 are the mean temperatures of each atmospheric region relating to the first, second and third terms of the equation (10), respectively. x_0 and x_{300} are the atmospheric depth of the sea level and the 300 milibar level along the path of the particles. As the intensity variation due to the change of T_1 becomes

$$\frac{\delta I}{I} = \frac{\int \frac{\partial}{\partial T_1} \mu(E, T_1, T_2, T_3, x_0) dE}{\int \mu dE} \delta T_1, \quad (12)$$

the equation (11) gives the value of the coefficient A_1 as follows:

$$A_1 = \int (\partial \mu / \partial T_1) dE / \int \mu dE. \quad (13)$$

In like manner,

$$\begin{aligned} A_2 &= \int (\partial \mu / \partial T_2) dE / \int \mu dE, \\ A_3 &= \int (\partial \mu / \partial T_3) dE / \int \mu dE, \end{aligned} \quad (14)$$

and

$$A_p = \int (\partial \mu / \partial p) dE / \int \mu dE,$$

respectively, where $p = x_0 \cos 80^\circ$.

(1) Positive temperature effect: A_1

This effect is attributed to the competition between the nuclear capture and the $\pi \rightarrow \mu$ decay of π -mesons. From the equation (13)

$$A_1 = \frac{\int \{ \mu / T_1 + (\partial f / \partial T_1) (\mu / f) \} dE}{\int \mu dE}, \quad (15)$$

in which

$$f = \frac{1}{1 + K_\pi T_1} + \sum_{n=1}^{\infty} (-1)^n (\lambda_\pi / \lambda')^n \frac{1}{n!} \frac{d^n}{dn} \left(\frac{i + K_\mu T_2}{n + 1 + K_\pi T_1} \right). \quad (16)$$

(2) Negative temperature effect in the upper atmosphere: A_2

This effect is, as discussed above, divided into two parts: intensity variation due to the change of the height of μ -meson production and that due to the change of the rate of ionization loss of particles which are corresponding to the first and the second terms in the following equation, respectively.

$$A_2 = \frac{1}{\int \mu dE} \left\{ -K_\mu \log \frac{x_{300}}{\lambda_\pi} + \left(\frac{\Gamma' (1 + K_\mu T_2)}{\Gamma (1 + K_\mu T_2)} K_\mu + \frac{1}{f} \frac{\partial f}{\partial T_2} \right) \right\} \mu dE$$

$$+ \int \frac{1}{\mu dE} \left[-K_\mu \log \frac{E}{E - \beta(x_{300} - x_1)} \right] \mu dE, \quad (17)$$

where $\Gamma''(x) = \partial \Gamma(x) / \partial x$.

(3) Negative temperature effect in the lower atmosphere: A_3

$$A_3 = \frac{1}{\int \mu dE} \left[-K_\mu \log \left(\frac{x_0}{x_{300}} \right) \right] \mu dE + \int \frac{1}{\mu dE} \left[-K_\mu \log \frac{E - \beta(x_{300} - x_1)}{E - \beta(x_0 - x_1)} \right] \mu dE. \quad (18)$$

(4) Barometric pressure effect: A_p

This effect grows with varying x_0 , and we obtain

$$A_p = \frac{1}{\int_{E_{\min} + \beta(x_0 - x_1)}^{\infty} \mu dE} \left[-K_\mu T_0 \left(\frac{1}{p} + \frac{\beta}{\{E - \beta(x_0 - x_1)\} \cos 80^\circ} \right) \right] \mu dE \\ - \frac{1}{\int_{E_{\min} + \beta(x_0 - x_1)}^{\infty} \mu dE} \lim_{\delta p \rightarrow 0} \left[\int_{E_{\min} + \beta(x_0 - x_1)}^{E_{\min} + \beta\{(x_0 + \delta x_0) - x_1\}} \frac{\mu dE}{\partial p} \right]. \quad (19)$$

The numerical values of these coefficients are presented as follows,

A_1	+0.0568 %/c		
A_2	-0.305 %/c	first term	-0.29 %/°c
		second term	-0.015 %/°c
A_3	-0.157 %/c	first term	-0.10 %/°c
		second term	-0.057 %/°c
A_p	-0.072 %/mb		

§ 5. Discussion

(A) Zenithal dependence and absolute intensity

In Fig. 3, the theoretical curve shows good accordance with the observed data. The normalization is made on the point (65° , $211 \text{ g/cm}^2 \text{ Fe}$). To examine the consistency of this normalization, some estimation of an absolute intensity of oblique μ -mesons will be necessary. For this purpose, we shall determine the value of I_0 in the equation (7). But unfortunately, the complicated aspect of nuclear cascade makes difficult to estimate I_0 from the knowledge of the primary flux at the top of the atmosphere. In this section, therefore, I_0 is estimated by the use of the experimental data on the energy spectrum of high energy μ -mesons at sea level.

We define the function $J(E_1, \theta)$ to be the sea level intensity at zenith distance θ of μ -mesons with energy larger than E_1 . Then, from the equation (7),

$$J(E_1, \theta) = I_0 \int_{E_2}^{\infty} F(E, \theta) dE, \quad (20)$$

where $F(E, \theta) = 1/I_0 \cdot \mu(E, \theta)$ and $E_2 = E_1 + \beta(x_0 - \lambda_p)$. The term $\beta(x_0 - \lambda_p)$ means the energy loss of μ -mesons from the points of production to the sea level. I_0 can be determined by the equation (20) when $J(E_1, \theta)$ is known from the experiment.

In the following, we shall estimate, for example, the intensity of μ -mesons observed by the telescope with 523g/cm² Fe at the zenith distance 75°. (The observed intensity under this condition fits the theoretical curve very well as shown in Fig. 3.) $J(E_1, \theta)$ is determined by the result of the measurement by Caro et al.¹²⁾ on the energy spectrum of vertical μ -mesons. Various values are to be obtained corresponding to the different values of E_2 . As a result of calculation, however, it is shown that the value of I_0 thus obtained is practically independent of E_2 in the energy interval from 10 Bev to 40 Bev and equals to about 0.36. (As shown in Fig. 1, most of the particles observed under the condition (75°, 523g/cm² Fe) have energy in this interval.) This fact means that the form of the energy spectrum of the observed vertical μ -mesons is similar to that deduced theoretically, and indicates the validity of our assumption $E^{-2.8}$, on the production spectrum of π -mesons. The absolute intensity of μ -mesons derived with this value of I_0 is 3.1×10^{-3} /cm². sterad. sec, which is in good agreement with the observed value 2.8×10^{-3} /cm². sterad. sec. This accordance seems to suggest that our present model on the cosmic ray is consistent.

(B) Temperature effect

In the preceding section, it is pointed out that the negative temperature effect of the μ -meson intensity is caused not only by the change of the production height of μ -mesons but also by that of the density distribution of the air. In the equations (17) and (18), the first terms correspond to the former and the second terms correspond to the latter. The necessity to take account of the second factor and to divide the atmosphere into three regions is clearly shown in the values of the correlation coefficients A_u and A_l of the negative temperature effect in the upper and the lower atmosphere. For A_u , the effect of the second term is about 5% of the total, while it is as large as 30% for A_l .

As an example, these values of the correlation coefficients of the atmospheric effect can be compared with the results obtained by Duperier. But it should be noticed that the equation (11) is not available because it contains an assumption that the μ -meson production is negligible under 300 milibar. The values of correlation coefficients of the temperature effects derived by the equation (7) instead of the equation (12) are shown in Table 1 with corresponding experimental results by Duperier.

Table 1

	positive temperature effect (%/°C)	negative temperature effect (%/Km)
Theoretical	+0.045	-4.07
Observed by Duperier	+0.124±0.32	-3.48±0.55
"	+0.228±0.045	-4.08±0.52

A good agreement is seen for the negative temperature effect, while the appreciably large value compared with the theoretical one is given by the experiment for the positive temperature effect, the reason of which is left to be searched for from some other viewpoint.

It has been shown in section 3 that the zenithal dependence of the μ -meson intensity is in good agreement with the result measured by Sekido et al. The observed intensity is also consistent with that estimated from the theoretical consideration as shown in section 5(A). From the result on the anisotropy of cosmic rays over the celestial sphere, Sekido et al. estimated that the average momentum of the primary cosmic rays which produced the μ -mesons observed by their counter telescope is about 2 Störmers (240 Bev/c). According to the theory, the energy distribution of the μ -mesons observed by this telescope is shown in Fig. 1. When the zenith distance of the telescope is 80° , the μ -meson energy corresponds to 21 Bev at the production layer. Although the strict determination of the energy of the primary cosmic rays which are most effective for the production of such μ -mesons is difficult, the estimation of Sekido et al. is fairly consistent with the theory.

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Theory of Mass Reversal in the Quantized Field Theory

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Detailed discussions of mass reversal in the quantized field theory are given. The construction of the mass reversal operator M is possible and the explicit form is given. It is shown that the mass reversal operation does not imply the hypothetical reversal of the sign of the intrinsic mass of the elementary particle, but the change of the direction of the intrinsic spin relative to the momentum for the spinor field. One result of this operation on the interaction Hamiltonian is to forbid the mixing of the couplings with and without derivatives for the pion-nucleon interaction. It seems natural to take the mass reversal as a new fundamental invariance (symmetry) principle other than those hitherto investigated.

§ 1. Introduction

Several attempts have been suggested to explain the seemingly paradoxical facts between the copious productions of the new particles and their metastabilities by introducing the new quantum numbers as many as required. As the other approach to this problem, we may try to discover the physical laws from all the possible invariance principles in the framework of the present field theory and then determine which physical laws, accordingly which invariance principles, should be discarded and which ones should be reserved or generalized. It will be worthwhile for this approach to deduce the selection rules, which restrict the interactions among the new particles, from the investigation of the symmetry properties of physical laws with respect to mirage, charge-conjugation, etc., or to find out the superselection rules, as Watanabe¹⁾ has summarized in his review articles. In this respect the mass reversal recently proposed by Tiomno²⁾ in c-number theory will be investigated in detail in the framework of the quantized field theory.

In Tiomno's paper, mass reversal is supposed to reverse the sign of the explicit masses and the masses of boson field operators and to interchange the large components with the small ones for the spinor field. As the spinor field has no classical analogue and further the mass reversal does not hold in the classical mechanics, we cannot determine whether his opinion is allowed in c-number theory or not. However, if we attempt to pass from c-number theory into q-number theory with Tiomno's interpretation of mass reversal, there appear some ambiguities concerning the mass reversal for the spinor field. This situation makes us to take another interpretation of mass reversal for spinor field than Tiomno's.

The charge conjugation is usually understood as the exchange of two real "charge

conjugate" states of the particle. However, the mass reversal could not be considered as the real exchange of two "mass reversed" states, because the negative mass state of the particle is non-existent. Thus we take it as the transformation from one description of the system to another one. Moreover, in q -number theory, mass has the same role as the arguments (xyz) of the field variables in that they are nothing but the parameters. For these reasons, we may assume that under mass reversal the spinor field undergoes the similar transformation to the mirage transformation. In other words, from the analogy of mirage, we say that two state functions $\mathcal{U}(\kappa)$ and $\mathcal{U}_M(-\kappa)$ represent mutually reversed states if the expectation values of the physical quantities in the states $\mathcal{U}(\kappa)$ and $\mathcal{U}_M(-\kappa)$ are equal. It will be shown that this mass reversibility (invariance under mass reversal) can be stated as follows: The probability of finding the physical system which was in the state $\mathcal{U}_i(\kappa)$ at the initial instant and in the state $\mathcal{U}_f(\kappa)$ at the final instant is equal to the probability of finding the system which was in the reversed state $\mathcal{U}_{iM}(-\kappa)$ at the initial instant and in the reversed state $\mathcal{U}_{fM}(-\kappa)$ at the final instant. It should be noted again that our concept of mass reversibility does not mean any hypothetical inversion of the intrinsic mass.

In Sec. 2, mass reversibility is formally defined in q -number theory and the unitary operator M for mass reversal is introduced. In Sec. 3, we demonstrate the existence of the unitary operator M for the various fields, completing the proof of the invariance of the theory under mass reversal. The form of the operator M is given explicitly in the operational expression. It will become clear through the discussions in Secs. 2 and 3 that the operation of the mass reversal for the particles with non-zero mass does not imply the reversal of the sign of the intrinsic mass, but the change of the direction of the intrinsic spin relative to the momentum for spinor field. In Sec. 4, we shall show, with the similar method used by Umezawa et al.¹⁾ in the study of time reversal, how many definitions for the fundamental mass reversal are possible. For the most simple interaction, we can take as the mass reversal two types: one is the proper mass reversal called M_1 in the previous note¹⁾ and the other (M_2) is the product of M_1 and charge-conjugation.

In Sec. 5, we deal with the various types of nucleon-pion interaction and nucleon-lepton interaction, taking into account the fact that in principle it will be permissible to reverse the sign of masses of the specified particles only among all the particles concerned. For the nucleon-pion interaction, the mixture of couplings with and without derivatives is generally forbidden, and for the nucleon-lepton interaction, Tiomno's result which states that either a linear combination of (V) and (PV) interaction or a linear combination of (S), (PS) and (T) interactions is allowed, is shown to be true only when two masses of the same families are reversed.

In the final section, we shall have the further discussions on the meaning of the mass reversal.

§ 2. Q -number theory of mass reversal

Throughout this paper we use the same notations as those in the review articles of Watanabe in order to compare the mass reversal with the conventional inversions and to

recognize the differences between them. We take the interaction representation in this paper, in which the time-development of the state function $\Psi(t)$ is governed by the interaction Hamiltonian $H(t, \kappa)$ and expressed by

$$\Psi^*(t_2) = U(t_2, t_1; \kappa) \Psi^*(t_1), \quad (2.1)$$

with

$$\left. \begin{aligned} dU(t_2, t_1; \kappa)/dt_2 &= -iH(t_2, \kappa)U(t_2, t_1; \kappa), \\ dU(t_2, t_1; \kappa)/dt_1 &= iU(t_2, t_1; \kappa)H(t_1, \kappa), \end{aligned} \right\} \quad (2.2)$$

and

$$U(t_1, t_1; \kappa) = 1, \quad U^{-1}(t_2, t_1; \kappa) = U^+(t_2, t_1; \kappa) = U(t_1, t_2; \kappa), \quad (2.3)$$

while the time-development of the physical quantities $Q(x, \kappa)$ is governed by the free Hamiltonian H_0 and expressed by

$$Q(t_2, \kappa) = U_0^{-1}(t_2, t_1; \kappa) Q(t_1, \kappa) U_0(t_2, t_1; \kappa), \quad (2.4)$$

with

$$\begin{aligned} dU_0(t_2, t_1; \kappa)/dt_2 &= -iH_0(t_2, \kappa)U_0(t_2, t_1; \kappa), \\ U_0(t_1, t_1; \kappa) &= 1, \text{ etc.}, \end{aligned} \quad (2.5)$$

the symbol κ representing the mass of the particle.

Now we shall proceed to define the "mass-reversed states" in the quantum-mechanical language. Since, in the quantized field theory, the arguments (xyz) of the field variables play simply a role of the parameters attached to particles and in this connection mass can be assumed to play the same role as these parameters, we can make the analysis of the mass reversibility in a similar way to that of the space inversion. Let us call $\Psi_M(t, -\kappa)$ the reversed state of $\Psi^*(t, \kappa)$ on the analogy of that of mirage, if the expectation values of physical quantity $Q(x, \kappa)$ for $\Psi^*(t, \kappa)$ and those for $\Psi_M(t_1, -\kappa)$ are related by

$$(\Psi_M(t, -\kappa), Q(x, -\kappa) \Psi_M(t, -\kappa)) = \rho_M(\Psi(t, \kappa), Q(x, \kappa) \Psi(t, \kappa)), \quad (2.6)$$

where ρ_M is the sign function for $Q(x, \kappa)$ under mass reversal and $Q(x, -\kappa)$ should be understood as obtained from $Q(x, \kappa)$ just replacing κ by $-\kappa$, whose precise form will be shown in Appendix, and the argument κ of the state vector is omitted below where no confusion will result.

We can now define mass reversibility as follows: If $\Psi_M(t)$ defined in eq. (2.6) is a solution of eqs. (2.1) and (2.2) on condition that $\Psi^*(t)$ is a solution of eqs. (2.1) and (2.2), then we speak of mass reversibility.

In order to verify the invariance in question, let us assume that a time-independent, unitary operator M is introduced by

$$Q(x, -\kappa) = \rho_M(MQ(x, \kappa)M^{-1}), \quad (2.7)$$

where M is supposed to operate on the field variables involved in $Q(x, \kappa)$. It is now easy to show that if M exists, we can build $\Psi_M(t)$ defined in eq. (2.6) from a given

$\Psi(t)$ as follows :

$$\Psi_M(t) = e^{i\alpha} M \Psi(t), \quad (2.8)$$

where $e^{i\alpha}$ is the arbitrary phase factor, but in the following discussions we assume it to be included in M . It will be shown in Sec. 4 that the operation of the mass reversal transforms the state vector $\Psi(t)$ into the reversed state $\Psi_M(t) = M\Psi(t)$, but not into the state $\Psi^*(t)M$.

To establish the reversibility, we must verify that $\Psi_M(t)$ given by eq. (2.8) is the solution of eqs. (2.1) and (2.2), when $\Psi(t)$ is a solution. Take the two transformation functions $U(t, 0; \kappa)$ and $U(0, -t; \kappa)$ which are given from eq. (2.2) by

$$\left. \begin{aligned} dU(t, 0; \kappa)/dt &= -iH(t, \kappa)U(t, 0; \kappa), \\ dU(0, -t; \kappa)/d(-t) &= +iU(0, -t; \kappa)H(-t, \kappa), \end{aligned} \right\} \quad (2.9)$$

and renaming κ as $-\kappa$ in eq. (2.9), we have

$$\left. \begin{aligned} dU(t, 0; -\kappa)/dt &= -iH(t, -\kappa)U(t, 0; -\kappa), \\ dU(0, -t; -\kappa)/d(-t) &= +iU(0, -t; -\kappa)H(-t, -\kappa), \end{aligned} \right\} \quad (2.10)$$

with

$$U(0, 0; \pm\kappa) = 1.$$

Applying M to eq. (2.9), we obtain

$$\left. \begin{aligned} d\{MU(t, 0; \kappa)M^{-1}\}/dt &= -iH(t, -\kappa)MU(t, 0; \kappa)M^{-1}, \\ d\{MU(0, -t; \kappa)M^{-1}\}/d(-t) &= +iMU(0, -t; \kappa)M^{-1}H(-t, -\kappa), \end{aligned} \right\} \quad (2.11)$$

with

$$MU(0, 0; \kappa)M^{-1} = 1,$$

in virtue of the assumption of the time-independence of M and $\rho_M = +1$ for the energy operator in eq. (2.7):

$$H(t, -\kappa) = +MH(t, \kappa)M^{-1}. \quad (2.12)$$

Comparing eq. (2.11) with eq. (2.10), we obtain

$$\left. \begin{aligned} MU(t, 0; \kappa)M^{-1} &= U(t, 0; -\kappa), \\ MU(0, -t; \kappa)M^{-1} &= U(0, -t; -\kappa). \end{aligned} \right\} \quad (2.13)$$

Combining the two equations of (2.13), we get for $U(t, -t; -\kappa) = U(t, 0; -\kappa) \times U(0, -t; -\kappa)$

$$U(t, -t; -\kappa) = MU(t, -t; \kappa)M^{-1}. \quad (2.14)$$

With this relation, we can easily verify that if

$$\Psi(t) = U(t, -t; \kappa) \Psi(-t), \quad (2.15)$$

then

$$\Psi_M(t) = U(t, -t; -\kappa) \Psi_M(-t). \quad (2.16)$$

Thus it has been shown that if M exists to satisfy eq. (2.7), mass reversibility is guaranteed. We shall show in Sec. 3 that such an M in fact exists. Finally, we shall give the proof that if the reversion operator M defined by eq. (2.7) exists, it automatically satisfies the requirement that it should be time-independent.

Taking a transformation function $U_0(t, 0; \kappa)$, $Q(t, \kappa)$ can be expressed from eq. (2.4)

$$Q(t, \kappa) = U_0^{-1}(t, 0; \kappa) Q(0, \kappa) U_0(t, 0; \kappa) \quad (2.17)$$

and here renaming κ as $-\kappa$, we have

$$Q(t, -\kappa) = U_0^{-1}(t, 0; -\kappa) Q(0, -\kappa) U_0(t, 0; -\kappa). \quad (2.18)$$

Then the defining relation for $M(t)$ will take the form, in virtue of eq. (2.7),

$$\begin{aligned} Q(0, -\kappa) &= \rho_M U_0(t, 0; -\kappa) M(t) U_0^{-1}(t, 0; \kappa) Q(0, \kappa) \\ &\times U_0(t, 0; \kappa) M^{-1}(t) U_0^{-1}(t, 0; -\kappa) \end{aligned} \quad (2.19)$$

which reduces to the relation

$$M(t) = U_0(t, 0; -\kappa) M(0) U_0^{-1}(t, 0; \kappa). \quad (2.20)$$

Differentiating eq. (2.20) with respect to t , it follows, on account of eqs. (2.12) and (2.5) and also of the time-independence of $M(0)$,

$$dM(t)/dt = i(M(t) H_0(t, \kappa) - H_0(t, -\kappa) M(t)) \equiv 0$$

which shows that M is time-independent.

§ 3. Mass reversion operator

In this section we shall consider mainly the transformation of spinor field $\psi(x, \kappa)$ because the behavior of the spinor field under mass reversal is expected to be rather different from that under mirage, etc. Under Lorentz transformation denoted by \mathfrak{S} the field quantity F will be transformed as

$$F'(x') = \mathfrak{S}F(x) = \mathfrak{S}F(\mathfrak{S}^{-1}x'). \quad (3.1)$$

As a consequence of our assumption, we can apply this relation to the mass reversal replacing x and \mathfrak{S} in eq. (3.1) by mass κ and the symbol of the transformation of mass reversal m . Therefore we have for the free spinor field

$$\begin{aligned} \psi'(x, \kappa') &= m\psi(x, \kappa) \equiv \lambda \gamma_5 \psi(x, \kappa) \\ &= m\psi(x, m^{-1}\kappa') \end{aligned} \quad (3.2)$$

with $\lambda = \pm 1, \pm i$.

Now we shall consider eq. (2.7) to deduce the relations between the operator M and the field variables. To avoid confusion about the meaning of the field variables $\psi(x, \kappa')$, we follow the deduction given by Watanabe, where the field equation is not explicitly taken into account. (See Appendix.) Then we get as the effect of mass reversal on a spinor

field $\psi(x, \kappa)$ and on a boson field $\phi(x, \kappa)$ respectively,

$$M\psi(x, \kappa)M^{-1} = e^{i\tau}\gamma_5\psi(x, -\kappa), \quad (3.3)$$

$$M\phi(x, \kappa)M^{-1} = \pm\phi(x, -\kappa), \quad (3.4)$$

$e^{i\tau}$ being the arbitrary phase factor.

We shall proceed to look for M satisfying eqs. (3.3) and (3.4).

We can expand any solution of Dirac wave equation for spinor field as

$$\begin{aligned} \psi(x, \kappa) &= (V)^{-1/2} \sum_{\mathbf{k}} \sum_{r=1,2} \{a_r(\mathbf{k}) u_r(\mathbf{k}, \kappa) e^{i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} + b_r^*(\mathbf{k}) v_r(\mathbf{k}, \kappa) e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0 t)}\}, \\ \psi^*(x, \kappa) &= (V)^{-1/2} \sum_{\mathbf{k}} \sum_{r=1,2} \{a_r^*(\mathbf{k}) u_r^*(\mathbf{k}, \kappa) e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} + b_r(\mathbf{k}) v_r^*(\mathbf{k}, \kappa) e^{i(\mathbf{k}\cdot\mathbf{x}-k_0 t)}\}, \end{aligned} \quad (3.5)$$

where $u(\mathbf{k}, \kappa)$ and $v(\mathbf{k}, \kappa)$ are the usual Dirac eigen-spinors, and $v_1(\mathbf{k}, \kappa) = -\gamma_5 u_2(\mathbf{k}, \kappa)$ and $v_2(\mathbf{k}, \kappa) = -\gamma_5 u_1(\mathbf{k}, \kappa)$. We can derive the following relations among Dirac eigen-spinors by a straightforward calculation

$$\begin{aligned} \sqrt{k^2} \begin{pmatrix} u_1(\mathbf{k}, -\kappa) \\ u_2(\mathbf{k}, -\kappa) \end{pmatrix} &= (\boldsymbol{\sigma} \cdot \mathbf{k}) \begin{pmatrix} v_2(\mathbf{k}, \kappa) \\ v_1(\mathbf{k}, \kappa) \end{pmatrix}, \\ \sqrt{k^2} \begin{pmatrix} v_1(\mathbf{k}, -\kappa) \\ v_2(\mathbf{k}, -\kappa) \end{pmatrix} &= (\boldsymbol{\sigma} \cdot \mathbf{k}) \begin{pmatrix} u_2(\mathbf{k}, \kappa) \\ u_1(\mathbf{k}, \kappa) \end{pmatrix}, \end{aligned} \quad (3.6)$$

where $\boldsymbol{\sigma}$ is Pauli's spin matrix vector and the factor $\sqrt{k^2}$ is resulted from the consideration of the normalization of eigen-spinors $u(\mathbf{k}, \kappa)$, $u(\mathbf{k}, -\kappa)$, $v(\mathbf{k}, \kappa)$, and $v(\mathbf{k}, -\kappa)$. Since the creation and annihilation operators $a^*(\mathbf{k})$, $b^*(\mathbf{k})$, ... are usually considered to be independent of the sign of the parameter κ , we have for $\psi(x, -\kappa)$ in virtue of eq. (3.5)

$$\begin{aligned} \psi(x, -\kappa) &= (V)^{-1/2} \sum_{\mathbf{k}} \sum_{r=1,2} \{a_r(\mathbf{k}) u_r(\mathbf{k}, -\kappa) e^{i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} + b_r^*(\mathbf{k}) v_r(\mathbf{k}, -\kappa) e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0 t)}\}, \\ \psi^*(x, -\kappa) &= (V)^{-1/2} \sum_{\mathbf{k}} \sum_{r=1,2} \{a_r^*(\mathbf{k}) u_r^*(\mathbf{k}, -\kappa) e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0 t)} + b_r(\mathbf{k}) v_r^*(\mathbf{k}, -\kappa) e^{i(\mathbf{k}\cdot\mathbf{x}-k_0 t)}\}. \end{aligned} \quad (3.7)$$

Substituting the expressions (3.5) and (3.7) in eq. (3.3), we obtain in virtue of (3.6)

$$M \begin{pmatrix} a_1(\mathbf{k}) \\ a_2(\mathbf{k}) \end{pmatrix} M^{-1} = -e^{i\tau} \frac{(\boldsymbol{\sigma} \cdot \mathbf{k})}{\sqrt{k^2}} \begin{pmatrix} a_1(\mathbf{k}) \\ a_2(\mathbf{k}) \end{pmatrix}, \quad M \begin{pmatrix} b_1(\mathbf{k}) \\ b_2(\mathbf{k}) \end{pmatrix} M^{-1} = -e^{-i\tau} \frac{(\boldsymbol{\sigma} \cdot \mathbf{k})}{\sqrt{k^2}} \begin{pmatrix} b_1(\mathbf{k}) \\ b_2(\mathbf{k}) \end{pmatrix}. \quad (3.8)$$

From these relations we can also verify directly that M is time-independent,

$$dM/dt = i[M, H_0] \quad (3.9)$$

with

$$H_0 = \sum_{\mathbf{k}} \sum_{r=1,2} \sqrt{k^2 + \kappa^2} (a_r^*(\mathbf{k}) a_r(\mathbf{k}) + b_r^*(\mathbf{k}) b_r(\mathbf{k})),$$

which was already proved formally.

Finally the mass reversal operator M satisfying eq. (3.8) can be written as

with

$$M = M' M_0$$

$$M_0 = \prod_k \prod_{r=1,2} (-e^{-i\tau}) N_r^+ (\mathbf{k}) (-e^{i\tau}) N_r^- (\mathbf{k}) \quad (3 \cdot 10)$$

and

$$\begin{aligned} M' = & \prod_k [1 - (1 - \alpha) N_1^+ (\mathbf{k}) - (1 + \alpha) N_2^+ (\mathbf{k}) + \beta a_1^* (\mathbf{k}) a_2 (\mathbf{k}) + \beta^* a_2^* (\mathbf{k}) a_1 (\mathbf{k})] \\ & \times [1 - (1 - \alpha) N_1^- (\mathbf{k}) - (1 + \alpha) N_2^- (\mathbf{k}) + \beta b_1^* (\mathbf{k}) b_2 (\mathbf{k}) + \beta^* b_2^* (\mathbf{k}) b_1 (\mathbf{k})], \end{aligned} \quad (3 \cdot 11)$$

where

$$N_r^+ (\mathbf{k}) = a_r^* (\mathbf{k}) a_r (\mathbf{k}), \quad N_r^- (\mathbf{k}) = b_r^* (\mathbf{k}) b_r (\mathbf{k}), \quad \alpha = \frac{k_x}{\sqrt{\mathbf{k}^2}}, \quad \beta = \frac{k_x - ik_y}{\sqrt{\mathbf{k}^2}}.$$

The arbitrary sign of the entire M thus defined is so chosen that

$$M \Psi_{vac} = \Psi_{vac} \quad (3 \cdot 12)$$

where Ψ_{vac} denotes the vacuum state vector.

Especially, if the direction of the intrinsic spin vector is parallel to that of the momentum vector of the particles, eq. (3.8) reduces to

$$M \begin{pmatrix} a_1 (\mathbf{k}) \\ a_2 (\mathbf{k}) \end{pmatrix} M^{-1} = -e^{i\tau} \begin{pmatrix} a_1 (\mathbf{k}) \\ a_2 (\mathbf{k}) \end{pmatrix}, \quad M \begin{pmatrix} b_1 (\mathbf{k}) \\ b_2 (\mathbf{k}) \end{pmatrix} M^{-1} = -e^{-i\tau} \begin{pmatrix} b_1 (\mathbf{k}) \\ b_2 (\mathbf{k}) \end{pmatrix}, \quad (3 \cdot 13)$$

which shows that M has the form by (3.12)

$$M = \prod_k \prod_r (-e^{-i\tau}) N_r^+ (\mathbf{k}) (-e^{+i\tau}) N_r^- (\mathbf{k}). \quad (3 \cdot 14)$$

As regards the charged boson field, similar treatment can be applied. Expressing the ϕ -field by its Fourier expression,

$$\begin{aligned} \phi(x, \kappa) &= \sum_k (2\omega_k V)^{-1/2} [a(\mathbf{k}) e^{i(kx - k_0 t)} + b^*(\mathbf{k}) e^{-i(kx - k_0 t)}], \\ \phi^*(x, \kappa) &= \sum_k (2\omega_k V)^{-1/2} [a^*(\mathbf{k}) e^{-i(kx - k_0 t)} + b(\mathbf{k}) e^{i(kx - k_0 t)}], \end{aligned} \quad (3 \cdot 15)$$

we can deduce, from eq. (3.4),

$$Ma(\mathbf{k})M^{-1} = \pm a(\mathbf{k}), \quad Mb(\mathbf{k})M^{-1} = \pm b(\mathbf{k}). \quad (3 \cdot 16)$$

Now the operator M satisfying the condition (3.12) can be expressed in the operator form as follows:

$$M = (\pm 1)^\theta, \quad \theta = \sum_k \{N_+(\mathbf{k}) + N_-(\mathbf{k})\}, \quad (3 \cdot 17)$$

with

$$N_+(\mathbf{k}) = a^*(\mathbf{k}) a(\mathbf{k}), \quad N_-(\mathbf{k}) = b^*(\mathbf{k}) b(\mathbf{k}). \quad (3 \cdot 18)$$

§ 4. Types of mass reversal

In this section we shall attempt to find how many types of transformation are pos

sible on the condition that the theory should be invariant under the mass reversal. For this purpose we can conveniently take the formulation similar to the one used by Umezawa et al. in the study of time-reversal.

In the canonical formalism, the fundamental equations are given by

$$-\partial_\mu u_\alpha(x, \kappa) = [u_\alpha(x, \kappa), T_\mu^0[u(x, \kappa)]], \quad (4.1)$$

$$[u_\alpha^+(x, \kappa), u_\beta(x', \kappa)]_\pm = iR_{\alpha\beta}(\partial, \kappa) \mathcal{A}(x-x', \kappa), \quad (4.2)$$

where T_μ^0 represents the free part of the total energy-momentum four vector and the symbol $R(\partial, \kappa)$ denotes a differential operator appropriately defined. Now, if these fundamental equations are proved invariant under mass reversal, we speak of mass reversibility. Then the restriction on the interaction Hamiltonian is derived from the requirement of the invariance of eq. (2.2).

In order to find out all the possible transformations of mass reversal, we now proceed to consider the following most extensive transformation of mass reversal;

$$\begin{aligned} \mathcal{V}(t) &\rightarrow \mathcal{V}_M(t) = \mathcal{V}^*(t)M + M\mathcal{V}(t), \\ u_\alpha(x, \kappa) &\rightarrow u'_\alpha(x, \kappa') = (\mathfrak{M})_{\alpha\beta} u_\beta^T(x, \kappa), \end{aligned} \quad (4.3)$$

where the unitary operators M and M' are supposed to operate on the state vector, and the suffices α, β, \dots of the unitary operator \mathfrak{M} should be considered to extend over both the field variables $(u_\alpha, u_\beta, \dots)$ and their hermite conjugates $(u_\alpha^+, u_\beta^+, \dots)$. As the phase of the state vector is not observable, either M or M' must vanish.

Incidentally we may consider the case $M' = 0$. The matrix element of any physical quantity $Q(x, \kappa)$ is rewritten as follows:

$$(\mathcal{V}(t_2), Q[u(x, \kappa)]\mathcal{V}(t_1)) = (\mathcal{V}_M(t_2), MQ[\mathfrak{M}^{-1}u'(x, \kappa')]M^{-1}\mathcal{V}_M(t_1)). \quad (4.4)$$

Now, let us assume that the field variable is transformed linearly;

$$Mu'(x, \kappa')M^{-1} = Nu'(x, \kappa') \quad (4.5)$$

or

$$\mathfrak{M}^{-1}Mu'(x, \kappa')M^{-1} = \mathfrak{M}^{-1}Nu'(x, \kappa'), \quad (4.6)$$

where the suffices of the operator N should be understood, like those of the operator \mathfrak{M} , to be extended over all the field variables $(u_\alpha, u_\beta, \dots, u_\alpha^+, u_\beta^+, \dots)$. From eqs. (4.4) and (4.5), we can conclude that the operator \mathfrak{M} appears always in the form $\mathfrak{M}^{-1}N$, which enables us to take \mathfrak{M} as that in eq. (3.2) and leave N arbitrary.

Applying the mass reversal transformation to eq. (4.1), we have

$$-\partial_\mu u'_\alpha(x, \kappa') = [u'_\alpha(x, \kappa'), T_\mu^0[Nu'(x, \kappa')]]. \quad (4.7)$$

On the other hand eq. (4.1) takes the form:

$$-\partial_\mu u_\alpha(x, \kappa) = [u_\alpha(x, \kappa), T_\mu^0[u'(x, \kappa')]], \quad (4.8)$$

since $u'_\alpha(x, \kappa')$ are the field variables in the new coordinate system. In order that eq. (4.1) is invariant, it is necessary that eq. (4.7) should be identical with eq. (4.8).

This gives the following relation

$$u'^+(x, \kappa') N^+ \mathcal{Q}^\mu(\partial, \kappa') N u'(x, \kappa') = u'^+(x, \kappa') \mathcal{Q}^\mu(\partial, \kappa') u'(x, \kappa'), \quad (4.9)$$

where we have used the fact that T_μ^0 can be expressed generally in the form

$$T_\mu^0[u(x, \kappa)] = \int u_\alpha^+(x, \kappa) \mathcal{Q}_{\alpha\beta}^\mu(\partial, \kappa) u_\beta(x, \kappa) d^3x, \quad (4.10)$$

with $\mathcal{Q}^\mu(\partial)$ denoting the differential operator.

Now for further discussions we separate the solutions of eq. (4.9) into two types M_1 and M_2 .

M_1 : Without any assumption concerning the commutation relations between field variables, we can derive the relation

$$N^+ \mathcal{Q}^\mu(\partial, \kappa') N = \mathcal{Q}^\mu(\partial, \kappa'), \quad (4.11)$$

from which a unique solution of (4.11) is shown to be $+1$ apart from the constant factor. The transformation which belongs to M_1 is the proper mass reversal transformation.

M_2 : Assuming the commutation and anti-commutation relations for boson field and spinor field variables respectively, we can reduce eq. (4.9) to

$$C^{-1} \mathcal{Q}^{\mu T}(\partial, \kappa') C = \sigma \mathcal{Q}^\mu(\partial, \kappa'), \quad (4.12)$$

where matrix C is supposed to be defined by

$$N u'(x, \kappa') = u'^+(x, \kappa') C^{-1} \quad (4.13)$$

and σ takes the values $+1$ or -1 according as the fields are boson or fermion fields. For the boson field, $C=1$ is a solution of eq. (4.12). For the fermion field, eq. (4.12) reduces in virtue of $\mathcal{Q}^i = \gamma_4(\gamma_k \partial_k + \kappa)$ to

$$C^{-1} \gamma_4^T C = -\gamma_4, \quad C^{-1} \gamma_k^T C = \gamma_k \quad (k=1, 2, 3), \quad (4.14)$$

which show that C is a well-known matrix of charge-conjugation. Thus we can see without proof that the mass reversal operator which belongs to type M_2 is equivalent to the product of the proper mass reversal operator and charge-conjugation operator.

There is no other possibility for mass reversal than M_1 and M_2 , since the field variables before and after the mass reversal cannot give such a correspondence to each other that

$$u(x, \kappa) \sim a u'(x, \kappa') + b u'^+(x, \kappa'), \quad (a \neq 0, b \neq 0). \quad (4.15)$$

The cases $a=0$ and $b=0$ correspond to M_2 and M_1 respectively.

To complete the mass reversibility, we must finally verify that the commutation relation (4.2) is invariant under mass reversal and, further, the free energy operators in the representation corresponding to $\Psi(t)$ and $\Psi_M(t)$ have the same sign. But we shall here only give the useful relations to this proof, for our aim is to look for the possibility for mass reversal.

For the differential operator $A(\partial)$ defined by

$$A_{\alpha\beta}(\partial, \kappa) R_{\beta\tau}(\partial, \kappa) = (\square - \kappa^2) \delta_{\alpha\tau}, \quad (4.16)$$

we can prove the relations :

$$\left. \begin{aligned} N^{-1}A(\partial, \kappa)N &= A(\partial, \kappa) && \text{for } M_1, \\ C^{-1}A^T(\partial, \kappa)A(x-x')C &= A(\partial, \kappa')A(x-x') && \text{for } M_2, \\ m^{-1}A(\partial, \kappa)m &= A(\partial, \kappa'), \end{aligned} \right\} \quad (4 \cdot 17)$$

and for the differential operator $R(\partial, \kappa)$ we can prove the relations :

$$\left. \begin{aligned} N^{-1}R(\partial, \kappa)N &= R(\partial, \kappa) && \text{for } M_1, \\ C^{-1}R^T(\partial, \kappa)A(x-x')C &= R(\partial, \kappa')A(x-x') && \text{for } M_2, \\ m^{-1}R(\partial, \kappa)m &= R(\partial, \kappa'). \end{aligned} \right\} \quad (4 \cdot 18)$$

Once the mass reversibility is established for the most typical system, it is clear that the similar discussions can be applied even to the case when the matrix N involves the unitary matrix of other transformation than the Lorentz transformation. For instance, in the theory of the nucleon and meson system, N may involve the unitary operator in the τ -space. In this way, there may exist various types of the mass reversal, which we shall not discuss further.

Finally we may consider the case $M=0$ in eq. (4.3). In this case, starting from eq. (4.3) with their transpose suffices, we can derive the condition for the invariance of the canonical eq. (4.1) with the same method above mentioned. But the transformation corresponding to $M=0$ in eq. (4.3) should be excluded for the following physical reason. In the direct calculation we find that the free energy operators in the representation corresponding to $\mathcal{U}(t)$ and $\mathcal{U}_M(t)$ have the opposite sign and that the commutation relation remains invariant against this transformation. Therefore we cannot keep the eigenvalue of the energy operator positive definite. Thus we should exclude this case.

§ 5. Restrictions for interactions

In this section we shall investigate to what extent we can restrict the forms of the interaction by the requirement of $\rho_M = +1$ in eq. (2.7), namely,

$$H(x, -\kappa) = MH(x, \kappa)M^{-1}. \quad (5 \cdot 1)$$

Our first step will be to determine the transformation properties of tensorial quantities of the spinor fields under M_1 and M_2 . With the help of eq. (2.7), they are determined by

$$\rho_M = (MQ(x, \kappa)M^{-1})/Q(x, -\kappa), \quad (5 \cdot 2)$$

and are listed in Table.

We now examine the effect of restriction eq. (5.1) on the typical interaction Hamiltonian in the following.

Case A. The boson field couples linearly with fermion source (for example, nucleon-pion interaction.):

$$\bar{f}\psi^a O_a \psi^b \phi^c + \text{hermite conj.} \quad (5 \cdot 3)$$

Table

	$Q(\kappa, \kappa)$	M_1	M_2
(S)	$\bar{\psi}\psi$	—	—
(V)	$\bar{\psi}\gamma_\mu\psi$	+	—
(T)	$\bar{\psi}\gamma_\mu\gamma_\nu\psi$	—	+
(PS)	$\bar{\psi}\gamma_5\psi$	—	—
(PV)	$\bar{\psi}\gamma_5\gamma_\mu\psi$	+	+
(PT)	$\bar{\psi}\gamma_5\gamma_\mu\gamma_\nu\psi$	—	+

It will be obvious from invariance requirements that ψ^a and ψ^b should be transformed simultaneously. When the source density consists of different fields (ψ^a, ψ^b), the coupling constant f may in general be complex, and thus the situations will become somewhat different according to the number of masses whose sign are reversed. Therefore we discuss separately two cases according to the number of masses in question.

(a) If we reverse all the masses of the fields, we find in virtue of eqs. (5.1), (3.3) and (3.4) the following relations between the coupling constant f and the phase factors $\rho_a's$,

$$\pm f \rho_a^* \rho_b \rho_c = f \quad \text{for } M_1, \quad (5.4)$$

where +sign should be taken for (V, PV) and —sign for (S, T, PS, PT). From (5.4) we can conclude that simultaneous couplings with and without derivatives are forbidden for nucleon-pion interaction in the standpoint of M_1 , as the phase factors are arbitrary but fixed. For the transformation M_2 , we get

$$\pm f \rho_a^* \rho_b \rho_c = f^* \quad \text{for } M_2, \quad (5.5)$$

where sign— is for (S, T, PV) and + is for (V, PS, PT). This relation expresses only the conditions which must be satisfied by the phase factors and the coupling constants, and put no restriction upon the mixture of the different types of interaction. However, the relation means that for the mixture of the interaction the ratio f/f' must be imaginary, where f and f' denote the coupling constants of interaction with and without derivatives respectively.

(b) If we perform the mass reversion only for spinor fields, only the standpoint M_1 is allowed by the charge conservation and we have the relations

$$\pm f \rho_a^* \rho_b = f \quad \text{for } M_1, \quad (5.6)$$

where the sign + or — is the same one as in eq. (5.4). Eq. (5.6) implies that we cannot mix both couplings with and without derivatives at the same time. Thus for the charged boson field the sources can be mixed only within the group (V, PV) or (S, T, PS, PT) and these results seem to conform with the prescription in five-dimensional theory of Corben.⁵⁾ For the neutral boson field (S, T, PS, PT) couplings are strictly forbidden if $\psi^a = \psi^b$. By this selection rule the specified mass reversal (b) may be supported or rejected according as the nucleon-pion system will be described by pv or ps coupling of ps meson theory.

Case B. The universal Fermi interaction (for example, nucleon-lepton interaction):

$$\sum_{\alpha} f_{\alpha} (\bar{\psi}^{\alpha} O_{\alpha} \psi^{\beta}) (\bar{\psi}^{\beta} O_{\alpha} \psi^{\alpha}) + \sum_{\alpha} f_{\alpha}^{*} (\bar{\psi}^{\beta} O_{\alpha} \psi^{\alpha}) (\bar{\psi}^{\alpha} O_{\alpha} \psi^{\beta}). \quad (5.7)$$

By reversing all of the masses we can obtain the following relations independent of the commutation relations between the two pairs of fermions ;

$$f_{\alpha} = f_{\alpha} \rho_{\alpha}^{*} \rho_b \rho_c^{*} \rho_d \quad \text{for } M_1, \quad (5.8)$$

$$f_{\alpha}^{*} = f_{\alpha} \rho_{\alpha}^{*} \rho_b \rho_c^{*} \rho_d \quad \text{for } M_2. \quad (5.9)$$

As the relative phase is expected to have a detectable physical effect even in the standpoint of M_2 for the same reason as in charge conjugation, from these relations we can say that the ratios $f_{\alpha}/f_{\alpha'}$ must be real for all combinations of α and α' for a mixture of the types in (5.7).

If we reverse the masses of one pair of the same family (for example $(\psi^{\alpha}, \psi^{\beta})$) only, we get,

$$\pm f_{\alpha} = f_{\alpha} \rho_{\alpha}^{*} \rho_b \quad \text{for } M_1. \quad (5.10)$$

This relation leads to the same conclusion for a mixture of the types in (5.7) as given by Tiomno, which expresses that either a linear combination of (V) and (PV) interaction or a linear combination of (S) , (PS) , and (T) interaction is allowed, but in the derivation of this conclusion there are some inadequate points in Tiomno's paper.

Case C. Pseudospinors are allowed to exist in the universal Fermi interaction in the form :

$$\sum_{\alpha} f_{\alpha} (\bar{\psi}^{\alpha} O_{\alpha} \psi^{\beta}) (\bar{\psi}^{\beta} \gamma_5 O_{\alpha} \psi^{\alpha}) - \sum_{\alpha} f_{\alpha}^{*} (\bar{\psi}^{\beta} O_{\alpha} \psi^{\alpha}) (\bar{\psi}^{\alpha} O_{\alpha} \gamma_5 \psi^{\beta}). \quad (5.11)$$

By reversing all the masses of fermions, we get the relations,

$$f_{\alpha} = f_{\alpha} \rho_{\alpha}^{*} \rho_b \rho_c^{*} \rho_d \quad \text{for } M_1, \quad (5.12)$$

$$-f_{\alpha}^{*} = f_{\alpha} \rho_{\alpha}^{*} \rho_b \rho_c^{*} \rho_d \quad \text{for } M_2. \quad (5.13)$$

These relations put no restriction upon the mixture of the different types of the interaction, but express the conditions which must be satisfied by the phase factors and the coupling constants. However, the relation (5.13) means that for a mixture of the types of the interaction in (5.11), the ratio $f_{\alpha}/f_{\alpha'}$ must be imaginary for all combinations of α and α' in the standpoint of M_2 .*

Under the transformation of one pair only, for example the transformation of masses of ψ^{α} and ψ^{β} , we get

$$f_{\alpha} = \pm f_{\alpha} \rho_{\alpha}^{*} \rho_b \quad \text{for } M_1, \quad (5.14)$$

where sign $-$ is for $(S) - (PS)$, $(PS) - (S)$, $(T) - (PT)$ and $(PT) - (T)$, and the sign $+$ is for $(V) - (PV)$ and $(PV) - (V)$. The mixture of the types of interaction is allowed only within each group.

So far we have discussed some possibilities of the mass reversal transformations with regard to the restrictions for the mixture of types of interaction, but it should be noted

*) In the previous note⁴⁾ there was a mistake with regard to the sign of eq. (5.13) and the results in the case M_2 should be corrected as the above.

that what types of the transformation could have the physical realization will be determined only by the comparison with the experimental evidences.

§ 6. Concluding remarks

In the preceding sections we have established the concept of mass reversal formally in the quantized field theory. Now we shall make further remarks in this connection that our mass reversal can be regarded as the fundamental operation. To make clear this point, we shall first investigate whether or not the unitary operator M for mass reversal can be built by the product of some of the mirage, reversion, and charge-conjugation operations, which are listed in the paper of Watanabe. We cannot, however, build the same operator as M by any product of the mirage, reversion, and charge-conjugation operators because the eq. (3·8) which determines the mass reversal operator M involves the direction of the momentum of the spinor field, while that for any product of the three operators does not involve the direction of the momentum. This may not necessarily mean that any product which gives the same selection from the mixture of the interactions as mass reversal does not exist. On this point further investigations will be required.

As is shown by eq. (3·8), if we apply the mass reversion to the state with a free particle having the positive energy and $+1/2 \hbar$ spin, we have a mixed state with free particles whose energy are positive and spin eigenvalue is $+1/2 \hbar$ or $-1/2 \hbar$. This fact shows that mass reversal has nothing to do with the real reversion of the intrinsic mass but with the transformation of the spin, which exhibits a striking contrast to charge-conjugation. This interpretation of mass reversal enables us to consider the mass reversal for the neutrino with zero mass.⁽⁶⁾

In Duffin-Kemmer formalism we shall be able to discuss the mass reversal for the boson field, in the similar way to that for the spinor field, and if we could have the similar relation between the mass reversal and the spin transformation, we might be able to speak of mass reversal for the photon field.

In conclusion the authors wish to thank Prof. K. Sakuma and Dr. S. Ogawa for their valuable discussions.

Appendix

We shall here deduce eqs. (3·3) and (3·4) from eq. (2·7) following the method given by Watanabe. Each physical quantity $Q(x, \kappa)$ has an expression in terms of the field variables $u(x, \kappa)$ and some operator which may depend on κ . Therefore we can write

$$Q(x, \kappa) = Q[u(x, \kappa); \kappa]. \quad (\text{A} \cdot 1)$$

For the "reversed process" we have to take u' at $m\kappa$ and transform the explicit κ in eq. (A·1) into $-\kappa$. Thus we have to compare with eq. (A·1) the following quantity :

$$Q[u'(x, m\kappa); m\kappa]$$

which becomes, in virtue of eq. (3.1)

$$Q[mu(x, \kappa); m\kappa].$$

Shifting the operation m from the parameter κ to the tensorial function u , the sign function ρ_M' which was in question in Section 2 must then be given by

$$Q[mu(x, \kappa); m\kappa] = \rho_M' Q[u(x, \kappa); \kappa], \quad (\text{A} \cdot 2)$$

or renaming $m\kappa$ as κ

$$Q[mu(x, m^{-1}\kappa); \kappa] = \rho_M' Q[u(x, m^{-1}\kappa); m^{-1}\kappa]. \quad (\text{A} \cdot 3)$$

Now let us consider eq. (2.7). The left-hand side of eq. (2.7) means in our notations,

$$Q(x, m\kappa) = Q[u(x, m\kappa); m\kappa],$$

there being the essential difference between our theory and Tiomno's c-number theory in which $u(x, -\kappa)$ is considered to be obtained from $u(x, \kappa)$ by exchanging the large components of $u(x, \kappa)$ into the small components. While the right-hand side of (2.7) means

$$Q[Mu(x, \kappa)M^{-1}; \kappa].$$

Therefore eq. (2.7) is equivalent to

$$Q[Mu(x, \kappa)M^{-1}; \kappa] = \rho_M' Q[u(x, m\kappa); m\kappa]. \quad (\text{A} \cdot 4)$$

Since $\rho_M = \rho_M'$, eq. (A.3) becomes

$$Q[mu(x, m\kappa); \kappa] = \rho_M' Q[u(x, m\kappa); m\kappa], \quad (\text{A} \cdot 5)$$

having the same right-hand side as eq. (A.4). Thus we can reduce the relation:

$$Mu(x, \kappa)M^{-1} = mu(x, m\kappa). \quad (\text{A} \cdot 6)$$

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Note added in proof. In a recent issue (*Prog. Theor. Phys.* **15** (1956) 81), Watanabe has presented a few comments concerning the relation between our mass reversal and the conventional space-time inversions. He has given also a possible interpretation for the mass reversal on the more general standpoint, where the sign change of some other natural constants is taken into account. The authors wish to thank Prof. S. Watanabe who kindly provided a preprint of his work prior to its publication.

Proposal for Experiments for Determination of Beta-Decay Interaction and Theory of Triple Cascade Transition

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Four new kinds of experiments on angular correlation in 3_- , $5_-(\beta \text{ 1st})$ or $6_+(\beta \text{ 2nd})$ $4_+(\gamma_1 \text{ 2})$ $2_+(\gamma_2 \text{ 2})$ 0_+ are proposed so as to determine definitely ST (or VT) as the β -decay interaction. Some other decay schemes also useful for this purpose are discussed.

Various angular correlation functions are given for the successive triple transitions of α , β , γ rays and also of the other kind of particles.

§ 1. Introduction

The type of the interaction for β -decay* seems to have been determined by the experiments on the $e-\nu$ angular correlations of $\text{He}^{6(2)}$ and $\text{Ne}^{19(3)}$. It is shown to be a linear combination of Scalar and Tensor interactions, (abbreviated as ST hereafter). To ascertain this conclusion by another method, we propose four new kinds of experiments. These experiments are the angular correlations**, I, II, III and IV, of a triple cascade $\beta-\gamma_1-\gamma_2$ transition in decay scheme 3_- , $5_-(\beta \text{ 1st})$ or $6_+(\beta \text{ 2nd})$ $4_+(\gamma_1 \text{ 2})$ $2_+(\gamma_2 \text{ 2})$ 0_+ *** as explained in § 2. If, besides these experiments, the measurements of the usual $\beta-\gamma$ angular correlation and of the β -spectrum are performed, the linear combination of β -decay interactions will be determined definitely. Some other decay schemes are also useful for this purpose.

The rapid development of instruments and methods in the field of β - and γ -ray spectroscopy of late years has facilitated to perform the measurements of $\gamma-\gamma$ angular correlation in many nuclei, and anisotropic $\beta-\gamma$ directional correlations have also been found in the elements $\text{Cl}^{38(4)}$, $\text{K}^{42(5)}$, $\text{As}^{76(6)}$, $\text{Rb}^{86(7)}$, $\text{Sb}^{122(8)}$, $\text{Sb}^{124(9)}$, $\text{I}^{126(10)}$, $\text{Tm}^{170(11)}$, $\text{Re}^{186(12)}$. In all these cases the differential correlations have been measured. The measurement of the angular correlation in $\beta-\gamma_1-\gamma_2$ cascade transition is, however, much more difficult than that in the above double cascade transitions, for the counting rate is strongly diminished in $\beta-\gamma_1-\gamma_2$ coincidence. In order to remedy this defect as much as possible, T. Hayashi¹³⁾ in-

*) Recently Konopinski¹⁾ has reviewed compactly the studies on the type of β -decay interactions.

**) In the present paper the angular correlation means only the directional correlation.

***) We denote the decay scheme as j_0 (forbiddenness of β -decay) j_1 (rank of γ_1 -interaction) j_2 (rank of γ_2 -interaction) j_3 , where j_0 , j_1 , j_2 and j_3 are the spins of the initial, the second, the third and the final nuclear states in a triple cascade transition, respectively. The sign (+ or -) of each spin indicates the parity of the relevant nuclear states, (see Figs. 1 and 3).

tended to measure the integrated $\beta-\gamma_1-\gamma_2$ angular correlation of Cs^{134} using the β -counter with an extraordinary wide window. This window subtended a solid angle of about $4\pi/3$ steradian with respect to the centre of the radiation source. Moreover, this β -counter did not discriminate the energy of β -particle. The arrangement of three counters was the same one as III in § 2. He obtained the angular correlation of $\beta-\gamma_1-\gamma_2$, which was different from his experimental angular correlation of $\gamma_1-\gamma_2$ without observing β -ray. The counting rate of $\beta-\gamma_1-\gamma_2$ coincidences was about 10/minute. The number of coincidences at each angle, θ equal to 90° , 135° , 157.5° or 180° , was of the order of 1×10^4 . Unfortunately, the data of his experiment are not sufficient* for our investigation of β -decay. However, it seems to us that such an experiment shows the possibility to perform our proposed experiments.

Among the four experiments, I, II, III and IV, the last one may be performed more easily than the other three, because in IV it is unnecessary to observe the intermediate γ_1 -ray. This helps to increase the coincidence counting rate by about factor 10^2 in comparison with that of I, II, III, if we use usual β - and γ -counters.

Similar experiments have been done in the cases of triple cascade gamma transitions on Pb^{204} by V. E. Krohn and S. Raboy¹⁵⁾ and on Fe^{56} and Cd^{110} by M. Sakai¹⁶⁾.

In § 2 we discuss a possibility of unambiguous determination of ST (or VT) as the β -decay interaction. In order to find the angular correlation functions in the new experiments, we extend the theory of the triple cascade gamma transition, which was developed by Biedenharn, Arfken and Rose¹⁷⁾, to the case of $\beta-\gamma_1-\gamma_2$ in § 3, and the results are given. The angular correlation functions are useful not only for the above purpose, but also for providing us some information concerning the model of nuclei, if the parameters x, y, z in these functions can be obtained by the analysis of experimental data. Further we calculate various angular correlation functions for successive triple cascade transitions of α, β, γ rays and the other kind of particles in § 4. Suitable nuclei for our experiments, the corrections for parameters $b_{LL'}^{(2)}$ of β -ray, etc., are given in § 5.

§ 2. A possibility of unambiguous determination of ST (or VT) as the β -decay interaction

A principal difficulty we encounter in the studies of β -decay is the appearance of several nuclear matrix elements in the transition probability. From the lack of our knowledge of nuclear wave functions, we cannot estimate the exact values of the nuclear matrix elements contained in the probability of the forbidden transitions when we take an arbitrary linear combination of five relativistic invariants as the β -decay interaction. In actual calculations, therefore, we are forced to treat the ratios among nuclear matrix elements more or less as phenomenological parameters. For ST, there are three such parameters x, y and z in the case of the first forbidden β -decay with $|JJ|=1$, (see reference 18).

*) Hayashi's attempt was to distinguish the excited state (or states), which is (or are) formed as the result of β -decay of 655 keV (and 683 keV) of Cs^{134} , with spin 4 (and/or 3) in $\text{Ba}^{134, 140}$.

$$G_S \mathcal{M}(\beta \mathbf{r}) / G_T \mathcal{M}(\beta \boldsymbol{\sigma} \times \mathbf{r}) = -ix, \quad \mathcal{M}(\beta \mathbf{a}) / \mathcal{M}(\beta \boldsymbol{\sigma} \times \mathbf{r}) = (\alpha Z / 2\rho) \gamma,$$

$$\mathcal{M}(B_{ij}^3) / \mathcal{M}(\beta \boldsymbol{\sigma} \times \mathbf{r}) = iz,$$

where x , y and z are all real. The notation is as follows: G_S and G_T are the coupling constants of Scalar and Tensor interactions, respectively. $\mathcal{M}(\beta \mathbf{r})$, $\mathcal{M}(\beta \mathbf{a})$, $\mathcal{M}(\beta \boldsymbol{\sigma} \times \mathbf{r})$ and $\mathcal{M}(B_{ij}^3)$ are the first forbidden reduced nuclear matrix elements. $(\alpha Z / \rho)$ is the Coulomb potential of the daughter nucleus. As the number of independent experimental data is less than that of the parameters, the theoretical analysis does not lead to a definite result without further assumptions. This situation prevails in the cases of β -decays for which the final states are the ground or the first excited states of the daughter nuclei.

Fortunately, the number of independent experiments exceeds that of the above parameters in the cases of β -decays for which the final states are the second excited or more highly excited states of the daughter nuclei. Consider, for example, the successive nuclear transitions, $5_-(\beta \text{ 1st}) 4_+(\gamma_1 \text{ 2}) 2_+(\gamma_2 \text{ 2}) 0_+$ (Fig. 1). In this decay scheme, one can observe the β -ray spectrum, the ft -value, the β - γ_1 angular correlation as usual, and furthermore the triple angular correlation of the β -ray and the two gamma rays. Theoretically, the last mentioned angular correlation depends on three angular variables (see (4)). In other words, the angular correlation function of two gamma rays changes its value, according to the arrangements of

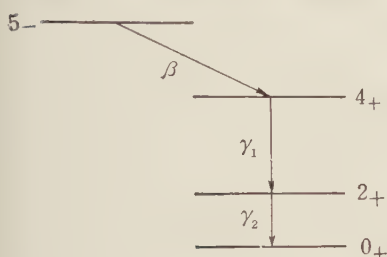
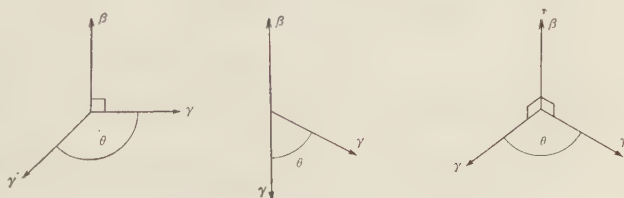


Fig. 1. Decay scheme,
 $5_-(\beta \text{ 1st}) 4_+(\gamma_1 \text{ 2}) 2_+(\gamma_2 \text{ 2}) 0_+$.

β - and γ -counters. We shall choose the following three cases as typical arrangements (Fig. 2).

- I. Three particles are emitted in a plane, β and one of γ 's being emitted in perpendicular directions.
- II. Three particles are emitted in a plane, β and one of γ 's being emitted in anti-parallel directions.
- III. β is emitted perpendicularly to the plane determined by the directions of two γ 's.

In all three cases, it is unnecessary to distinguish the names of two γ 's in experiments. We give the angular correlation functions, $W(\theta)$, which have different forms for respective cases as shown in § 3. Together with these experiments, one can observe β - γ_2 angular correlation without observing γ_1 .



I. $W(\theta : \text{I})$. II. $W(\theta : \text{II})$. III. $W(\theta : \text{III})$
Fig. 2. Arrangements of β - and γ -counters for measurement of β - γ_1 - γ_2 angular correlation in $j_0(\beta) 4(\gamma_1 \text{ 2}) 2(\gamma_2 \text{ 2}) 0$. In all three cases, it is unnecessary to distinguish the names of two γ 's in experiments.

- IV. β - γ_2 angular correlation without observing γ_1 .

If these new experiments, I, II, III, IV, and the measurements of a usual β - γ angular correlation and of a β -spectrum are performed, we can obtain the values of the parameters x, y, z using some of the data and check these values by other data. If a certain set of x, y, z can fit all the data consistently, we have a definite proof to distinguish between ST and VT as the interaction for β -decay.

For our purpose the following decay schemes are also useful: $3_-, 4_-(\beta \text{ 1st})$ or $6_+(\beta \text{ 2nd}) 4_+(\gamma_1 2) 2_+(\gamma_2 2) 0_+$, and $2_+, 3_+, 4_+(\beta \text{ 1st})$ or $5_-(\beta \text{ 2nd}) 3_-(\gamma_1 2) 1_-(\gamma_2 1) 0_+$.

§ 3. Theory of triple cascade transition

In order to deduce the various angular correlation functions of a β - γ_1 - γ_2 transition (Fig. 3), we extend the theory of a triple cascade gamma transition developed by Biedenharn, Arfken and Rose¹⁷⁾, (referred to as BAR, hereafter). This theory is restricted to gamma transitions with a pure multipolarity*. However, this restriction can be removed easily. Thus we consider the successive transitions $j_0, m_0 \rightarrow j_1 \rightarrow j_2 \rightarrow j_3, m_3$ emitting three gamma rays of multipolarities $2^{L_0}, 2^{L_1}$ and 2^{L_2} , respectively (Fig. 4). The relative probability for this cascade transition is given by (3) of BAR.

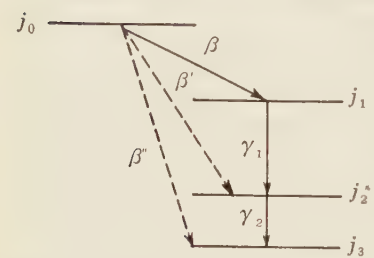


Fig. 3. Successive β - γ_1 - γ_2 transitions. β' and/or β'' may or may not exist.

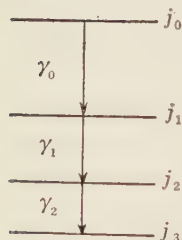


Fig. 4. Triple cascade gamma transition.

$$P(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) = \left| \sum_{m_1 m_2} \langle j_0 L_0 m_0 m_1 - m_0 | j_1 m_1 \rangle D_{m_1 - m_0 \gamma_0}^{(L_0)}(\alpha_0 \beta_0 0) \right.$$

$$\times \langle j_1 L_1 m_1 m_2 - m_1 | j_2 m_2 \rangle D_{m_2 - m_1 \gamma_1}^{(L_1)}(\alpha_1 \beta_1 0)$$

$$\times \langle j_2 L_2 m_2 m_3 - m_2 | j_3 m_3 \rangle D_{m_3 - m_2 \gamma_2}^{(L_2)}(\alpha_2 \beta_2 0) \left. \right|^2. \quad \text{BAR}(3)^{**}$$

Notation is: p_i and \mathbf{k}_i are the polarization and the direction of the i -th gamma ray, $i=0, 1, 2$. α_i, β_i and γ_i are the Euler angles of \mathbf{k}_i . $D_{M'M}^{(L)}(\alpha \beta \gamma)$ is the M th, M' th component of the $2L+1$ dimensional representation of a three-dimensional rotation group. $W(abcd:ef)$ is the Racah coefficient¹⁹⁾. $(j_1 j_2 m_1 m_2 | j m)$ is the Clebsch-Gordan coefficient.

It is easily justified that the Euler angles γ_i may be set equal to zero. As usual, we define the angular distribution function of the 0-th gamma ray with multipolarity 2^{L_0} as follows,

$$F_{L_0}^{m_1 - m_0}(\beta_0) \delta_{m_1 m_1'} = (1/4\pi) \sum_{\gamma_0} \int D_{m_1 - m_0 \gamma_0}^{(L_0)*}(\alpha_0 \beta_0 0) D_{m_1' - m_0 \gamma_0}^{(L_0)}(\alpha_0 \beta_0 0) d\alpha_0. \quad (1)$$

More generally, $F_{L, l'}^M(\theta)$ for a certain particle is defined by¹⁸⁾

*) They applied their theory also to the double cascade gamma emission following the capture of particles with nonzero orbital momentum.

**) Of course, we assume that the nuclei are not disturbed by extranuclear fields

$$F_{LL'}^M(\theta) = S \sum_{T_L(X_i)} \sum_{T_{L'}(X_j)} \varepsilon \{a^*[T(X_i)]d[T(X_j)]\mathfrak{M}^*(X_i)\mathfrak{M}(X_j) \\ \times \mathcal{Y}_{LM}(A_i) \mathcal{Y}_{L'M}^*(A_j) + \text{c.c.}\}, \quad \text{M(3)}$$

with $\varepsilon = \begin{cases} 1/2 & \text{for square terms,} \\ 1 & \text{for cross terms,} \end{cases}$

and $L \leq L'$.

The notation is as follows: X_i 's are the vector operators for the nucleus, A_i 's are the argument vectors for the emitted particle, $a[T(X_i)]$'s are the numerical coefficients in the interaction Hamiltonian, $\mathfrak{M}(X_i)$'s are the reduced nuclear matrix elements and \mathcal{Y}_{LM} 's are the polarized solid harmonics. S indicates the sums over the directional arguments (spin etc.) except the colatitude angle θ . $\sum_{T_L(X_i)}$ denote the sums over the various irreducible tensors retaining the rank L constant. The last summations are unnecessary when we consider only the coexistence of the magnetic (electric) 2^L pole and the electric (magnetic) $2^{L'}$ pole radiations. In the case of the forbidden β -decay, however, these summations become important taking into account the various irreducible tensors with the same rank L , e. g., $\mathfrak{M}(\beta \mathbf{r})$, $\mathfrak{M}(\beta \alpha)$ and $\mathfrak{M}(\beta \boldsymbol{\sigma} \times \mathbf{r})$ terms, simultaneously. Then $(j_i L m_i M | j_f m_f) (j_n L' m_n M | j_n m_n) F_{LL'}^M(\theta)$ can be understood as the relative and partial transition probability from j_i, m_i to j_f, m_f caused by L, M and L', M terms in the interaction Hamiltonian. Here we assume an isotropic distribution of magnetic substate m_a^* .

(M 3) is reduced to

$$F_{LL'}^M(\theta) = \sum_n (-)^M (L' L - M M | 2n 0) b_{LL'}^{(2n)} P_{2n}(\cos \theta). \quad \text{M(4)}$$

with $L \leq L'$ and $L' - L \leq 2n \leq L' + L$.

Here $b_{LL'}^{(2n)}$'s are parameter depending upon the properties of the emitted particle. $b_{LL'}^{(2n)}$'s for the β -ray** and for the i -ray up to quadrupole radiations are given in reference 18, and more generally in (34), with $b_{LL'}^{(2n)}$'s for the particles of zero or nonzero spin in (39).

Comparing (1) with M(3) and substituting M(4) in BAR (3), we obtain

$$P(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) = \sum_{L_0 \leq L_0'} \sum_{L_0' \leq n} \sum_{m_1} (j_0 L_0 m_0 m_1 - m_0 | j_1 m_1) (j_0 L_0' m_0 m_1 - m_0 | j_1 m_1) \\ \times (-)^{m_1 - m_0} (L_0' L_0 - m_1 + m_0 m_1 - m_0 | 2n 0) b_{L_0 L_0'}^{(2n)} P_{2n}(\cos \beta_0) \\ \times \left| \sum_{m_2} (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2) D_{m_2 - m_1 p_1}^{(L_1)}(\alpha_1 \beta_1 0) \right. \\ \left. \times (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) D_{m_3 - m_2}^{(L_2)}(\alpha_2 \beta_2 0) \right|^2. \quad (2)$$

Since m_0, m_3, p_1, p_2 are not measured in experiments, the angular correlation function of the triple cascade transition is obtained by summing over these quantum numbers.

$$W(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) = \sum_{m_0 m_3 p_1 p_2} P(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2). \quad (3)$$

*) For an anisotropic distribution of magnetic substates, e.g., in an aligned nucleus, the investigation was done by Rose, Steenberg, Tolhoek et al.⁽²⁰⁾ and is being carried on by us⁽²¹⁾.

**) $b_{LL'}^{(2n)}$'s are equal to $b_{L_1 L_2}^{(2n)}$'s in reference 18.

After summing over m_0 , we choose k_0 as z -axis, i.e., we put $\beta_0=0$. Then we have

$$\begin{aligned} W(k_0, k_1, k_2) &= W(\theta_1, \theta_2, \varphi) \\ &= \sum_{L_0 \leq L_0'} \sum_n \sum_{m_1 m_2 p_1 p_2} (-1)^{j_1-j_0} \sqrt{(2j_1+1)(4n+1)} \\ &\quad \times b_{L_0 L_0'}^{(2n)}(j_1 2n m_1 0 | j_1 m_1) W(j_1 j_1 L_0 L_0'; 2n j_0) \\ &\quad \times \left| \sum_{m_2} (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2) D_{m_2 - m_1 p_1}^{(L_1)}(0 \theta_1 0) \right. \\ &\quad \left. \times (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) D_{m_3 - m_2 p_2}^{(L_2)}(\varphi \theta_2 0) \right|^2. \end{aligned} \quad (4)$$

In (4), the 0-th particle is not restricted to the γ -ray. The m_2 dependent factors $D_{m_2 - m_1 p_1}^{(L_1)}(0 \theta_1 0) D_{m_3 - m_2 p_2}^{(L_2)}(\varphi \theta_2 0)$ make the summing over the magnetic quantum numbers very laborious. However, it is quite easy to perform it in special cases indicated by Fig. 2. In the following calculations we do not specify the parity of each state, because the formulae do not depend on whether gamma rays are electric or magnetic. We specify the emitted particles or the arrangements of the counters in the parenthesis of $W(\theta)$, as $W(\theta: \alpha-\gamma_e)$ or $W(\theta: I)$.

In the decay scheme, $j_0(\beta^-) 4(\gamma_1 2) 2(\gamma_2 2) 0$, it is unnecessary to distinguish the names of two gamma rays in experiments.

1) $j_0(\beta^- 1st) 4(\gamma_1 2) 2(\gamma_2 2) 0$.

$W(\theta: I, II, III)$

$$\begin{aligned} &= (-1)^{j_1-j_0} [\{ W(4 4 0 0; 0 j_0) b_{00}^{(0)} + W(4 4 1 1; 0 j_0) b_{11}^{(0)} + W(4 4 2 2; 0 j_0) b_{22}^{(0)} \} \\ &\quad \times \{ 24 + 3 \cos^2 \theta + \cos^4 \theta \} \\ &\quad + \{ W(4 4 1 1; 2 j_0) b_{11}^{(2)} + W(4 4 0 2; 2 j_0) b_{02}^{(2)} + W(4 4 1 2; 2 j_0) b_{12}^{(2)} \\ &\quad + W(4 4 2 2; 2 j_0) b_{22}^{(2)} \} W_2(\theta: I, II, III)]. \end{aligned} \quad (5)$$

$$W_2(\theta: I) = (5/2 \sqrt{77}) \{ -24 + 69 \cos^2 \theta - 13 \cos^4 \theta \}. \quad (6)$$

$$W_2(\theta: II) = (5/2 \sqrt{77}) \{ -24 - 57 \cos^2 \theta + 17 \cos^4 \theta \}. \quad (7)$$

$$W_2(\theta: III) = (10/\sqrt{77}) \{ 12 - 3 \cos^2 \theta - \cos^4 \theta \}. \quad (8)$$

Example. $W(\theta: I)$ for $5_-(\beta^- 1st) 4_+(\gamma_1 2) 2_+(\gamma_2 2) 0_+$.

From (5) and (6) we obtain $W(\theta: I)$.

$$\begin{aligned} W(\theta: I) &= \{ -(1/\sqrt{3}) b_{11}^{(0)} + (1/\sqrt{5}) b_{22}^{(0)} \} \{ 24 + 3 \cos^2 \theta + \cos^4 \theta \} \\ &\quad - \{ (1/11 \sqrt{6}) b_{11}^{(2)} + (\sqrt{3}/22) b_{12}^{(2)} + (5/22 \sqrt{14}) b_{22}^{(2)} \} \{ -24 + 69 \cos^2 \theta - 13 \cos^4 \theta \}. \end{aligned} \quad (5')$$

Substituting the $b_{LL'}^{(2n)}$'s of the first forbidden β -transition¹⁸⁾ in (5'), it becomes in the case of ST as follows:

$W(\theta: I)$

$$\begin{aligned}
&= [(\alpha Z/2\rho)^2 (x-y+1)^2 \\
&\quad + (\alpha Z/2\rho) \{-K(x-1) + (p^2/W)(x+1)\} (2/3)(x-y+1) \\
&\quad + \{(1/12)(K^2+p^2)(4x^2+z^2+2) + (2Kp^2/9W)(1-x^2)\}] \\
&\quad \times \{24+3\cos^2\theta+\cos^4\theta\} \\
&\quad - (1/22)[(\alpha Z/2\rho)(p^2/3W)(4x+3\sqrt{3}z-2)(x-y+1) \\
&\quad + \{-Kp^2/9W)(4x+3\sqrt{3}z-2)(x-1) \\
&\quad + (p^2/24)(16x^2-5z^2+12\sqrt{3}z-4)\}] \\
&\quad \times \{-24+69\cos^2\theta-13\cos^4\theta\}, \tag{5''}
\end{aligned}$$

where x , y and z are the parameters defined in § 1. $(\alpha Z/\rho)$ is the Coulomb potential of the daughter nucleus, (unit of ρ is a Compton wave length of the electron). W , p and $K=W_0-W$ are the energy (unit mc^2), momentum (unit mc) of the electron and the energy of the neutrino respectively. Suffix zero of W indicates its maximum value. (5'') is the final result to be compared with the experimental data. Similar procedures are applicable to other cases. Here we do not write the explicit dependence on x , y , z for them to spare the space.

$$2) \quad j_0(\beta \text{ 2nd}) 4(\gamma_1 \text{ 2}) 2(\gamma_2 \text{ 2}) 0.$$

$W(\theta : \text{I, II, III})$

$$\begin{aligned}
&= (-)^{j_0} [\{W(4 \ 4 \ 2 \ 2 ; 0 \ j_0) b_{32}^{(0)} + W(4 \ 4 \ 3 \ 3 ; 0 \ j_0) b_{33}^{(0)}\} \{24+3\cos^2\theta+\cos^4\theta\} \\
&\quad + \{W(4 \ 4 \ 2 \ 2 ; 2 \ j_0) b_{32}^{(2)} + W(4 \ 4 \ 2 \ 3 ; 2 \ j_0) b_{23}^{(2)} + W(4 \ 4 \ 3 \ 3 ; 2 \ j_0) b_{33}^{(2)}\} \\
&\quad \times W_2(\theta : \text{I, II, III}) \\
&\quad + \{W(4 \ 4 \ 2 \ 2 ; 4 \ j_0) b_{32}^{(4)} + W(4 \ 4 \ 2 \ 3 ; 4 \ j_0) b_{23}^{(4)} + W(4 \ 4 \ 3 \ 3 ; 4 \ j_0) b_{33}^{(4)}\} \\
&\quad \times W_4(\theta : \text{I, II, III})]. \tag{9}
\end{aligned}$$

$$W_2(\theta : \text{I}) = (6),$$

$$W_4(\theta : \text{I}) = (3/2 \sqrt{2002}) \{-421 + 1283 \cos^2\theta - 919 \cos^4\theta\}. \tag{10}$$

$$W_2(\theta : \text{II}) = (7),$$

$$W_4(\theta : \text{II}) = (3/2 \sqrt{2002}) \{-421 + 1038 \cos^2\theta - 769 \cos^4\theta\}. \tag{11}$$

$$W_2(\theta : \text{III}) = (8),$$

$$W_4(\theta : \text{III}) = (9/2 \sqrt{2002}) \{-67 + 36 \cos^2\theta + 12 \cos^4\theta\}. \tag{12}$$

In the decay scheme, $j_0(\beta) 3(\gamma_1 \text{ 2}) 1(\gamma_2 \text{ 1}) 0$, it is necessary to distinguish the names of two gamma rays in experiments, except in III (Fig. 5).

I. Three particles are emitted in a plane, β and

IA: γ_1 being emitted in perpendicular directions,

IB: γ_2 being emitted in perpendicular directions.

- II. Three particles are emitted in a plane, β and
 IIA: γ_1 being emitted in antiparallel directions,
 IIB: γ_2 being emitted in antiparallel directions.
 III. β is emitted perpendicularly to the plane determined by the directions of two γ 's.



IA. $W(\theta: \text{IA})$. IB. $W(\theta: \text{IB})$. IIA. $W(\theta: \text{IIA})$. IIB. $W(\theta: \text{IIB})$. III. $W(\theta: \text{III})$.

Fig. 5. Arrangements of β - and γ -counters for measurement of β - γ_1 - γ_2 angular correlation $j_0(\beta)3(\gamma_1 2)1(\gamma_2 1)0$. It is necessary to distinguish the names of two γ 's in experiments, except in III.

3) $j_0(\beta \text{ 1st})3(\gamma_1 2)1(\gamma_2 1)0$.

$W(\theta: \text{I, II, III})$

$$= (-)^{j_1-j_0} [\{ W(3 \ 3 \ 0 \ 0; 0j_0) b_{00}^{(0)} + W(3 \ 3 \ 1 \ 1; 0j_0) b_{11}^{(0)} + W(3 \ 3 \ 2 \ 2; 0j_0) b_{22}^{(0)} \} \\ \times \{ 29 - 3 \cos^2 \theta \} \\ + \{ W(3 \ 3 \ 1 \ 1; 2j_0) \{ b_1^{(2)} \} + W(3 \ 3 \ 0 \ 2; 2j_0) b_{02}^{(2)} + W(3 \ 3 \ 1 \ 2; 2j_0) b_{12}^{(2)} \\ + W(3 \ 3 \ 2 \ 2; 2j_0) b_{22}^{(2)} \} W_2(\theta: \text{I, II, III})]. \quad (13)$$

$$W_2(\theta: \text{IA}) = (6\sqrt{3}/\sqrt{7}) \{ 5 - 4 \cos^2 \theta \}. \quad (14)$$

$$W_2(\theta: \text{IB}) = (6\sqrt{3}/\sqrt{7}) \{ -5 + 5 \cos^2 \theta + \cos^4 \theta \}. \quad (15)$$

$$W_2(\theta: \text{IIA}) = (6\sqrt{3}/\sqrt{7}) \{ -5 + 3 \cos^2 \theta \}. \quad (16)$$

$$W_2(\theta: \text{IIB}) = (6\sqrt{3}/\sqrt{7}) \{ 5 - 6 \cos^2 \theta - \cos^4 \theta \} \quad (17)$$

$$W_2(\theta: \text{III}) = (6\sqrt{3}/\sqrt{7}) \cos^2 \theta. \quad (18)$$

4) $j_0(\beta \text{ 2nd})3(\gamma_1 2)1(\gamma_2 1)0$.

$W(\theta: \text{I, II, III})$

$$= (-)^{j_1-j_0} [\{ W(3 \ 3 \ 2 \ 2; 0j_0) b_{22}^{(0)} + W(3 \ 3 \ 3 \ 3; 0j_0) b_{33}^{(0)} \} \{ 29 - 3 \cos^2 \theta \} \\ + \{ W(3 \ 3 \ 2 \ 2; 2j_0) b_{22}^{(2)} + W(3 \ 3 \ 2 \ 3; 2j_0) b_{23}^{(2)} + W(3 \ 3 \ 3 \ 3; 2j_0) b_{33}^{(2)} \} W_2(\theta: \text{I, II, III}) \\ + \{ W(3 \ 3 \ 2 \ 2; 4j_0) b_{22}^{(4)} + W(3 \ 3 \ 2 \ 3; 4j_0) b_{23}^{(4)} + W(3 \ 3 \ 3 \ 3; 4j_0) b_{33}^{(4)} \} W_4(\theta: \text{I, II, III})]. \quad (19)$$

$$W_2(\theta: \text{IA}) = (14),$$

$$W_4(\theta: \text{IA}) = (3/\sqrt{22}) \{ -2 - 11 \cos^2 \theta \}. \quad (20)$$

$$W_2(\theta: \text{IB}) = (15),$$

$$W_4(\theta: \text{IB}) = (3/\sqrt{22}) \{ -12 + 89 \cos^2 \theta - 90 \cos^4 \theta \}. \quad (21)$$

$$W_2(\theta: \text{IIA}) = (16),$$

$$W_4(\theta: \text{IIA}) = (2\sqrt{2}/\sqrt{11}) \{ -9 - 17 \cos^2 \theta \}. \quad (22)$$

$$W_2(\theta : \text{IIB}) = (17),$$

$$W_4(\theta : \text{IIB}) = (\sqrt{2}/\sqrt{11}) \{-3 + 96 \cos^2 \theta - 145 \cos^4 \theta\}. \quad (23)$$

$$W_2(\theta : \text{III}) = (18),$$

$$W_4(\theta : \text{III}) = (3/\sqrt{22}) \{-7 - 6 \cos^2 \theta\}. \quad (24)$$

To find $W(\theta)$ for the experiment IV, we generalize (4) further to the case of gamma rays with mixed multiplicities $2^{L_i}, 2^{L'_i}, \dots (i=1, 2)$.

$$\begin{aligned} W(\theta_1, \theta_2, \varphi) = & \sum_{L_0 \leq L_0', L_1 L_1' L_2 L_2'} \sum_n \sum_{\delta_1 \delta_1' \delta_2 \delta_2'} \sum_{p_1 p_2} \sum_{m_0 M m_1 m_2 m_3} \\ & b_{L_0 L_0'}^{(2n)} \alpha_1 \alpha_1'^* \alpha_2 \alpha_2'^* i^{L_1 + L_2 - L_1' - L_2' + \delta_1 + \delta_2 - \delta_1' - \delta_2'} \\ & \times p_1^{\delta_1 + \delta_1'} p_2^{\delta_2 + \delta_2'} \sqrt{(2L_1 + 1)(2L_1' + 1)(2L_2 + 1)(2L_2' + 1)} \\ & \times (-)^M (L_0' L_0 - MM | 2n 0) (j_0 L_0 m_1 - MM | j_1 m_1) (j_0 L_0' m_1 - M M | j_1 m_1) \\ & \times (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2) (j_1 L_1' m_1 m_2' - m_1 | j_2 m_2') \\ & \times (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) (j_2 L_2' m_2' m_3 - m_2' | j_3 m_3) \\ & \times D_{m_3 - m_1 p_1}^{(L_1)} (0 \theta_1 0) D_{m_2' - m_1 p_1}^{(L_1')*} (0 \theta_1 0) D_{m_3 - m_2 p_1}^{(L_2)} (\varphi \theta_2 0) D_{m_3 - m_2' p_2}^{(L_2')*} (\varphi \theta_2 0), \end{aligned} \quad (25)$$

$$\text{with } \delta_i = \begin{cases} 0 & \text{for magnetic radiation,} \\ 1 & \text{for electric radiation.} \end{cases}$$

α_i is the reduced matrix element of γ -ray transition*.

5) β - γ_2 angular correlation, γ_1 unobserved (Experiments IV, Fig. 3).

We average (25) over the polarization and the angle variables of the first gamma ray and put $\varphi=0, \theta_2=\theta$. Then, (25) becomes:

$$\begin{aligned} W(\theta : \beta - \gamma_2) = & \sum_{L_0 \leq L_0', L_1 L_2 L_2'} \sum_n \sum_{\delta_2 \delta_2'} \sum_{p_2} \sum_{m_0 M m_1 m_2 m_3} \\ & b_{L_0 L_0'}^{(2n)} |\alpha_1|^2 \alpha_2 \alpha_2'^* i^{L_2 - L_2' + \delta_2 - \delta_2'} p_2^{\delta_2 + \delta_2'} \sqrt{(2L_2 + 1)(2L_2' + 1)} \\ & \times (-)^M (L_0' L_0 - MM | 2n 0) (j_0 L_0 m_1 - MM | j_1 m_1) (j_0 L_0' m_1 - MM | j_1 m_1) \\ & \times (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2)^2 (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) (j_2 L_2' m_2 m_3 - m_2 | j_3 m_3) \\ & \times D_{m_3 - m_2 p_2}^{(L_2)} (0 \theta 0) D_{m_3 - m_2 p_2}^{(L_2')*} (0 \theta 0). \end{aligned} \quad (26)$$

After summing over the magnetic quantum numbers and the polarization p_2 , we have

$$\begin{aligned} W(\theta : \beta - \gamma_2) = & \sum_{L_0 \leq L_0', L_1 L_2 L_2'} \sum_n b_{L_0 L_0'}^{(2n)} |\alpha_1|^2 \alpha_2 \alpha_2'^* i^{L_2 - L_2' + \delta_2 - \delta_2'} \\ & \times (-)^{-L_1 + 1 + 2j_1 + 2j_2 - j_0 - j_3} \sqrt{(2L_2 + 1)(2L_2' + 1)} (2j_1 + 1)(2j_2 + 1)(2j_3 + 1) \\ & \times (L_2 L_2' 1 - 1 | 2n 0) W(j_1 j_1 L_0 L_0'; 2n j_0) W(j_1 j_1 j_2 j_2; 2n L_1) \\ & \times W(j_2 j_2 L_2 L_2'; 2n j_3) P_{2n}(\cos \theta), \end{aligned} \quad (27)$$

*) α_i is equal to, e.g., α or β of M(7).

where $\sum_{\delta, \dots}$ is omitted, because the parity condition, $L_i + L_i' + \delta_i + \delta_i' = \text{even}$, is always taken into account. Now we take the reduced matrix element of $\text{BR}^{(2)}$, $(j_i \| L_i \| j_{i+1})$, which is connected by the relation, $(-)^{j_i} (j_{i+1} \| L_i \| j_i) = \alpha_i i^{j_i + \delta_i}$. These $(j_i \| L_i \| j_{i+1})$'s are chosen as real numbers simultaneously²³. We also use the F coefficient²³

$$F_N(L L' j_a j_b) = F_N(L' L j_a j_b) \\ = (-)^{j_b - j_a - 1} \sqrt{(2j_b + 1)(2L + 1)(2L' + 1)(LL' + 1)} W(j_b j_b LL'; N j_a). \quad (28)$$

(27) becomes,

$$W(\theta : \beta - \gamma_2) = \sum_n \left[\left\{ \sum_{L_0 \leq L_0'} (-)^{j_1 - j_0} b_{L_0 L_0'}^{(2n)} W(j_1 j_1 L_0 L_0'; 2nj_0) \sqrt{2j_1 + 1} \right\} \right. \\ \times \left\{ \sum_{L_1} (j_1 \| L_1 \| j_2)^2 W(j_1 2n L_1 j_2; j_1 j_2) \sqrt{(2j_1 + 1)(2j_2 + 1)} \right\} \\ \left. \times \left\{ \sum_{L_2 L_2'} (-)^{L_2 + L_2'} (j_2 \| L_2 \| j_3) (j_2 \| L_2' \| j_3) F_{2n}(L_2 L_2' j_3 j_2) \right\} \right] P_{2n}(\cos \theta), \quad (29)$$

where we put the same factors $\sqrt{2j_1 + 1}$ into the first and the second curly brackets.

6) $\beta - \gamma_1$ angular correlation, γ_2 unobserved^{6,7)}, (Fig. 3).

This angular correlation is the usual double cascade correlation of β - and γ -rays. Averaging over the polarization, the angle variables of the second gamma ray and m_{β} , (25) becomes,

$$W(\theta : \beta - \gamma_1) = \sum_{L_0 \leq L_0', L_1 L_1'} \sum_n \sum_{\delta_1 \delta_1'} \sum_{p_1} \sum_{m_0 M m_1 m_2} \\ b_{L_0 L_0'}^{(2n)} \alpha_1 \alpha_1'^* i^{L_1 - L_1' + \delta_1 - \delta_1'} p_1^{\delta_1 + \delta_1'} \sqrt{(2L_1 + 1)(2L_1' + 1)} \\ \times (-)^M (L_0' L_0 - MM | 2n 0) (j_0 L_0 m_1 - MM | j_1 m_1) (j_0 L_0' m_1 - MM | j_1 m_1) \\ \times (j_1 L_1 m_1 m_2 - m_1 | j_1 m_2) (j_1 L_1' m_1 m_2 - m_1 | j_1 m_2) \\ \times D_{m_2 - m_1 p_1}^{(L_1)}(0 \theta 0) D_{m_2 - m_1 p_1}^{(L_1')*}(0 \theta 0), \quad (30)$$

where $\theta = \theta_1$.

After summing over the magnetic quantum numbers and the polarization p_1 ,

$$W(\theta : \beta - \gamma_1) = \sum_{L_0 \leq L_0', L_1 L_1'} \sum_n b_{L_0 L_0'}^{(2n)} \alpha_1 \alpha_1'^* i^{L_1 - L_1' + \delta_1 - \delta_1'} \\ \times (-)^{1 + 2j_1 - j_0 - j_2} \sqrt{(2L_1 + 1)(2L_1' + 1)(2j_1 + 1)(2j_2 + 1)} \\ \times (L_1 L_1' 1 - 1 | 2n 0) W(j_1 j_1 L_0 L_0'; 2nj_0) W(j_1 j_1 L_1 L_1'; 2nj_2) P_{2n}(\cos \theta). \quad (31)$$

This is equal to M(6) or Formula I of M. Moreover (31) is expressed by

$$W(\theta : \beta - \gamma_1) = \sum_n \left[\left\{ \sum_{L_0 \leq L_0'} (-)^{j_1 - j_0} b_{L_0 L_0'}^{(2n)} W(j_1 j_1 L_0 L_0'; 2nj_0) \sqrt{2j_1 + 1} \right\} \right. \\ \left. \times \left\{ \sum_{L_1 L_1'} (-)^{L_1 + L_1'} (j_1 \| L_1 \| j_2) (j_1 \| L_1' \| j_2) F_{2n}(L_1 L_1' j_2 j_1) \right\} \right] P_{2n}(\cos \theta). \quad (32)$$

§ 4. Miscellaneous angular correlations

In this section we calculate the angular correlation functions for various decay schemes without observing one of the triple cascade particles.

In order to symmetrize $F_{LL'}^M(\theta)$ with respect to L and L' , we introduce a new definition:

$$F_{LL'}^M(\theta) = \sum_{T_L(\mathbf{x}_i)} \sum_{T_{L'}(\mathbf{x}_j)} a^*[T(\mathbf{x}_i)] a[T(\mathbf{x}_j)] \mathcal{M}^*(\mathbf{x}_i) \mathcal{M}(\mathbf{x}_j) \\ \times \mathcal{J}_{LM}(\mathbf{A}_i) \mathcal{J}_{L'M}^*(\mathbf{A}_j), \quad \text{M}(3')$$

where the restriction, $L \leq L'$, is removed.

With this definition, $F_{LL'}^M(\theta)$ for γ -ray is

$$F_{LL'}^M(\theta) = \sum_n (j \| L \| j') (j \| L' \| j') (-)^{M+1} \\ \times \sqrt{(2L+1)(2L'+1)} (L L - M M | 2n 0) (L L' 1 - 1 | 2n 0) P_{2n}(\cos \theta). \quad (33)$$

From M(4) and (33), $b_{LL'}^{(2n)}$ of γ -ray is

$$b_{LL'}^{(2n)} = b_{L'L}^{(2n)} = - (j \| L \| j') (j \| L' \| j') \sqrt{(2L+1)(2L'+1)} (L L' 1 - 1 | 2n 0). \quad (34)$$

1) $\gamma_0 - \gamma_2$ angular correlation, γ_1 unobserved, (Fig. 4).

Inserting (34) in (29), we obtain

$$\mathcal{W}(\theta; \gamma_0 - \gamma_2) = \sum_n \left[\sum_{L_0 L_0'} (j_0 \| L_0 \| j_1) (j_0 \| L_0' \| j_1) F_{2n}(L_0 L_0'; j_0 j_1) \right. \\ \times \left\{ \sum_{L_1} (j_1 \| L_1 \| j_2)^2 \mathcal{W}(j_1 2n L_1 j_2; j_1 j_2) \sqrt{(2j_1+1)(2j_2+1)} \right\} \\ \times \left. \left\{ \sum_{L_2 L_2'} (-)^{L_2+L_2'} (j_2 \| L_2 \| j_3) (j_2 \| L_2' \| j_3) F_{2n}(L_2 L_2'; j_2 j_3) \right\} \right] P_{2n}(\cos \theta) \quad (35)$$

which is the generalization of BAR (24).

2) $\gamma_0 - \gamma_1$ angular correlation, γ_2 unobserved, (Fig. 4).

Inserting (34) in (32), we obtain $\mathcal{W}(\theta; \gamma_0 - \gamma_1)$ which is equal to $\mathcal{W}(\theta)$ of the usual double cascade gamma transition, BR(64)²².

$$\mathcal{W}(\theta; \gamma_0 - \gamma_1) = \sum_n \left[\sum_{L_0 L_0'} (j_0 \| L_0 \| j_1) (j_0 \| L_0' \| j_1) F_{2n}(L_0 L_0'; j_0 j_1) \right. \\ \times \left. \left\{ \sum_{L_1 L_1'} (-)^{L_1+L_1'} (j_1 \| L_1 \| j_2) (j_1 \| L_1' \| j_2) F_{2n}(L_1 L_1'; j_1 j_2) \right\} \right] P_{2n}(\cos \theta). \quad (36)$$

3) $\gamma_1 - \gamma_2$ angular correlation, γ_0 unobserved, (Fig. 4).

$$\mathcal{W}(\theta; \gamma_1 - \gamma_2) = (36), \text{ in which each suffix } i \text{ except } n \text{ is replaced by } i+1. \quad (37)$$

In (35), (36) and (37) the sign functions $(-)^{L_i+L_i'}$ are preferable in analysing the three gamma rays in connection with $\gamma_0 - \gamma_1$, $\gamma_1 - \gamma_2$, $\gamma_0 - \gamma_2$.

In the case of the particle with spin zero, e.g., α -particle, π meson, etc., $F_{LL'}^M(\theta)$ is given by

$$F_{LL'}^M(\theta) = \sum_n (j_1 \| L \| j') (j_1 \| L' \| j') (-)^M \times \sqrt{(2L+1)(2L'+1)} (L'L-MM|2n\ 0) (LL' 00|2n\ 0) P_{2n}(\cos \theta), \quad (38)$$

where L and L' indicate the L -th and L' -th partial waves.

$$b_{LL'}^{(2n)} \equiv b_{LL'}^{(2n)} = (j_1 \| L \| j') (j_1 \| L' \| j') \sqrt{(2L+1)(2L'+1)} (LL' 00|2n\ 0). \quad (39)$$

Comparing (34) with (39), the equations for γ -ray are also useful for the particle with spin zero if we replace $(LL' 1-1|2n\ 0)$ by $-(LL' 00|2n\ 0)$ for relevant γ -ray. When the particle has nonzero spin, we calculate the equations with channel spin formalism. For example, in the capture of the particle with nonzero spin, we should replace $(L_0 L_0' 1-1|2n\ 0) W(j_1 j_1 L_0 L_0'; 2n j_0)$ by $-\sum_s |A(s)|^2 (L_0 L_0' 00|2n\ 0) W(j_1 j_1 L_0 L_0'; 2n s)$, where $A(s)$ is the reduced matrix element for the capture process of channel spin s .

4) α - γ angular correlation (Fig. 6).

$W(\theta: \alpha-\gamma)$

$$= \sum_n \left[\left\{ \sum_{L_0 L_0'} (-)^{j_1-j_0} (j_0 \| L_0 \| j_1) (j_0 \| L_0' \| j_1) \sqrt{(2L_0+1)(2L_0'+1)(2j_1+1)} \right. \right. \\ \times (L_0 L_0' 00|2n\ 0) W(j_1 j_1 L_0 L_0'; 2n j_0) \} \\ \times \left. \left\{ \sum_{L_1 L_1'} (-)^{L_1+L_1'} (j_1 \| L_1 \| j_2) (j_1 \| L_1' \| j_2) F_{2n}(L_1 L_1' j_2 j_2) \right\} \right] P_{2n}(\cos \theta). \quad (40)$$

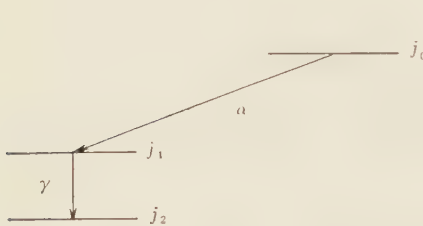


Fig. 6. $W(\theta: \alpha-\gamma)$.

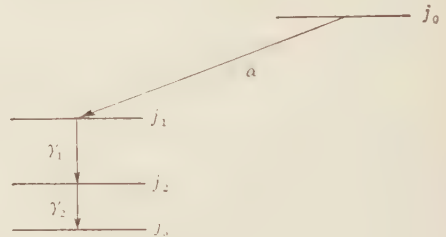


Fig. 7. $W(\theta: \alpha-\gamma_2)$.

Detailed relation between the nuclear transition and the nuclear or π meson reaction is discussed elsewhere^{22,24)}.

5) α - γ_2 angular correlation, γ_1 unobserved (Fig. 7).

$W(\theta: \alpha-\gamma_2)$

$$= \sum_n \left[\left\{ \sum_{L_0 L_0'} (-)^{j_1-j_0} (j_0 \| L_0 \| j_1) (j_0 \| L_0' \| j_1) \sqrt{(2L_0+1)(2L_0'+1)(2j_1+1)} \right. \right. \\ \times (L_0 L_0' 00|2n\ 0) W(j_1 j_1 L_0 L_0'; 2n j_0) \} \\ \times \left\{ \sum_{L_1} (j_1 \| L_1 \| j_2)^2 W(j_1 2n L_1 j_2; j_1 j_2) \sqrt{(2j_1+1)(2j_2+1)} \right\} \\ \times \left. \left\{ \sum_{L_2 L_2'} (-)^{L_2+L_2'} (j_2 \| L_2 \| j_3) (j_2 \| L_2' \| j_3) F_{2n}(L_2 L_2' j_3 j_3) \right\} \right] P_{2n}(\cos \theta). \quad (41)$$

6) $\beta-\alpha$ angular correlation (Fig. 8).

$W(\theta: \beta-\alpha)$

$$\begin{aligned}
 &= \sum_n \left[\left\{ \sum_{j_0 \leq L_0} (-)^{j_1-j_0} b_{L_0 L_0'}^{(2n)} W(j_1 j_1 L_0 L_0'; 2n j_0) \sqrt{2j_1+1} \right\} \right. \\
 &\times \left\{ \sum_{L_1 L_1'} (-)^{L_1+L_1'+j_1-j_2} (j_1 \| L_1 \| j_1) (j_1 \| L_1' \| j_2) \right. \\
 &\times \left. \sqrt{(2L_1+1)(2L_1'+1)(2j_1+1)} (L_1 L_1' 00 | 2n 0) W(j_1 j_1 L_1 L_1'; 2n j_2) \right\} \left. \right] P_{2n}(\cos \theta). \quad (42)
 \end{aligned}$$

7) $\beta-\alpha$ angular correlation, γ unobserved (Fig. 9).

$W(\theta: \beta-\alpha, \gamma \text{ unobserved})$

$$\begin{aligned}
 &= \sum_n \left[\left\{ \sum_{L_0 \leq L_0'} (-)^{j_1-j_0} b_{L_0 L_0'}^{(2n)} W(j_1 j_1 L_0 L_0'; 2n j_0) \sqrt{2j_1+1} \right\} \right. \\
 &\times \left\{ \sum_{L_1} (j_1 \| L_1 \| j_2)^2 W(j_1 2n L_1 j_0; j_1 j_2) \sqrt{(2j_1+1)(2j_2+1)} \right\} \\
 &\times \left\{ \sum_{L_2 L_2'} (-)^{L_2+L_2'+j_2-j_3} (j_2 \| L_2 \| j_2) (j_2 \| L_2' \| j_3) \right. \\
 &\times \left. \sqrt{(2L_2+1)(2L_2'+1)(2j_2+1)} (L_2 L_2' 00 | 2n 0) W(j_2 j_2 L_2 L_2'; 2n j_3) \right\} \left. \right] P_{2n}(\cos \theta). \quad (43)
 \end{aligned}$$

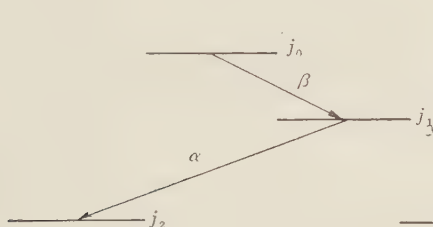


Fig. 8. $W(\theta: \beta-\alpha)$.

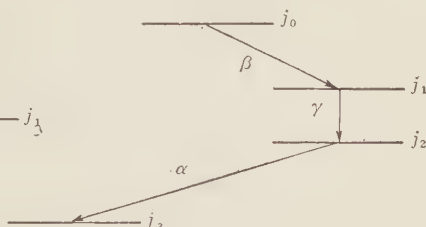


Fig. 9. $W(\theta: \beta-\alpha, \gamma \text{ unobserved})$.

§ 5. Concluding remarks

A set of the experiments on the angular correlation $\beta-\gamma_1-\gamma_2$, I, II, III, IV, together with the measurements of $\beta-\gamma_1$ angular correlation and of β -ray spectrum, is expected to determine the linear combination of β -decay interaction definitely. $W(\theta)$'s given in § 3 and $b_{L,L'}^{(2n)}$'s for β -ray¹⁸⁾ may be useful in analysing the experimental data theoretically. It seems that the β -decays of odd-odd nuclei are suitable for our purpose, especially, 3_- or $5_- (\beta \text{ 1st}) 4_+ (\gamma_1 \text{ 2}) 2_+ (\gamma_2 \text{ 2}) 0_+$, because the even-even nucleus has a well-known rotational level structure $4_+ - 2_+ - 0_+$. Unfortunately, there has not been tried any experiment on 3_- or $5_- (\beta \text{ 1st}) 4_+ (\gamma_1 \text{ 2}) 2_+ (\gamma_2 \text{ 2}) 0_+$ so far. It is also possible to use the decay schemes, $4_- (\beta \text{ 1st})$ or $6_+ (\beta \text{ 2nd}) 4_+ (\gamma_1 \text{ 2}) 0_+$; 2_+ , 3_+ , $4_+ (\beta \text{ 1st})$ or $5_- (\beta \text{ 2nd}) 3_- (\gamma_1 \text{ 2}) 1_- (\gamma_2 \text{ 1}) 0_+$ and any other $\beta-\gamma_1-\gamma_2$ process, for which, however, more laborious calculation is needed. We will pick up suitable nuclei for our purpose from references 25, 26 and 27 and show them in the following.

1) ${}_{41}\text{Nb}^{94} \rightarrow {}_{42}\text{Mo}^{94}$: $\log ft = 12.1$, $W_0 = 0.50 \text{ Mev.}^{25)}$

If it is certain that the second excited state of Mo^{94} is $4_{-}^{25)}$, $6_{-}(\beta 2\text{nd}) 4_{-}(\gamma_1 2) 2_{-}(\gamma_2 2) 0_{+}$ is available. But, if it is 3_{+} as described in reference 26 and 27, this decay scheme is useless, for β -decay is caused by $S_{4/2}^{\beta}$ term only.

2) ${}_{53}\text{I}^{130} \rightarrow {}_{54}\text{Xe}^{130}$: $\log ft = 6.5$, $W_0 = 1.02 \text{ Mev.}^{25)}$

$6_{-}(\beta 1\text{st}) 6_{-}(\gamma_1 2) 4_{-}(\gamma_2 2) 2_{+}$ is available. 6_{-} state is the third excited state of Xe^{130} .

3) ${}_{53}\text{Cs}^{134} \rightarrow {}_{56}\text{Ba}^{134}$: $\log ft = 8.9 (4 \rightarrow 4_{+})^{25)}$,

$W_0 = 655 \text{ kev} (4 \rightarrow 4_{+})$ and $683 \text{ kev} (4 \rightarrow 3_{-})^{28)}$.

The ground state of Cs^{134} has spin 4, but its parity is not certain. If its parity is plus, $4_{+}(\beta 1\text{st}) 3_{-}(\gamma_1 2) 1_{-}(\gamma_2 1) 0_{-}$ is available. On the contrary, if its parity is minus, $4_{-}(\beta 1\text{st}) 4_{+}(\gamma_1 2) 2_{+}(\gamma_2 2) 0_{+}$ is available.

4) ${}_{81}\text{Tl}^{208} \rightarrow {}_{82}\text{Pb}^{208}$: $\log ft = 5.6$, $W_0 = 1.79 \text{ Mev.}^{25)}$

$5_{+}(\beta 1\text{st}) 5_{-}(\gamma_1 2) 3_{-}(\gamma_2 3) 0_{+}$ is available.

$W(\theta)$'s for the decay schemes, 2) and 4), are not yet given here. They are deducible by the same procedure described in § 3 and will be calculated if the experiments are possible.

If the window of the β -counter is circularly symmetric with respect to z -axis in colatitude angle θ to the centre of the radiation source, then $b_{ll'}^{(n)}$'s for β -ray in $W(\theta)$'s should be replaced by $b_{ll'}^{(n)} \int_0^{\theta} P_{2n}(\cos \theta) \sin \theta d\theta$. If the energy of β -ray is not resolved, $b_{ll'}^{(n)}$'s should be replaced by $\int_{W_{min}}^{W_0} b_{ll'}^{(2n)} F(Z, W') K^2 p W' dW'$. In the energy region, where the Fermi function $F(Z, W')$ times $p W'$ is nearly constant, we can replace this integration by $\int_{W_{min}}^{W_0} b_{ll'}^{(2n)} K^2 W'^2 dW'$. This reduces the labour of computation very much. For example, in $Z=42$,

$$(p/W) F(Z, W) = \begin{cases} 3.475 & : p=0.0, \\ 3.449 & : p=1.8, \end{cases}$$

which are given by Rose³⁰⁾. In $0.0 \leq p \leq 1.8$, $(p/W) F(Z, W)$ may be regarded as constant with an error of about 0.2 percent.

When some drastic cancellation occurs in $b_{ll'}^{(2n)}$'s, we have to take into account the finite nuclear size correction and the finite de Broglie wave length effect. However, the main terms of these corrections to $b_{ll'}^{(2n)}$'s are expressible as the renormalization of the Coulomb potential ($\alpha Z/\mu$), i.e., the effects of the corrections are equivalent to the effective change of Z and/or $\mu^{30)}$. This guarantees the use of $b_{ll'}^{(2n)}$ 's¹⁸⁾ in the usual approximation with great accuracy³¹⁾. Unfortunately, this fact is verified only for the first forbidden transition with $\alpha Z \ll 1^*$. We are now engaged in removing this restriction.

*) This approximation is not so bad even in the case of RaE.

It is necessary to consider the correlation of $b_{l,l'}^{(2n)}$'s ($n \neq 0$) in $W(\theta : \text{I, II, III})$, when the β -spectra are measured by $\beta-\gamma_1-\gamma_2$ coincidence. If one wants to exclude this correlation in experiments, the β -counter has to be rotated over the whole solid angle.

To our regret, we cannot determine the relative sign of coupling constants of Scalar and Tensor interactions from our experiments. However, we may obtain a set of accurate and definite values of x, y, z ; therefore, this relative sign is deducible from these values with a least assumption on the model of nuclei. Further, these values of x, y, z may give us some information concerning the model of nuclei if they are accumulated in a sufficient number for various nuclei. Recent progress of the experiments may make this possible. The author hopes that the experiments will be performed in this field.

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Remarks on the Pion-Pion Interaction*

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The Cosmotron data on pion-nucleon reactions suggest the presence of a fairly strong pion-pion interaction near 1 Gev. The cosmic ray experiment performed recently by the Fristol group, when analyzed according to the Landau theory of multiple production, also leads to a similar conclusion.

§ 1. Introduction and summary

Recent progress in the experiment and the theory of meson reactions affords us a fairly clear image for the low energy—up to 200 Mev, say—behaviour of the nucleon-meson system: an extended heavy nucleon clothed with the meson cloud of relatively low frequency interacts rather strongly with the p -wave of the incident meson.

A new feature seems to appear, however, at somewhat higher energy. According to the Cosmotron experiments the incident pion of the energy near 1 Gev behaves as if it interacts not directly with the core of the target nucleon but with the virtual meson in the surrounding field, thus indicating the presence of an appreciable pion-pion interaction.

Another indirect support to the hypothesis of strong meson-meson interaction is given by the analysis of cosmic ray jet showers. It should be noticed that certain data of these events, though they are induced by primary particles of extremely high energy (at least $\gtrsim 10^3$ Gev), essentially reflect the properties of an assembly of interacting mesons at a definitely lower energy. Indeed, as will be seen in § 4, the energy concerned turns out to be lower than 1 Gev.

Thus the non-linear character of the pion field at this energy region seems—at least from the phenomenological point of view—remarkable and one will be able to estimate its order of magnitude rather easily, in contrast to the case of the quantum electrodynamics, where the corresponding effect, the Delbrück scattering** for example, is extremely small.

In the following I shall make rough estimation of this effect at the energy near 1 Gev and show that the values derived from accelerator experiments and cosmic ray evidences are not inconsistent with each other.

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A brief account appeared in Prog. Theor. Phys. 15 (1956), 294. The main contents of §§ 2-4 have been distributed in Japanese, Soryûsiron Kenkyu (mimeographed circular) 10 (1956), 491.

** I am indebted to Prof. S. Hayakawa for his comment on this effect.

In § 2 the arguments for the pion-pion interaction model suggested from the Cosmotron data are recapitulated and the order of magnitude of this interaction is conjectured. In § 3 it is shown that the cosmic ray data of the Bristol group can supply us with an information on the pion-pion interaction, and in § 4 the actual evaluation of the cross section is carried out. Assuming the energy-dependence of this pion-pion cross section, possible consequences are discussed in § 5.

§ 2. Inference from the Cosmotron data

The experimental data^{1),2)} on the reaction: $\pi + N \rightarrow \pi + \pi + N$ (N being a nucleon), induced by the 1.4~1.5 Gev pion beams from the Cosmotron, have been analyzed by Ito and Minami.³⁾ With regard to the character of the relevant transition matrix element in the center of mass system, they put the following two postulates:

i) it should have an appreciable value when and only when the nucleon suffers little deflection, and

ii) it should be nearly independent of the directions of the emitted pions, and confirm that these can explain the experimental angular distributions and energy spectra of pions and nucleons fairly well.

Then they examine existing "theories" and find that the hypothesis of strong pion-pion interaction, first proposed by Kovacs,⁴⁾ fulfils these requirements, while others, such as neutral $ps(ps)$ or charge-independent $ps(pv)$ theory, completely fail.

Based on their results, one can put, to a rough approximation,*

$$\sigma(\pi + N \rightarrow (n+1)\pi + N) \approx P \cdot \sigma(\pi + \pi \rightarrow \pi + n\pi) \quad (2.1)$$

$(n=1, 2, \dots),$

if one denotes by P the effective number of slow pions in the surrounding field of a nucleon.

It will not be unreasonable to assume, say, $P \approx 0.2-1.0$, so that we can conjecture, for the energy region concerned, as

$$\sigma_{\pi-\pi} \sim (\hbar/m_{\pi}c)^2, \quad (2.2)$$

where $\sigma_{\pi-\pi}$ means the total cross section of the $\pi-\pi$ interaction, because the total cross section for the inelastic $\pi-N$ scattering at 1.4 Gev can be estimated to be about $(25 \pm 3) \cdot 10^{-27} \text{ cm}^2$.

Another argument for the probable existence of such an interaction has been presented by Dyson⁵⁾ and Takeda⁶⁾ independently of each other. The pronounced maximum of the $\pi^- - p$ scattering cross section at about 900 Mev ($\sim 50 \text{ mb}$) and the absence of the corresponding peak in the $\pi^+ - p$ scattering cannot be explained as a resonance in a single state of the pion-proton system. Thus they introduce a strongly interacting two-pion state at this energy, the second pion being loosely bound in the field of the proton.

Dyson assumes a resonance state of isotopic spin 0, while Takeda chooses the state

*For example, we neglect the different factors for various charge states.

of isotopic spin 1. Independently of any such details, however, their models lead to the same order of magnitude as (2) for the *total* pion-pion cross-section in the energy region, say, 600 Mev–1.2 Gev, only if a reasonable value is given to the dissociation probability P of a nucleon into a pion and a nucleon. (Takeda takes $P=0.4$).

§ 3. Inference from the multiple meson production

—*The theory of Landau and the experiment of the Bristol group—*

We shall consider, according to the Fermi-Landau picture, the production process of a large number of mesons when a nucleon of extremely high energy collides with another nucleon or a nucleus.

Fermi's theory of multiple production in its original form⁷⁾ (thermodynamical approximation in the case of extremely high energy) describes the whole process as follows: in the Lorentz-contracted volume of the colliding nucleons (or a certain part of the nucleus) a thermodynamical equilibrium state will be realized at the instant of collision for a mixed gas of nucleons, antinucleons and various mesons, and then this state will be, so to speak, immediately "frozen up" in momentum space, so that what we shall observe in the stars, for example, recorded in emulsion plates is related just to this initial equilibrium. That is to say, the ratio of various kinds of particles produced, energy spectra and angular distributions of these particles are completely determined by the initial high temperature, which in turn depends on the incident energy, the angular momentum, and other constants of motion of the system.

Landau's modified version,⁸⁾ on the other hand, emphasizes the strong interaction between the produced particles after the initial equilibrium is attained: the nucleon-meson gas of high temperature and high pressure will spurt out of the above mentioned Lorentz-contracted volume according to the laws of relativistic hydrodynamics, keeping local thermal equilibrium in its individual parts, and while it is spreading out to a certain distance, the various parts of the cloud will successively cool down to a critical temperature, thus producing free particles which fly away to the outer space. Therefore our observational data (ratio of various kinds of particles, their energy and angular distributions) will now reflect the properties of the nucleon-meson gas in its critical state, i.e., at a definitely lower temperature than in Fermi's original treatment.

Owing to this essential difference, one can discriminate between these two theories experimentally. Indeed such an investigation has been projected and performed by the Bristol group.⁹⁾ They made use of a large emulsion-stack and carried out a detailed analysis on the nature of the secondary particles produced in jet showers (19 events) which had been induced by primary particles of energy $10^3 \sim 10^4$ Gev, and obtained

$$N_y/N_{\pi^0} = 0.65 \pm 0.25, \quad (3.1)$$

where N_y denotes the number of secondary stars produced by neutral secondaries and N_{π^0} denotes the number of neutral pions (detected by electron pairs and subsequent cascade showers). The above value should give an upper limit to the ratio N_{π^0}/N_{π^0} , N_{π^0} being

the number of neutral heavy mesons, since besides θ^0 's neutrons and antineutrons could act as star-producing secondary rays.

In order to derive theoretical predictions of this ratio, it is necessary to know the statistical weight of heavy mesons. Little can be said about the spins of these particles. As for their isotopic spin and strangeness, however, one can with some confidence adopt Nishijima¹⁰⁾-Gell-Mann¹¹⁾'s scheme: $I(\theta) = 1/2$, charge of $\theta = 1/2 + I_z(\theta)$, since not only the data of the same experiment of the Bristol group are consistent with it, but also no experimental evidence has so far been obtained that would contradict with it. Then the statistical weight g_{θ^0} of neutral heavy meson cannot be less than 2.

Now, even if we take $g_{\theta^0} = 2$, Fermi's theory gives, for the incident energy $10^3 \sim 10^4$ Gev and the target nucleus of $A \lesssim 108$,

$$N_y/N_{\pi^0} \gtrsim 3.5, \quad (3.2)$$

or subtracting the calculated contribution of neutrons and antineutrons,

$$N_{\theta^0}/N_{\pi^0} \gtrsim 1.5, \quad (3.3)$$

in definite disagreement with the experimental value (3.1).

Thus Fujimoto¹²⁾ could suggest that the modification of the Fermi theory proposed by Landau would indeed be necessary.

Another argument in favour of Landau's theory has been more recently given by Nishimura.¹³⁾ He has analyzed the distribution of transverse momenta carried by secondary particles emitted from the jet stars and has shown that they are of the order of several hundreds Mev, in apparent contradiction with what could have been expected from the high-temperature equilibrium hypothesis of Fermi. This analysis is of particular importance, because it will rule out the possibility of fitting the experimental value (3.1) into the frame of Fermi's theory by the help of an assumption that the heavy mesons were composite particles.

Now that we are to agree to Landau's view, we should picture the process of meson formation in the following way: While the nucleon-meson gas of high temperature and high pressure gradually cools down, the mean free paths of nucleons and various mesons, which are primarily determined by the mutual interaction of mesons, gradually increase until they become comparable with the dimension of the system ($\sim h/m_\pi c$), which instant should just correspond to the critical state and formation of free mesons. Thus we have a means of evaluating the order of magnitude of the pion-pion interaction for the average energy of pions in the cloud of critical state, since the experimental value (3.1) allows us to determine, within a certain limit of accuracy, the value of the critical temperature T_c , the latter, in turn, giving energy density and number density of the critical state.

A detailed calculation along the line of the above reasoning will be carried out in the next section.

§ 4. Inference from the multiple meson production (continued)

—Calculation and results—

A system composed of a large number of π -mesons and heavy mesons, which are

incessantly interacting with each other and keeping local equilibrium, can be approximately described as an ideal Bose gas. The particle density of this gas ρ is expressed by¹⁾

$$\rho = \sum_i \rho_i, \quad (4.1)$$

$$\begin{aligned} \rho_i &= \frac{g_i}{2\pi^2} \left(\frac{m_i c}{\hbar} \right)^3 \int_0^\infty \frac{x^2 dx}{\exp(m_i c^2 / kT \cdot \sqrt{1+x^2}) - 1} \\ &= (g_i / 2\pi^2) (kT / \hbar c)^3 F(m_i c^2 / kT), \end{aligned} \quad (4.2)$$

with

$$F(x) = x^2 \sum_{\nu=0}^\infty \frac{K_\nu((1+\nu)x)}{1+\nu}. \quad (4.3)$$

Here the suffix i represents various kinds of mesons (π, θ, \dots), m_i and g_i are the mass and the statistical weight of the corresponding meson, respectively. $K_\nu(x)$ is the modified Bessel function of the second kind.

It should be here remarked that the conservation law for the strangeness (or η -charge in Nishijima's terminology¹⁾) does not affect the distribution function (4.2) in this case, because the initial condition for the total strangeness (η -charge) is just zero.

First we shall assume that the spin of the heavy meson is zero; then we have

$$g_{\theta^0} = 2, \quad g_{\pi^0} = 1, \quad (4.4)$$

as was explained in the previous section. Further we make use of the experimental value of the mass ratio,

$$m_\theta / m_\pi \approx 3.5. \quad (4.5)$$

Then the production ratio of θ^0 - and π^0 -particles will be expressed by

$$\frac{N_{\theta^0}}{N_{\pi^0}} = \frac{g_{\theta^0}}{g_{\pi^0}} \frac{F(3.5x_c)}{F(x_c)}, \quad (4.6)$$

with

$$x_c = \frac{m_\pi c^2}{kT_c}. \quad (4.7)$$

With the help of Table 1, one compares (4.6) with the experimental value (3.1) and obtains

$$kT_c \approx (1.2 \pm 0.3) m_\pi c^2. \quad (4.8)$$

Now the average energy of pions in the cloud at this critical temperature can be easily estimated. The energy density $\varepsilon(T)$ of the Bose gas at temperature T is given by¹⁾

$$\begin{aligned} \varepsilon(T) &= mc^2 \left(\frac{mc}{\hbar} \right)^3 \frac{g}{2\pi^2} \int_0^\infty \frac{\sqrt{1+x^2} x^2 dx}{\exp(mc^2 / kT \cdot \sqrt{1+x^2}) - 1} \\ &= \frac{g}{2\pi^2} \frac{(kT)^4}{(\hbar c)^3} G\left(\frac{mc^2}{kT}\right), \end{aligned} \quad (4.9)$$

Table 1

x	$F(x)$	$F(3.5x)$	$N_{\theta 0}/N_{\pi 0}=2F(3.5x)/F(x)$
0.0	2.4	2.4	2.0
0.1	2.4	2.3	1.9
0.2	2.3	2.0	1.7
0.3	2.3	1.7	1.5
0.4	2.2	1.5	1.3
0.5	2.2	1.2	1.1
0.6	2.1	1.0	0.95
0.7	2.0	0.80	0.79
0.8	1.9	0.64	0.66
0.9	1.9	0.51	0.55
1.0	1.8	0.40	0.45
1.1	1.7	0.31	0.37
1.2	1.6	0.24	0.30
1.3	1.5	0.19	0.24
1.4	1.5	0.14	0.20
1.5	1.4	0.11	0.16

where

$$G(x) = x^2 \sum_{\nu=0}^{\infty} \frac{1}{(1+\nu)} \left\{ \frac{3K_2((1+\nu)x)}{1+\nu} + xK_1((1+\nu)x) \right\}. \quad (4.10)$$

The energy per particle becomes, therefore,

$$E_{\pi} = \frac{\varepsilon_{\pi}(T_c)}{\rho_{\pi}(T_c)} = kT_c \frac{G(m_{\pi}c^2/kT_c)}{F(m_{\pi}c^2/kT_c)}, \quad (4.11)$$

which turns out, using the value (4.8) for T_c ,

$$E_{\pi}(T_c) \approx (4 \pm 1)m_{\pi}c^2. \quad (4.12)$$

(See Table 2.) This is somewhat lower than the energy of the Cosmotron beam discussed in § 2. (Notice that (4.12) includes also the rest energy.)

Let us next conjecture the order of magnitude of the pion-pion interaction at this energy region. At a temperature so low that one can neglect the contribution of the nucleon-antinucleon pairs, the mean free paths l_{π} , l_{θ} of π -mesons and θ -mesons are determined by

$$\begin{aligned} l_{\pi}(T) &= \{\rho_{\pi}(T)\sigma_{\pi-\pi}(T) + \rho_{\theta}(T)\sigma_{\pi-\theta}(T)\}^{-1}, \\ l_{\theta}(T) &= \{\rho_{\pi}(T)\sigma_{\pi-\theta}(T) + \rho_{\theta}(T)\sigma_{\theta-\theta}(T)\}^{-1}. \end{aligned} \quad (4.13)$$

ρ_{π} and ρ_{θ} here include neutral as well as charged particles, that is to say, one has to put $g_{\pi}=3$ and $g_{\theta}=4$ in (4.2).

By the definition of the critical temperature in Landau's theory of multiple production,*

* A more detailed analysis indicates¹⁵⁾ that the dimension of the system and consequently $l(T_c)$ depends very slightly on the incident energy ($\sim E^{1/2}$); but here I shall neglect this dependence.

Table 2

x	$G(x)$	$E_\pi/m_\pi c^2 = (1/x) \langle G(x)/F(x) \rangle$
0.0	6.5	
0.1	6.5	27.
0.2	6.5	14.
0.3	6.4	9.3
0.4	6.4	7.2
0.5	6.3	5.8
0.6	6.2	5.0
0.7	6.1	4.3
0.8	6.0	3.9
0.9	5.9	3.5
1.0	5.8	3.3
1.1	5.7	3.0
1.2	5.5	2.8
1.3	5.4	2.7
1.4	5.2	2.6
1.5	5.1	2.4

$$l_\pi(T_c) \approx l_0(T_c) \approx \hbar/m_\pi c. \quad (4.14)$$

It will not be unreasonable to assume that $\sigma_{0-\pi}$ or σ_{0-0} cannot be extraordinarily larger than $\sigma_{\pi-\pi}$. Then one has

$$l_\pi(T_c) \lesssim \{\rho_\pi(T_c) \sigma_{\pi-\pi}(T_c)\}^{-1}, \quad (4.15)$$

whence

$$\begin{aligned} \sigma_{\pi-\pi}(T_c) &\lesssim \frac{m_\pi c}{\hbar} \frac{1}{\rho_\pi(T_c)} \\ &\approx (2.5 \pm 2.0) (\hbar/m_\pi c)^2 \\ &\sim (\hbar/m_\pi c)^2. \end{aligned} \quad (4.16)$$

This conclusion is not inconsistent with the value derived in § 2. Such a circumstance may be regarded as an indirect support both to Landau's point of view and to the assumption spin 0 for heavy mesons.

Indeed if we take the case of spin 1 heavy mesons, i.e., $g_0=6$, the same argument as above yields the following results:

The critical temperature is given by

$$kT_c = (0.75 \pm 0.10) m_\pi c^2. \quad (4.17)$$

The average energy of pions at this temperature becomes

$$E(T_c) = (2.6 \pm 0.3) m_\pi c^2, \quad (4.18)$$

and the pion-pion cross section for this energy region is estimated to be

$$\begin{aligned}\sigma_{\pi-\pi} &\approx (14 \pm 7) (\hbar/m_{\pi}c)^2 \\ &\sim 10 \cdot (\hbar/m_{\pi}c)^2.\end{aligned}\quad (4 \cdot 19)$$

It appears that this value may be a little too large not to affect the low energy (220 Mev or so) nucleon-pion scattering, though, of course, more detailed examination is required on this point.

The assumption of spin $\frac{1}{2}$ would yield $\sigma_{\pi-\pi} \gtrsim 15 (\hbar/m_{\pi}c)^2$ and perhaps may be excluded.

So far only one kind of heavy mesons has been taken into account. The existence of many kinds of mesons, such as suggested at first sight by various decay modes, leads naturally to a large statistical weight. One might say that two kinds of spinless heavy mesons ($g_{\theta\theta}=4$) is the upper limit that could be consistent with the above-analyzed experimental data. It can, however, be that other heavy mesons appear as composite particles, since such particles would occupy a very limited region of the phase space.

§ 5. Energy dependence of the meson-meson interaction

As was described in § 2, Dyson¹⁾ and Takeda¹⁰⁾ assumed a resonance state of two pion system at about 900 Mev (in the rest system of one pion), but this was related to a particular value of the isotopic spin ($T=0$ and 1 respectively). If one averages over all possible charge states, the assumed pion-pion cross section shows only a weak energy-dependence.

The analysis in the foregoing two sections was meant, on the other hand, only to indicate the probable existence of the pion-pion interaction and was obviously too crude to say anything on its energy-dependence.

In this section, on the contrary, let us first assume the energy-dependence of the average pion-pion interaction in a simple form

$$\sigma_{\pi-\pi}(E) = (\hbar/m_{\pi}c)^2 (E/E_0)^n, \quad (5 \cdot 1)$$

where E denotes the total energy of the incident pion in the rest system of the target pion, E_0 being its value at the critical temperature, and see what consequences can be expected from such an assumption. (The Cosmotron data seem to favour the case $n \approx 0$. The simplest non-linear equation of meson field with the term $(\varphi_a \varphi_a)^2$, φ being the meson wave function, leads to $n=-1$ for $E \gg m_{\pi}c^2$.)

In the following, for simplicity, the nucleon-meson gas will be treated as if it were composed of pions only,* so that the results obtained will be meant only for suggesting the qualitative features of the problem.

First the variation of the mean free path of the pions in the cloud is evaluated. The curves for $n=-2, -1, 0, 1, 2$ and 3 are given in Fig. 1. They are normalized to $\hbar/m_{\pi}c$ at $m_{\pi}c^2/kT=0.9$. As is easily expected, a positive large value for n defines a

* The effect of heavy mesons is taken into account as far as the critical temperature and the corresponding mean free path is normalized so as to agree with the results of the preceding sections.

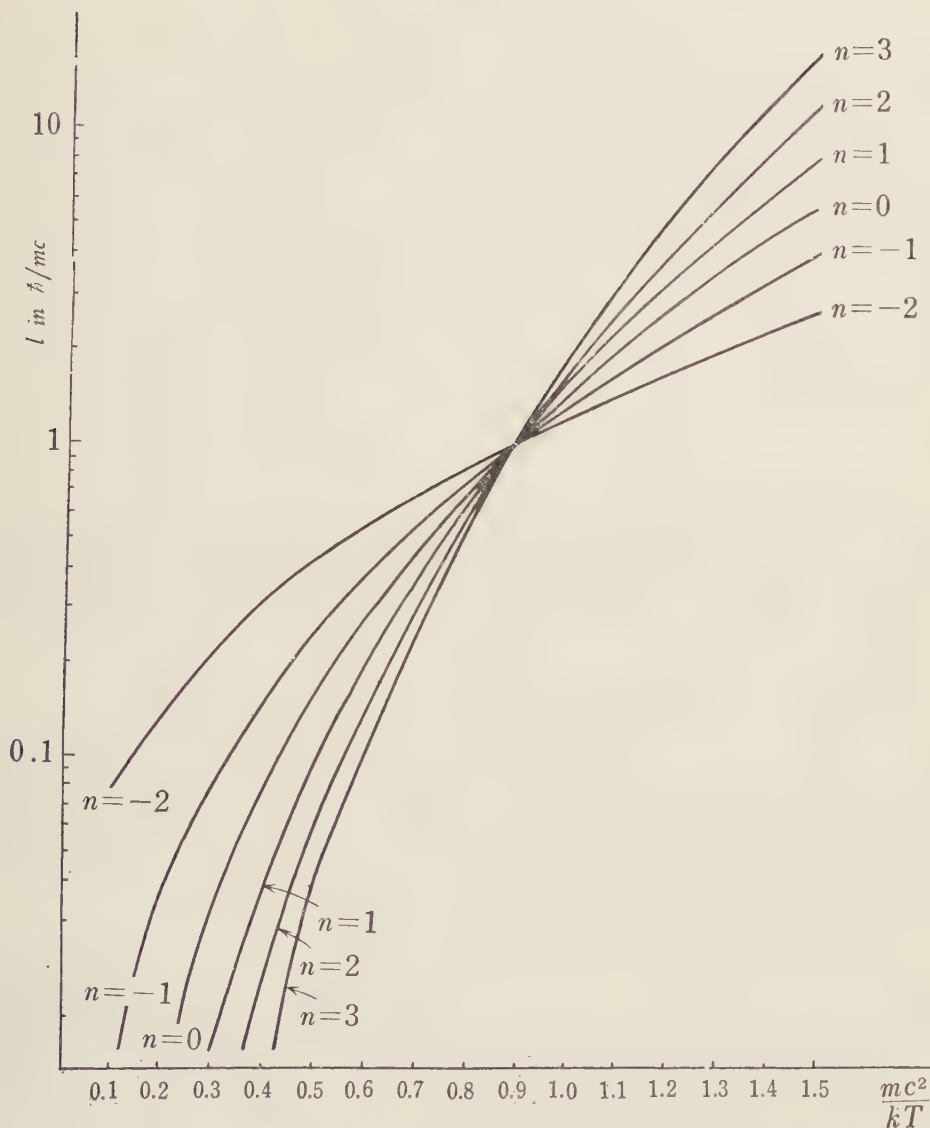


Fig. 1. Mean free path in meson cloud
 $\sigma \propto E^n$, $n = -2, -1, 0, 1, 2, 3$
 normalized at $mc^2/kT = 0.9$

clear-cut critical temperature, while for a large negative n the cloud dissolves rather sluggishly.

Suppose, for instance, that free particles fly out when l becomes $0.3 \sim 1.1$ times $\hbar/m_\pi c$, then we get the region of the allowed critical temperature and the corresponding fluctuations in the number ratio of θ^0 - to π^0 - mesons,* which are shown in Table 3

* Cf. the footnote of the preceding page.

(spin 0 for θ is assumed). It appears as if the experimental data would not favour negative n -values, though nothing definite can be said at present.

Next let us make a rough estimate of the viscosity of meson gas and the work done by viscous forces during the process of multiple production. In his famous work⁽¹⁵⁾ Landau solved the hydrodynamical problem neglecting all dissipative processes. If the

Table 3

n	kT_0 in (mc^2)	$N_{\theta 0}/N_{\pi 0}$
-2	0.8 ~ 2.5	0.25 ~ 1.3
-1	0.85 ~ 1.8	0.3 ~ 1.0
0	0.9 ~ 1.6	0.35 ~ 0.95
1	0.95 ~ 1.5	0.3 ~ 0.8
2	0.95 ~ 1.4	0.4 ~ 0.8
3	1.0 ~ 1.3	0.45 ~ 0.75

viscous forces should overwhelm the inertial forces during a greater part of the expansion process, Landau's basic assumption of isentropic motion would break down, and one would expect a larger multiplicity,⁽¹⁶⁾ somewhat similar to the results of Heisenberg's theory.⁽¹⁷⁾ As we shall see in the following, however, Landau's treatment is valid for the case $n \geq -1$ if the incident energy is large enough.

The viscosity η in the relativistic region can be defined by the relation:

$$\text{momentum transport} \propto \eta (\partial u / \partial x), \quad (5.2)$$

where u is a certain component of the "macroscopic" four-velocity (Here I shall not discriminate between longitudinal and transverse viscosities, because we are interested only in the order of magnitude.). Then η is related to the mean free path of the constituent particles through the following expression:

$$\eta \approx \rho v l m, \quad (5.3)$$

where v is the mean velocity of the thermal motion of particles and can be regarded as $\approx c$ in our case.

A measure for the role played by dissipative effects* is given by the well-known Reynolds number, which represents the ratio of the effects of inertial to those of viscous forces and is expressed, in our relativistic case, by**

* In the extremely relativistic case, where no essential distinction between mass flow and energy flow is possible, the effect of heat conduction can be disregarded. Indeed, if the 4 velocity of the fluid is properly defined, this effect turns out to be proportional to the chemical potential,⁽¹⁸⁾ which one can neglect here.

** The energy momentum tensor for the viscous fluid is given by⁽¹⁸⁾

$$\begin{aligned} T_{ik} = & (p + \varepsilon) u_i u_k + p g_{ik} \\ & - \eta c^2 \left(\frac{\partial u_i}{\partial x^k} + \frac{\partial u_k}{\partial x^i} + u_k u^l \frac{\partial u_i}{\partial x^l} + u_i u^l \frac{\partial u_k}{\partial x^l} \right) \\ & - \left(\zeta - \frac{2}{3} \eta \right) c^2 \frac{\partial u^l}{\partial x^l} (g_{ik} + u_i u_k). \end{aligned}$$

The inertial effect may be represented by εu^2 and the viscous effect by $c^2 \eta (\partial u / \partial x) \sim c^2 \eta u / L$, whence (5.4).

$$R = \varepsilon u L / \eta c^2. \quad (5.4)$$

Here L is a length that characterizes the problem under consideration (More precisely, L should be taken equal to the length, during which the value of u changes appreciably: $\Delta u \sim u$.)

Putting (5.3) into (5.4) and making use of (4.2), (4.9), (4.15) and (5.1), one obtains

$$R \approx \frac{\eta}{2\pi^2} \frac{u L \hbar^2}{m^3 c^5} \frac{(kT)^4}{(\hbar c)^3} \left(\frac{T}{T_c} \right)^n \frac{G^{n+1}}{G_c^n} \cdot \frac{F_c^n}{F^n}, \quad (5.5)$$

where the suffix c indicates the values at the critical temperature T_c . One has, according to Landau,⁹⁾ at the last stage of the hydrodynamical motion (stage of conical flight),

$$u = u_1 (T_1/T),$$

and

$$t = t_1 (T_1/T), \quad (5.6)$$

where the suffix 1 means the values corresponding to the limit of one-dimensional motion. One can put here $L \approx ct$ and also regard F 's and G 's as slowly varying quantities, and then, by the help of (5.6), gets ultimately***

$$R \propto T^{n+2} \quad (5.7)$$

Now we see that R decreases with decreasing temperature through the whole process of expansion if $n \geq -1$. When the value of R at T_c is sufficiently large, therefore, we need no longer bother about the effect of viscosity. But this is certainly the case when the incident energy W in the centre of mass system is sufficiently large, because

$$\begin{aligned} R_c &= \varepsilon_c u_c L_c / \eta_c c^2 t_c m \\ &= (E_c / mc^2) u_c / c, \end{aligned} \quad (5.8)$$

and the average value of u_c , which is roughly proportional to the mean energy of emitted mesons and so proportional to the square root of W , increases with W . We have therefore, at energy $\gtrsim 10^3$ Gev, in the laboratory system, say, quite a large value of R , which justifies the neglect of dissipative effects.

In conclusion I should like to express my sincere gratitude to Prof. Y. Fujimoto and to Dr. J. Nishimura for their kind information on the results of their analysis before publication and for their discussions and also to Dr. S. Minami for his discussions on Gev phenomena.

*** The dependence of R on T at the earlier stage (one-dimensional motion) is complicated, but is not weaker than $\propto T^{n+1}$.

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1) The experimental value (3.1) has been obtained with the assumption that only π^0 -mesons contribute to γ -rays. If secondary particles contain hyperons and heavy mesons which immediately decay with emission of γ -rays the ratio will somewhat increase. On the other hand there are some experimental informations* which imply that this ratio has a lower value than (3.1).

2) In deriving (3.2), (3.3) and (4.6) — (4.16), the statistical weight of θ^0 -mesons is taken to be 2. That is to say, we have θ^0 's and $\bar{\theta}^0$'s which differ from each other by their strangeness quantum numbers, and we have assumed that they will be equally detected experimentally. If it should turn out, for instance, that one of them were very hard to detect at the energy concerned, the effective value of g_{θ_0} should have been taken equal to 1 and this should have led to the results: $kT \approx 2.5m_{\pi}c^2$, $E_{\pi}(T_c) \approx 7m_{\pi}c^2$, $\sigma_{\pi-\pi} \approx 0.2(\hbar/m_{\pi}c)^2$.

3) The determination of the critical temperature (4.8) by the number ratio of heavy to light mesons is fairly accurate. It should not be overlooked, however, that at the critical state the concept of the temperature itself becomes also critical, so that the numerical values concerned should not be taken too seriously.

4) Recent experimental evidences** seem to suggest that the pion-pion interaction, if it exists, will radically depend on the charge states and the energy. Thus the assumption (5.1) may be perhaps too simple for lower energy region (~ 1 Gev).

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On the Relativistic Thomas-Fermi Atom

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The method of March for obtaining the boundary values of the Thomas-Fermi function at high pressures has been extended to the relativistic case with and without temperature perturbation. The boundary values of the non-relativistic temperature perturbed Thomas-Fermi function, obtained by the same method, are compared with the semi-empirical formulae of Gilvarry, thus obtaining a theoretical basis for the latter. The results of this paper may have some astrophysical applications.

§ 1. Introduction

The method of Thomas¹⁾ and Fermi²⁾, (hereafter referred to as T.F.), devised to find the potential and electron distribution around an atomic nucleus, has been employed to the study of metals by Slater and Krutter³⁾, and to the study of equation of state of elements at high pressure by Feynman, Metropolis and Teller⁴⁾. The boundary values of the T.F. function are sufficient to determine the equation of state of the atom; and hence the numerical data of Slater and Krutter, and of Feynman, Metropolis and Teller, for the boundary values have been fitted by Gilvarry⁵⁾ and Gilvarry and Peebles⁶⁾ into semi-empirical analytical expressions with proper limiting forms, so that one can extrapolate them beyond the existing numerical data. The arguments by which Gilvarry arrived at the analytical expressions are essentially intuitive in character, and are based on the observation that the results of the T.F. method must go over into those appropriate for a Fermi-Dirac gas of completely degenerate electrons when pressures are sufficiently high. Recently March⁷⁾ has given a theoretical basis for Gilvarry's expressions, and in addition has indicated the way in which the non-relativistic equation of state of elements is modified at high pressures. But at high pressures such as those that occur in the interior of a white dwarf like the companion of Sirius B, the volume available for the electrons of an atom is so small that the maximum Fermi momentum is larger than $m_e c$, so that relativistic effects become important. It is therefore necessary to investigate the relativistic T.F. equation, when one deals with extremely high pressures. In this paper, we shall use March's method to the investigation of the relativistic T.F. equation, and also include the temperature perturbation. We shall confirm Gilvarry's choice of the limiting form of T.F. function for the case of temperature perturbed non-relativistic T.F. equation.

Our results may be useful in determining the equation of state of matter under very extreme physical conditions and may find some astrophysical applications.

§ 2. Relativistic T.F. equation

The temperature perturbed relativistic T.F. equation can be deduced by using the general expression for the temperature dependent electron concentrations as given by Chandrasekhar⁽¹⁾ in defining the charge density used in the Poisson equation for the atomic potential. If we retain terms up to order T^2 only, we have for the electron concentration,

$$n = \frac{8\pi^3 m^3 c^3}{3h^3} \gamma^3 \left[1 + \left(\frac{\pi k T}{mc^2} \right)^2 \frac{2\gamma^2 + 1}{\gamma^4} \right], \quad (1)$$

$$\text{where} \quad \gamma = (\hat{\xi}_0^2 + 2mc^2\hat{\xi}_0)^{1/2}/mc^2,$$

and $\hat{\xi}_0$ is the energy corresponding to the top of the Fermi level, k is the Boltzman constant, and the other symbols have their usual meaning. The charge density will be modified in the presence of a potential V , with $\hat{\xi}_0$ replaced by $\hat{\xi}_0 + eV$. One can show, following Singh⁽²⁾, that the temperature perturbed relativistic T.F. equation takes the form

$$d^2\phi/dx^2 = \alpha (\phi^3/x^2) (1 + \beta x/\phi)^{3/2} \times \left[1 + (\pi k T)^2 \frac{\left(\frac{0^2}{x^2} \right) \left(1 + \frac{\beta x}{\phi} \right) + \frac{\hat{\xi}_0^2}{8}}{\frac{(2mc^2)^2}{\hat{\xi}_0^2} \cdot \frac{\phi^4}{x^4} \left(1 + \frac{\beta x}{\phi} \right)^2} \right] \quad (2)$$

where ϕ is defined by

$$(Ze^2/r)\phi(r) = \hat{\xi}_0/e + V(r),$$

$$\text{and} \quad \alpha = 32\pi^2 Z^2 e^8 / 3 (hc)^3; \beta = 2mc^2 a / Ze^2; a = \hbar^2 / me^2, \quad (2a)$$

$$\text{and} \quad r = ax.$$

In writing equation (2) we have neglected terms arising on account of exchange effects which would be small for atoms with high atomic number. Let us consider the temperature independent relativistic T.F. equation. If we neglect the temperature term in (2), we have the equation

$$d^2\phi/dx^2 = \alpha (\phi^3/x^2) (1 + \beta x/\phi)^{3/2}. \quad (3)$$

In applying the T.F. equation for high pressures, we impose the following two boundary conditions on the solution :

$$\phi(0) = 1 \quad (4a)$$

$$(d\phi/dx)_{x_b} = \phi(x_b)/x_b, \quad (4b)$$

where x_b defines the boundary of the atom. Equation (3) reduces to the non-relativistic form if we neglect 1 in comparison with $\beta x/\phi$. We then have

$$d^2\phi/dx^2 = \alpha \beta^{3/2} \cdot \phi^{3/2}/x^{1/2}. \quad (5)$$

If we now write

$$\alpha^{2/3} \beta x = x^*, \quad (6)$$

equation (5) reduces to the more familiar form

$$d^2\phi/dx^{*2} = \phi^{3/2}/x^{*1/2}, \quad (7)$$

where distances are measured in units of $(a/\alpha^{2/3}\beta) = (9\pi^2/128Z)^{1/3} \cdot a$

$$\text{i.e.} \quad r = (9\pi^2/128Z)^{1/3} ax^*.$$

Following March, we solve equation (3) for $\phi(x)$ as a Taylor series about x_b , subject to the boundary conditions (4a, b).

$$\phi(x) = \phi(x_b) + \sum_{n=1}^{\infty} t_n h^n, \quad (8)$$

where

$$t_n = (-1)^n/n! \cdot (d^n\phi/dx^n)_{x_b},$$

and

$$h = x_b - x.$$

Substituting (8) in (3) we get for t_n 's

$$\left. \begin{aligned} t_1 &= -\left(\frac{d\phi}{dx}\right)_{x_b} = -\frac{\phi'(x_b)}{x_b} \\ t_2 &= (\alpha/2) \cdot \frac{\phi^3(x_b)}{x_b^3} \left(1 + \frac{\phi''(x_b)}{\phi'(x_b)}\right)^{3/2} \\ t_3 &= -\frac{\alpha}{6} \frac{\phi^3(x_b)}{x_b^3} \left(1 + \frac{\beta x_b}{\phi(x_b)}\right)^{3/2} \\ t_4 &= \frac{\alpha^2}{8} \left(1 - \frac{\eta}{2}\right) \frac{\phi^5(x_b)}{x_b^4} \left(1 + \frac{\beta x_b}{\phi(x_b)}\right)^3 \\ t_5 &= -\frac{\alpha^2}{40} \left(1 - \frac{\eta}{2}\right) \frac{\phi^5(x_b)}{x_b^5} \left(1 + \frac{\beta x_b}{\phi(x_b)}\right)^3 \\ &\dots \dots \dots \end{aligned} \right\} \quad (9)$$

(coefficients of Taylor series solution of equation (3)).

where

$$\eta = \{\beta x_b/\phi(x_b)\} / \{1 + \beta x_b/\phi(x_b)\}, \quad (10)$$

which tends to unity in the non-relativistic limit.

Within the limitations as stated in March's paper, we can use the five coefficients of (9) to determine the form of $\phi(x_b)$ in the limit $x_b \rightarrow 0$.

The boundary condition (4a) is automatically satisfied by the solution (8). The other boundary condition (4a) —if we take terms up to t_5 , gives for $\phi(x_b)$ the expression

$$\phi(x_b) = (\beta x_b/2) [\{1 + 4 \cdot 3^{2/3}/\alpha^{2/3}\beta^2 x_b^2\}^{1/2} - 1], \quad (11)$$

as $x_b \rightarrow 0$ and which in the non-relativistic limit reduces to

$$\phi(x_b) = 3^{3/2}/\alpha^{2/3}\beta x_b, \quad (12)$$

agreeing with the 1st term of equation (15) of March⁷ if we note that our units are related to those of March by equation (6). Let us call $\phi(x_b)$ given by equation (11) as $\phi^{(1)}(x_b)$. Taking t_4 and t_5 coefficients of the series, we can show that

$$\phi(x_b) = \frac{\beta x_b}{2} \left[\left\{ 1 + \frac{4 \cdot 3^{2/3}}{\alpha^{2/3} \beta^2 x_b^2} \left(1 - \frac{9(1-\eta/2)}{10 \phi^{(1)}(x_b)} \right)^{2/3} \right\}^{1/2} - 1 \right], \quad (13)$$

which in the non-relativistic limit reduces to

$$\phi(x_b) \xrightarrow{\text{N.R.}} \frac{3^{2/3}}{\alpha^{2/3} \beta x_b} \left(1 - \frac{3^{1/3} \alpha^{2/3} \beta x_b}{10} \right), \quad (14)$$

and which agrees with equation (15) of March⁷.

So the correct expression in the limiting values of x_b should be (13). It is, however, too complicated to be of much use in practice. Let us consider the extreme relativistic case. In this limit we have $\beta x / \phi \ll 1$, and one can show that in this limit

$$\phi(x_b) \xrightarrow{\text{E.R.}} (3/\alpha)^{1/3} \left[1 - (\alpha^{1/3} \beta x_b / 2 \cdot 3^{1/3}) + (\alpha^{2/3} \beta^2 x_b^2 / 8 \cdot 3^{2/3}) (1 - (3/10)(\alpha/3)^{1/3}) + \dots \right] \quad (15)$$

The series (15) is valid only if

$$\alpha^{1/3} \beta x_b \ll 1. \quad (16)$$

$$\text{From (2a),} \quad \alpha^{1/3} \beta = \hbar c / e^2 (32/3\pi Z)^{1/3} < \hbar c / e^2$$

for cases of large Z .

Therefore condition (16) becomes

$$x_b \leq e^2 / \hbar c. \quad (17)$$

Since we expressed distances in units of $b^* m c^2 = 0.528 \times 10^{-8}$ cm (17) shows that the volume available for electrons is a sphere of radius $< 5 \times 10^{-11}$ cm.

In writing the equation of state based on the T.F. model, one needs only the electron concentration at the boundary of the atom. Expressed in terms of the boundary values, the electron concentration is

$$n(x_b) = \frac{Z}{4\pi} \frac{\alpha}{(ax_b)^3} \phi^3(x_b) \left\{ 1 + \frac{\beta x_b}{\phi(x_b)} \right\}^{3/2}. \quad (18)$$

Using the limiting form $\phi(x_b)$ (E.R. case) from equation (15), it can easily be shown that

$$n(x_b) = \frac{Z}{(4\pi/3)(ax_b)^3} \left[1 - \frac{\hbar c}{e^2} \frac{x_b^2}{5\pi} \right]. \quad (19)$$

Since $x_b \leq e^2 / \hbar c$, (19) shows that $n(x_b)$ is merely the average concentration of Z electrons enclosed in a sphere of radius ax_b minus a correction term of the order less than the fine structure constant. Hence in the extreme relativistic limit the equation of state differs from that of a Fermi-Dirac gas by a term of the order less than the fine structure constant.

§ 3. Temperature perturbed T.F. equation

To investigate the temperature perturbed equation (2) we assume that the volume of the atom to be the same as that of temperature unperturbed atom. Following the same procedure as before, we write the solution of (2) in the form of a Taylor's series (8), and get for t_n 's, the following values:

$$\begin{aligned}
 t_1 &= -(d\phi/dx)_{x_b} \equiv -\phi(x_b)/x_b \\
 t_2 &= \frac{\alpha \beta^{3/2} \phi^{3/2}(x_b)}{2x_b^{1/2} \eta^{3/2}} \left[1 + \frac{(\pi k T \beta)^2}{\mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}} \right] \\
 t_3 &= -\frac{\alpha \beta^{3/2} \phi^{3/2}(x_b)}{6x_b^{3/2} \eta^{3/2}} \left[1 + \frac{(\pi k T \beta)^2}{\mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}} \right] \\
 t_4 &= \frac{\alpha^2 \beta^3 \phi^3(x_b)}{24x_b \eta^3} \left[1 + \frac{(\pi k T \beta)^2}{\mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}} \right]^2 \\
 &\quad \times [(1 - \eta/2) \{ 3 + 4\lambda(\mu/2 - 1) \}] \\
 t_5 &= -\frac{\alpha^2 \beta^3 \phi^3(x_b)}{120x_b^2 \eta^3} \left[1 + \frac{(\pi k T \beta)^2}{\mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}} \right]^2 \\
 &\quad \times [(1 - \eta/2) \{ 3 + 4\lambda(\mu/2 - 1) \}], \\
 &\dots\dots\dots
 \end{aligned} \tag{20}$$

(coefficients of Taylor series solution of (2))

where

$$\mu = \frac{(\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b))}{(\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) + \beta^2/8},$$

and

$$\lambda = \frac{(\pi k T \beta)^2 / \mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}}{1 + [(\pi k T \beta)^2 / \mu (2mc^2)^2 \{ (\phi^2(x_b)/x_b^2) (1 + \beta x_b/\phi(x_b)) \}]}.$$

In the non-relativistic limit when $\beta x/\phi \gg 1$

$$\left. \begin{aligned}
 &\mu \rightarrow 8\phi(x_b)/\beta x_b \ll 1 \\
 \text{and} \quad &\lambda \rightarrow \frac{(\pi k T a)^2 x_b^2 / 8 Z^2 e^4 \phi^2(x_b)}{1 + (\pi k T a)^2 x_b^2 / 8 Z^2 e^4 \phi^2(x_b)} \Bigg\}.
 \end{aligned} \right\} \tag{21}$$

By taking coefficients up to t_3 and proceeding as before we can show that in the extreme relativistic limit,

$$\phi(x_b, T) = \left(\frac{3}{\alpha}\right)^{1/3} \left[1 - \frac{\alpha^{1/3} \beta x_b}{2 \cdot 3^{1/3}} - \frac{1}{6} \left(\frac{\alpha}{3}\right)^{2/3} \frac{(\pi k \beta)^2}{(2mc^2)^2} x_b^2 T^2 + \dots \right]. \tag{22}$$

Since x_b is restricted by (17), equation (22) puts the following condition on temperature

$$kT/2mc^2 \sim 1 \quad \text{or} \quad T \sim 10^5 K. \quad (23)$$

In the general case, it has not been possible to obtain explicit form of $\phi(x_b)$ as $x_b \rightarrow 0$ due to the complexity of coefficients (20).

§ 4. Temperature perturbed non-relativistic T.F. equation

In the non-relativistic limit equation (2) has the form

$$d^2\phi/dx^2 = \alpha\beta^{3/2} \frac{\phi^{3/2}}{x^{1/2}} \left(1 + \frac{(\pi kTa)^2}{8Z^2e^4} \cdot \frac{x^2}{\phi^2} \right), \quad (24)$$

which on using equation (6) becomes

$$d^2\phi/dx^{*2} = \phi^{3/2}/x^{*1/2} \cdot (1 + \xi T^2 \cdot x^{*2}/\phi^2), \quad (25)$$

where

$$\left. \begin{aligned} \xi &= \frac{1}{8} (\pi \epsilon k / Z e^2)^2, \\ \text{and} \quad \epsilon &= a / \alpha^{2/3} \beta. \end{aligned} \right\} \quad (26)$$

Equation (25) has been solved by previous investigators (Feynman, Metropolis and Teller⁴) by numerical methods and semi-empirical expressions for the boundary values have been give by Gilvarry^{5,6}. Our aim is to confirm by Taylor's expansion method the choice of the limiting form of the formula of Gilvarry. In doing so we can start with equation (25) directly or reduce the relativistic temperature perturbed coefficients given by (20) to the non-relativistic limit. In either way it can be shown that the coefficients t_n 's of the Taylor's series for the solution of equation (25), are

$$\left. \begin{aligned} t_1 &= -(d\phi/dx)_{x_b} \equiv -\phi(x_b)/x_b \\ t_2 &= \frac{1}{2} \phi^{3/2}(x_b) / x_b^{1/2} \cdot (1 + \xi T^2 \cdot x_b^2 / \phi^2(x_b)) \\ t_3 &= -\phi^{3/2}(x_b) / 6x_b^{3/2} \cdot (1 + \xi T^2 \cdot x_b^2 / \phi^2(x_b)) \\ t_4 &= \phi^2(x_b) / 24x_b \cdot (1 + \xi T^2 \cdot x_b^2 / \phi^2(x_b))^2 (\frac{3}{2} - 2\theta) \\ t_5 &= -\phi^2(x_b) / 120x_b^2 \cdot (1 + \xi T^2 \cdot x_b^2 / \phi^2(x_b)) (\frac{3}{2} - 2\theta) \\ &\dots\dots\dots \end{aligned} \right\} \quad (27)$$

(Coefficients of Taylor's series solution of equation (25). We have used x_b to denote the boundary value for consistency of notation.)

where

$$\theta = \{ \xi T^2 x_b^2 / \phi^2(x_b) \} / \{ 1 + \xi T^2 x_b^2 / \phi^2(x_b) \}. \quad (28)$$

In deducing these expressions for t_n 's we have ignored the variation of x_b due to temperature. This procedure corresponds to the method of Marshak and Bethe^{5,1} and our results can then be compared with those of Gilvarry. Proceeding as before, we can show that the form of $\phi(x_b)$ in the non-relativistic case when temperature perturbation is taken into account, is

$$\phi(x_b, T) = 3^{2/3} / x_b \cdot (1 - 3^{1/3} x_b / 10) - 2\xi T^2 x_b^3 / 3^{5/3}. \quad (29)$$

In (29), the temperature independent term is just the expression obtained by March, and the second term gives the correction due to temperature. If we write the temperature correction term of (29) as $F(x_b)\xi T^2$, then we see that $F(x_b)$ corresponds to χ of equation (47) of Gilvarry⁵⁾ in the limit $x_b \rightarrow 0$. As stated in the introduction, the above limiting form is obtained by Gilvarry by demanding that the results of T.F. method must go over into those appropriate for a Fermi-Dirac gas of completely degenerate electrons when pressures are sufficiently high. It may be mentioned that Gilvarry modified the form of χ slightly⁶⁾ without of course changing the leading term. It must be noted that although the limiting form of χ is justified, its actual use for small x_b is unreliable since relativistic effects become important, and one has to use the more exact form obtained from t_n 's of (20), but it has not been possible to get an explicit form of $\phi(x_b)$ as a function of x_b owing to the complicated form of t_n 's.

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The Statistical Mechanical Aspect of the H -Theorem, II

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This paper is devoted to the discussion of the properties of the H -function, especially concerning its irreversible character somewhat quantitatively, using the same model as that used in the previous paper. As the result, it is concluded that the more roughly the macroscopic observation is done, the more rapidly the H -function decreases to an equilibrium value as shown in Fig. 2. Thus it is clarified that H -function of a given system which is in a non-equilibrium situation should be defined in connection with the accuracy of macroscopic observations. In our calculations, we have not used any assumptions such as "Stoßzahlansatz" or the coarse-graining in every moment of the process, except only once in setting up the ensemble which represents a given non-equilibrium initial state. Our result gives a justification of statistical aspect of the H theorem suggested by Ehrenfest without proof.

§ 1. Introduction

In the previous paper,¹⁾ which will be referred to as Part I, we discussed the simple diffusion process of particles contained in a one-dimensional box of length $2l$. At the initial time, N particles, which have equal mass ($m=1$) and collide with each other completely elastically, are contained in the left room of the box, which is divided into two rooms by a wall in the middle. If the wall is removed at time $t=0$, these particles diffuse into the right room and the number of particles $n(t)$ in the left room changes, decreases generally, as a function of time t . We have investigated the change of function $n(t)$ in this process, using a simple graphical method.

If our knowledge obtained about the initial condition is only that all particles exist in the left room of the box (such a condition is controllable and seems to be actually realized in a certain way), an exact mechanical condition of this system cannot be specified and it is necessary to introduce a certain statistical procedure to make up for this lack of informations. Thus at $t=0$, we construct an ensemble of systems which satisfies the given macroscopic initial condition that all particles exist in the left room of the box, in other words, we consider a flux of mechanical trajectories in l -space each of which starts with the same macroscopic initial condition. Along these trajectories, the function $n(t)$ changes in various ways, and this was already investigated by a schematical method in Part I.

The mean value $\langle n(t) \rangle$, where the mean is effected over all trajectories, was evaluated, and it was shown that the mean value $\langle n(t) \rangle$ which is expected in our macroscopic observation behaves in such a way that it approaches an equilibrium value $N/2$, and it was also clarified that the distribution of values of $n(t)$ in the flux has a sharp maximum at its mean value $\langle n(t) \rangle$, so that it is very unlikely for us to encounter such a mechanical

path that it gives a value of $n(t)$ appreciably deviated from its mean value.

As for the ensemble constructed at the initial time, we have considered two cases, i.e., a uniform distribution, in which any one of the particles can take every point of μ -space equally probably as its initial state, so long as it lies in a rectangular region of μ -space where the conditions $0 \leq x_0 \leq l$, $|v_0| \leq \sqrt{2E}$ are satisfied, and the Maxwell-Boltzmann distribution in which all states are equally probable under the condition that the total energy of particles is constant E and all particles are contained in the left room of the box. In both cases, the statistical properties of function $n(t)$ are much the same, namely the function $n(t)$ almost always approaches the equilibrium value $N/2$ in an irreversible manner.

In Part I, we have discussed only the diffusion process of particles, putting aside the problem about the properties of the H -function. This paper is devoted to discussion of the properties of the H -function, especially its irreversible character, somewhat quantitatively by using the same model as that used in Part I. In § 2 some arguments on the definition of the distribution function in μ -space are given, and two kinds of distribution function, namely fine-grained and coarse-grained distribution functions, are defined. In § 3, using these definitions of distribution function, the behaviors of the H -function are examined with the use of our one-dimensional model, and it is shown that if we adopt the coarse-grained distribution function, which is properly defined corresponding to the accuracy of our macroscopic observations, the H -function almost always decreases to attain the equilibrium value substantially in a monotonous way, and that the more roughly the macroscopic observation is done, the more rapidly the H -function approaches the equilibrium value. Lastly the general discussions on the statistical mechanical aspect of irreversible processes are given.

§ 2. Distribution function

Boltzmann's H -function is given by the equation

$$H = \int \cdots \int f(x, y, z, v_x, v_y, v_z) \log f(x, y, z, v_x, v_y, v_z) dx dy dz, dv_x dv_y dv_z \quad (1)$$

where $f dx dy dz dv_x dv_y dv_z$ is the number of particles which have coordinates in the ranges $(x - \frac{1}{2} dx, x + \frac{1}{2} dx)$, $(y - \frac{1}{2} dy, y + \frac{1}{2} dy)$, $(z - \frac{1}{2} dz, z + \frac{1}{2} dz)$ and velocities with components in $(v_x - \frac{1}{2} dv_x, v_x + \frac{1}{2} dv_x)$, $(v_y - \frac{1}{2} dv_y, v_y + \frac{1}{2} dv_y)$, $(v_z - \frac{1}{2} dv_z, v_z + \frac{1}{2} dv_z)$. The above mentioned definition of the distribution function, which we shall call hereafter the fine-grained distribution function, can be understood as follows. Consider a small six-dimensional parallelepiped $\partial x \partial y \partial z \partial v_x \partial v_y \partial v_z$ with a point (x, y, z, v_x, v_y, v_z) as its center, its volume $\partial v = \partial x \partial y \partial z \partial v_x \partial v_y \partial v_z$ being sufficiently small but microscopically large enough to contain a large number of representative points of particles in it.* Then we can define the value

* If the volume of this parallelepiped has atomic dimension, the number of particles whose representative points are in this cell becomes very small, for instance zero or one, then the distribution function becomes an irregularly discontinuous function.

of distribution function f at the point (x, y, z, v_x, v_y, v_z) with the ratio of the number of particles whose representative points are in this parallelepiped and its volume, $\partial N / \partial v$. Thus the fine-grained distribution function is usually given by a continuous function defined at all points in μ -space and it gives us sufficiently detailed knowledge about the state of the system, but still this definition includes a statistical character, because discretely distributed representative points are described by a continuous function f , which is obtained by a kind of statistical procedure mentioned above.

In the next, we shall consider a definition of another kind of distribution function, which we shall call hereafter coarse-grained distribution function. In this case μ -space is divided into a number of cells $\mathcal{J}v_1, \mathcal{J}v_2, \dots, \mathcal{J}v_i, \dots$. These cells are constructed in accordance with the nature of our macroscopic observation in such a manner that any more detailed specification of state of a particle in each of these cells cannot be done by this macroscopic observations, so that the volumes and shapes of these cells are decided according to what kind of observation is carried out. As the number of particles, n_i , whose representative points are in the cell $\mathcal{J}v_i$ is macroscopically observable, though the exact positions of these representative points in this cell are unknown for us, we can define the distribution function f in this i -th cell with a constant value $f_i = n_i / \mathcal{J}v_i$. Thus if we adopt the coarse-grained distribution function, the H -function must be rewritten in the form

$$H = \sum_i f_i \log f_i. \quad (2)$$

Obviously the coarse-grained distribution function agrees with the fine-grained distribution function when the perfectly precise observations are made, by which the coordinates and velocities of all particles at each instant are completely measured. This is true in principle in classical mechanics, but actually it seems hopeless to carry out such a precise observation for the macroscopic system which has an enormous degrees of freedom. Thus usually the volume of the cell in the case of fine-grained distribution function is much smaller than that of coarse-grained distribution function, and the coarse-grained distribution function corresponding to usual macroscopic observations is generally different entirely from the fine-grained one of that system.

§ 3. The behaviors of the H -function

We shall consider a one-dimensional box of length $2l$ in which N independent particles move freely without collisions between them and reflected elastically at the boundaries of the box. The state of this system can be indicated by specifying the positions of N representative points in two-dimensional μ -space. We shall assume that at the initial time all these particles are situated in the left half of the box; for instance, all particles are shut up in the left half of the box by another wall which is removed at $t=0$. This macroscopic initial condition can be satisfied by a number of microscopic initial states so it is natural to introduce an ensemble of systems each of which satisfies this macroscopic initial condition.

As for this ensemble constructed at the initial time, we assume here a uniform

distribution, in which any one of the particles can take every point of μ -space equally probably as its initial state, so long as it lies in the rectangular region of μ -space in which the conditions $0 \leq x_0 \leq l$, $|v_0| \leq \sqrt{2E}$ are satisfied. Corresponding to systems of this ensemble, an infinite number of trajectories in I -space are constructed, each of which starts from a different microscopic initial state represented by N representative points distributed randomly in the initial rectangular region in μ -space, as shown in Fig. 1 (a).

It is easily seen that with the lapse of time these representative points change their positions as shown in Fig. 1 (b), (c), in which the group of the representative points forms bands. As time passes, the number of these bands increases and they become narrower in width and gentler in slope, but the total area of these bands does not change and is always equal to the area of the initial rectangular region $2l\sqrt{2E}$.

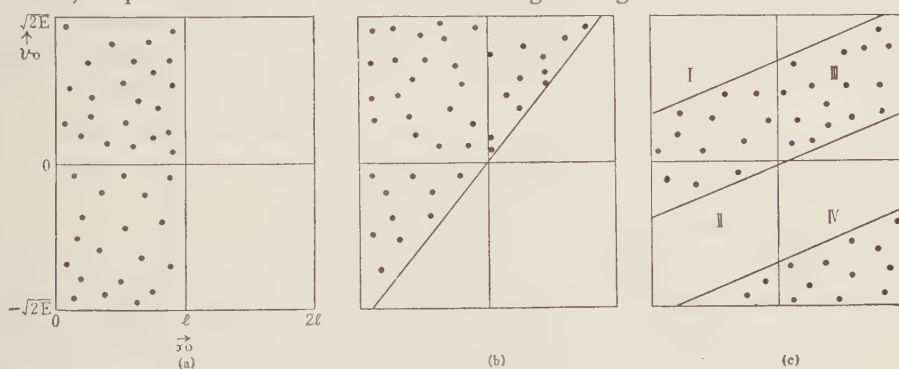


Fig. 1.

Using this diagram, we can calculate the *H*-function as a function of time t . According to the Liouville's theorem, the density of representative points does not change under the natural motion of the system, and so is the fine-grained distribution function except in the vicinity of the boundaries of the bands. Thus it is easily seen that, if we adopt the fine-grained distribution function, the value of the *H*-function does not change appreciably and maintains approximately its initial value until the gaps of the bands become so narrow that its width can be covered by the linear dimension of the cell ∂v . When, after a sufficiently long time, the width of the bands becomes the order of magnitude of the linear dimension of the cell ∂v , the value of the *H*-function will change, decreases generally. It must be also remarked that the initial value of the *H*-function presented in terms of fine-grained distribution function can not be defined uniquely by the knowledge of macroscopic observation of its initial state but it is possible to attribute various initial values to this *H*-function according to which microscopic initial state is assumed in our calculations.

On the other hand, if we adopt the coarse-grained distribution function the desirable irreversible character of the *H*-function can be obtained. We shall begin with a very rough observation by which only the numbers of particles n and $N-n$ respectively in the left and right half of the box can be measured. In this case two cells are taken into considerations, which correspond to the left and the right half of the region in Fig. 1.

Then the H -function can be written in the form

$$H = n \log \{n/\mathcal{A}v\} + (N-n) \log \{(N-n)/\mathcal{A}v\} \quad (3)$$

where $\mathcal{A}v = 2l\sqrt{2E}$ and the time dependence of the number of particles in the left room n has various forms according to the choice of microscopic initial state and so does the H function also. But the initial value of the H -function in this case can be decided uniquely from the macroscopic information about its initial state and is given by $H(0) = N \log(N/\mathcal{A}v)$. The equilibrium value of H which is expected to be reached in the final state is given by inserting the value $n = N/2$ into the equation (3); hereafter we shall write this quantity as $H_0 (= N \log(N/2\mathcal{A}v))$.

The change of H in the course of diffusion process is determined thoroughly by the change of n , which is already investigated in Part I. The probability that, at time t , n particles are contained in the left room is given by the Bernoulli distribution function (eq. 2 in Part I)

$$P_N(n) = N! / n! (N-n)! \cdot r^{N-n} (1-r)^n, \quad (4)$$

where r is given as a function of time in the form

$$\begin{aligned} r(t) &= 2lm / \sqrt{2E}t + (\sqrt{2E}t/4l) (1 - 4ml / \sqrt{2E}t)^2 \\ &\quad \text{for } 4ml / \sqrt{2E} \leq t \leq (4m+2)l / \sqrt{2E}, \\ r(t) &= 1 - (2m+1)l / \sqrt{2E}t - (\sqrt{2E}t/4l) (1 - (4m+2)l / \sqrt{2E}t)^2 \\ &\quad \text{for } (4m+2)l / \sqrt{2E} \leq t \leq 4(m+1)l / \sqrt{2E}, \\ &\quad m = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (5)$$

This probability can be understood as the probability of realization of such microscopic initial conditions that at time t just n particles are in the left room of the box. Making use of this distribution function (4), we can calculate the mean value of H at time t which seems to be most probable value of H in our macroscopic observation at time t . As we have seen in Part I, if the number of particles N is sufficiently large the probability distribution of n has a sharp maximum at its mean value $\langle n \rangle$, so we can approximately replace the mean value $\langle H \rangle$ with

$$\langle H \rangle \doteq \langle n \rangle \log \{ \langle n \rangle / \mathcal{A}v \} + (N - \langle n \rangle) \log \{ (N - \langle n \rangle) / \mathcal{A}v \}, \quad (6)$$

when the number of particles N is sufficiently large. n has been evaluated in Part I as follows.

$$\begin{aligned} \langle n \rangle &= N(1-r) = N\{1/2 + T(1-m/T)((2m+1)/2T-1)\} \\ &\quad \text{for } m \leq T \leq m+1/2, \\ \langle n \rangle &= N\{1/2 + T(1-(m+1)/T)(1-(2m+1)/2T)\} \\ &\quad \text{for } m+1/2 \leq T \leq m+1, \end{aligned} \quad (7)$$

where $T = t\sqrt{2E}/4l$.

Thus we can obtain the $\langle H \rangle$ -curve as an explicit function of time t which is indicated by curve A in Fig. 2.

From Fig. 2 it is seen that almost always H decreases to attain the equilibrium value H_0 in a monotonous way, and then it practically maintains the value H_0 for ever. In an actual system, of course, we cannot deny the possibility of reappearance of the macroscopic initial state, but, as has been discussed in Part I, it is very seldom when the system contains sufficiently a large number of particles, and it seems to be acceptable to regard the value of $\langle H \rangle$ as the macroscopic quantity corresponding to the given macroscopic observation by which we can measure only the numbers of particles in the left and right rooms of the box.

In the next, we consider another observation by which not only the numbers of particles in the left and right rooms but also the numbers of particles which have respectively positive and negative directions of velocities can be measured. In this case four cells in μ -space should be taken into considerations as shown in Fig. 1 (c), and the H -function is given by the equation

$$H = \sum_{i=1}^{1V} n_i \log \left(n_i / \frac{\Delta v}{2} \right) \quad (8)$$

The mean value of H as a function of time t can be calculated in quite the same manner as in the first case. The result is plotted in Fig. 2 by a curve B, in which such an initial state is assumed that all particles are contained in the left room and half of them have the positive velocities and the remainder have the negative velocities. Under this condition the initial value of H clearly agrees with that of the first case.



Fig. 2.

The results show the fact that the more precisely observations are made, the longer

becomes the time required for the H function to attain to its equilibrium value H_0 ; in our example this time in the second case is just twice that in the first case. Therefore such a situation may occur that for one observer the system is in equilibrium state but for another observer it has not yet reached the equilibrium state. However, so long as one makes the macroscopic observation the system will be, after a sufficiently long time, almost always found to approach the equilibrium state at which all kinds of macroscopic quantities have the equilibrium value.

§ 4. Summary

The H -function is given by using the coarse-grained distribution function which is defined in connection with the accuracy of macroscopic observations. We consider all possible microscopic initial states which have the same initial non-equilibrium value of H , then along the trajectories starting from these states the H -value almost always decreases to attain the equilibrium value H_0 and the change of mean value of H effecting over all these trajectories is calculated just as in Fig. 2. It is also clarified that the more precisely observations are done, the longer becomes the time required for the system to attain the equilibrium state. Thus, in a non-equilibrium state, it becomes necessary to assign a properly defined H -function to the system according to the accuracy of macroscopic observations. But at the final equilibrium state which is expected to realize after a sufficiently long time, H reaches a unique value H_0 which has the well-known relation with the thermodynamical entropy.

The example presented here gives a justification of the statistical aspect of the H -theorem suggested by Ehrenfest without proof.²⁾ Ehrenfest's suggestion, in somewhat generalized form, is as follows. We take a mean value of some phase function $A(p_1, p_2, \dots, q_1, q_2, \dots)$ at time t in the form

$$\langle A \rangle = \int \dots \int A(p_1, p_2, \dots, q_1, q_2, \dots) \rho(p_1^0, p_2^0, \dots, q_1^0, q_2^0, \dots) dp_1^0 dp_2^0 \dots dq_1^0 dq_2^0 \dots$$

where $p_1, p_2, \dots, q_1, q_2, \dots$ are generalized coordinates and momenta of the given isolated system, are functions of time t and their initial values $p_1^0, p_2^0, \dots, q_1^0, q_2^0, \dots$, and satisfy the Hamilton equation of motion, and $\rho(p_1^0, p_2^0, \dots, q_1^0, q_2^0, \dots) dp_1^0 dp_2^0 \dots dq_1^0 dq_2^0 \dots$ is the probability that at initial time the system is in the state given by a representative point in the range $(p_1^0, p_1^0 + dp_1^0)$, $(p_2^0, p_2^0 + dp_2^0)$, \dots , $(q_1^0, q_1^0 + dq_1^0)$, $(q_2^0, q_2^0 + dq_2^0)$, \dots in μ space under the condition that the quantity A has the initial non-equilibrium value A^0 . Then, for some classes of physical quantities A and of Hamiltonians of the system, $\langle A \rangle$ has a limit A_0 when time t tends to infinity which is equal to the equilibrium value. Such an approach to the foundation of statistical mechanics is desirable one, but the proof of this supposition in the general form is very difficult, so we have treated only a very simple system and investigated the irreversible process on this line.

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On the Universality of the Weak Interaction

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With the aim to understand the weak interactions occurring in nature from the unified point of view, we shall propose one possible model in this note. In our model, new charged Bose fields are introduced and are assumed to interact universally with all Fermi particles. We shall also discuss about the difference between ours and the other view-point in which all weak interactions are assumed to come from the universal Fermi interaction.

§ 1. Introduction

Recently, Pais-Nishijima-Gell-Mann¹⁾ theory has given a clear explanation on the transition processes of the particle families which have both the strong and the weak interaction, including new unstable particles. According to this theory, the strongly interacting particles are all characterized by their isotopic spins and γ -charges which are recognized by their conservation in the strong interaction processes. On the other hand, there has been found no strong interaction in the transition processes accompanied with the lepton family including electron, neutrino and μ -meson. Owing to this fact, we cannot anticipate a useful meaning of the isotopic spin or γ -charge in the lepton family.

Now, paying our attention only to weak interactions realized in nature, we shall find some peculiar regularity among them which has already been noticed in a previous paper.²⁾ The coupling constants of these interactions take a value of the unique order of magnitude ($\simeq 10^{-14}$) in the natural unit*, independently of the coupling type or the sorts of the particles concerned. From this feature, we may expect the following:

(α) The weak interaction occurs not only among the nucleon and π -meson families³⁾ which have the strong interaction too, but also among the lepton family. Within the frame work of Pais-Nishijima-Gell-Mann theory, the weak interaction comes out destroying the conservation of the isotopic spin or γ -charge, while the weak interaction itself originates from another root with wider validity.

(β) The unique value of the coupling constants above mentioned indicates that all weak interactions come from a single base.

If these conjectures are correct, we may also anticipate the clue to the weak interaction in the processes participated by the lepton family. As the key to it, we shall take the following fact:

* Through the paper, we take this unit as $\hbar=c=r_0(\simeq 10^{-13} \text{ cm})=1$, where \simeq means the equality in the order of magnitude.

(i) When the lepton-family takes part in the decay (or capture) processes, we can always find neutrino, namely, there is no lepton process without neutrino.*⁽¹⁾

The purpose of this paper is to introduce one possible model, in order to understand the above characteristic features of the weak interactions in a unified manner. In this section, we shall present some outline of this idea. Paying our attention to the universal character of the weak interaction, we shall follow the example of the electromagnetic interaction

$$eA_\mu \sum_a j_\mu(a) \quad (I)$$

which appears most universally among all families. In the expression (I), e is the charge constant and $j_\mu(a)$ is the current of particle a . After the electromagnetic field, we shall introduce charged Bose fields B 's, which have the following type of interaction

$$g \sum_\alpha \sum_{ab} \bar{\psi}_\alpha O_j(a, b, \alpha) \psi_b \cdot K_j(a, b, \alpha) B_\alpha \quad (II)$$

with all Fermi particles. This is the basic idea of this paper. The interaction (II) is assumed to conserve charge and nucleon-number and also to be Lorentz invariant. Further, from our view of the family,⁽³⁾ B -fields are assumed to belong to the separate family from the π -meson family, namely, π , η and τ , etc., because of the difference in their characteristic interactions.

Compared with (I), the interaction (II) has some defect in the simplicity and symmetry. As we shall see below, B -fields cannot be restricted in one type, and all combinations (a, b) of Fermions do not interact in common with a B -field. Apart from these points, the interaction (II) can qualitatively satisfy the above features (a), (i) and (j) of the weak interactions. Leaving the precise discussion to the following sections, we shall give some brief remarks. First, all weak interactions come out through the interaction (II) with the coupling constant g . From this fact we can understand the universality and the regularity of the weak interaction. Secondly, because B -fields are charged, the decay interactions originated from (II) are necessarily charge-dependent, and in the coupled source with B -field, there are one neutral and one charged Fermion. This last feature is compatible with (j).

Now, in the resultant expression, the interaction (II) can be brought into Fermi interaction among Fermi particles. In this sense, there is no phenomenological difference between ours and the view-point of "primary Fermi interaction" about weak interactions. Under these circumstances, some detailed discussion will be necessary as to our standpoint. As was already noted,⁽³⁾ to understand all weak interactions on a single base, the view point of "primary Fermi interaction" will also serve to the purpose. Then, where does the difference between ours and the view-point of "primary Fermi interaction" lie? In precise analysis of the transition processes, there will be found some differences in the selection rule. However, in view of the present stage far from the final determination by the experimental test, we should like to emphasize rather the difference in the standpoint of the theory. If the

* Such higher order processes as double β -decay are excluded.

view of Fermi interaction succeeds in the clarification of the phenomena, then it will be necessary to clarify the reason why Fermi interaction destroys the charge independence and why it maintains such feature as (γ). In the present stage of our theory, however, there occurs no such problem, while the problems to be inquired are, for instance, the distinction of three characteristic Boson families, that is, the electromagnetic field, π -meson fields (including θ , τ) and B -fields, and also the origin of their interactions. Further, we are following the view-point in which *the interactions among the particles (Fermions) are all assumed to be mediated by the existence of the fields (Bosons).*

§ 2. The nature of B -fields

In this paragraph, we shall present a brief summary of the nature of B -fields and its interactions. First, all weak interactions are mediated by B -fields. For instance, the natural decay of free neutron takes place through

$$N \rightarrow P + B (\rightarrow e + \nu) \rightarrow P + e + \nu.$$

The scalar type Fermi interaction thus brought about is given by

$$g^2 (\bar{\psi}_P \psi_N) (\bar{\psi}_e \psi_\nu) (1/m_B)^2 \quad (1)$$

where m_B is the mass of B -fields. Accordingly, our coupling constant g is correlated with Fermi coupling constant f_s as

$$(g/m_B)^2 \simeq f_s = 10^{-49} \text{ erg cm}^3 \quad (2)$$

and in the natural unit

$$(g/m_B)^2 \simeq 10^{-7}. \quad (2')$$

As is clear from this, the strength of the coupling of Fermions with B -fields lies between the strong interaction ($G^2 \simeq 1$) and the weak interaction ($\simeq 10^{-14}$). For this reason, it is very improbable to detect B -particles in the nuclear interactions, and the observed K -mesons cannot be identified with B -particles. On the other hand, when B -particles are created, then they decay into leptons in very short interval ($\simeq 10^{-17}$ sec).

Now, if the mass m_B of B -particles were less than that of the unstable K -meson or the mass difference between hyperon's and nucleon's, the decay of K -meson or hyperon would be very fast. To avoid this difficulty, the minimum of m_B is restricted as

$$m_B > m_K \sim 1000 m_e. \quad (3)$$

The mass value of B -particles would, in other words, designate the maximum mass of the K -mesons, whose life-time lies within the present technics of the experiment ($\sim 10^{-10}$ sec). Further, if there exists another heavy meson with mass $> m_B$, we shall observe such phenomenon as the direct production of leptons in the nuclear interaction, because this heavy meson created will rapidly decay into B -particle and B -particle also into leptons.

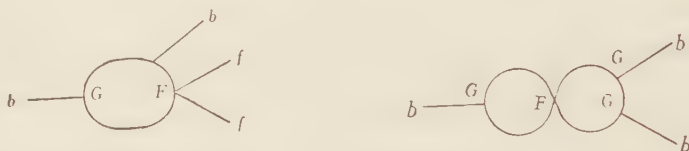
Because of their electric charge, B -fields are also accompanied with the electromagnetic interaction. However, the effect of B -field to the ordinary electromagnetic phenomena

would not be so large, in virtue of its large mass. The pair creation of B -particles by the high energy γ -ray is also hardly detectable, because the pair creation of electron and μ -meson would rise up simultaneously and cover the B -field phenomena.

§ 3. Decay interactions mediated by B -fields

In our theory, all weak interactions are brought about through the interaction (II) by two steps. In the resultant expression for weak interactions, however, there is no distinction from Fermi interaction. Therefore we shall take, for a moment, Fermi interaction as the primary of the weak interactions.

Now, about the weak interaction realized in nature, we know three kinds, that is, Fermi interaction, Boson-Fermion and Boson-Boson interactions. To derive the latter two from Fermi interaction, the strong Boson-Fermion (e. g., π - N) interaction is necessary. They are represented, for instance, by the following diagrams;



where b and f indicate Boson and Fermion respectively, and F is Fermi interaction and G the strong Boson-Fermion interaction. When all weak interactions are derived from a single weak interaction as above, an important suggestion is presented by the regularity of coupling constants mentioned in § 1.

Let τ_a be the lifetime of particle a , then the general form of τ_a is given, in the unit $\hbar=c=1$, as

$$1/\tau_a = (2\pi)^4 \frac{\rho_a}{E_a} |I|^2 \prod_i \frac{dk_i}{(2\pi)^3 E_i} \delta(E_a - \sum_i E_i) \delta(\mathbf{k}_a - \sum_i \mathbf{k}_i) \quad (4)$$

where E_i and \mathbf{k}_i are the energy and momentum of final particle i , and

$$I = \sqrt{E_a}(\mathbf{k}_a | H' | \mathbf{k}_1 \cdots \mathbf{k}_n) \prod_i^n \sqrt{E_i}$$

is an invariant transition matrix. Extracting Fermi coupling constant f , we may express I as

$$I = f M(G, m_s)$$

where M would be a complicated function of, for instance, the strong coupling constant G and the masses of the particles in virtual and real state. Under this preliminary, rewriting the expression (4) in the unit $r_0 (\simeq 10^{-13} \text{ cm}) = 1$, we shall obtain

$$1/\tau_a = 3 \times 10^{23} (\text{sec}^{-1}) \cdot f^2 \cdot |M|^2 \cdot \rho_F^* \quad (5)$$

from the dimensional consideration. ρ_F is the final density including $(2\pi)^4$ in (4). As

* The numerical factor is nothing but (c/r_0) .

is clear from the expression (5), if $\rho_F \simeq |M|^2 \simeq 1$ and $f^2 \simeq 10^{-14}$ in the natural unit, it results $\tau_a \simeq 10^{-10}$ sec. For the various unstable particles now observed, namely, $\tau (\rightarrow 3\pi)$, $\theta (\rightarrow 2\pi)$ and $\Lambda^0 (\rightarrow P + \pi^-)$, etc., we shall see $\rho_F \simeq 1$ and we are also informed of the lifetime of these particles to be $10^{-9} - 10^{-10}$ sec. These facts indicate that $|M|^2$ is not seriously dependent on the transition scheme and takes a value $\simeq 1$ in the natural unit. In virtue of this, we shall take the following assumption as a first step:

(A) Except in the case where some particular selection rules play a role, the contribution $M(G, m_s)$ of the strong interaction to the decay matrix takes a value $\simeq 1$ in the natural unit.

This assumption is not so drastic because the practical calculations so far have not found the serious deviations.⁶⁾ In view of the difficulty to perform the correct estimation of the strong interaction, we shall rather intend to leave the problem to be studied in future on the above assumption. This assumption will be used without notices in the following.

When the above assumption is accepted, the absolute lifetime of each decay process is not a serious problem to be inquired. We shall only mention that the lifetimes of various unstable particles now observed are all elucidated in its order of magnitude. More precise estimation of the lifetime, —for instance the difference between $\theta^0 \rightarrow \pi^+ + \pi^-$ and $\theta^0 \rightarrow \pi^0 + \pi^0$ — will necessitate more profound analysis on the strong interactions. The problems to be studied in our theory will rather be the following:

- (a) the selection rules for transition processes under the existence of the various interactions,
- (b) the dynamics due to the interaction with B -fields and
- (c) the final density.

These problems play an important role in the determination of lifetime or the competition among the various decay modes.

About the problem (c), only remembering the fact that the large difference of the lifetime between $\mu - e$ decay and free neutron decay is due to the final density, we shall rather draw our attention to (a) and (b). These problems will be studied with a special notice to the difference between ours and the view-point of primary Fermi interaction. For this example, we shall take the decay process of K -meson into leptons. From each view-point, the decay process will be given by the following diagrams:



As is already pointed out by K. Nishijima,¹⁾ the theorem of Fukuda-Miyamoto⁷⁾ is not valid in the transition processes accompanied with unstable K -mesons or hyperons. Accordingly, when Fermi interaction is of scalar and vector type, scalar K -meson cannot be

* Because of the mediation of charged B -field, we can expect the existence of the difference in their decay life time.

forbidden to decay into lepton.* In our theory, on the other hand, scalar K -meson can decay, only when B -field is of scalar type. The above mentioned distinction lies just in this point, namely, that in the view of "primary Fermi interaction", the selection rule in the transition process is controlled by the interaction type, while in our case, it is rather ruled by the type of B -field concerned. At this moment, however, this distinction cannot be examined because the established facts are too meagre to find the distinction in selection rule successfully.

§ 4. Decay processes

In this section, to show a way of application of our theory to the decay phenomena, we shall give some explanations, although the following discussion is not necessarily decisive. To apply our theory to the decay phenomena, we need the identification of the particles. Because τ -meson is seemingly of pseudo-scalar type, it cannot be identified with θ -meson which decays into two π -mesons. In the following, assuming θ -meson to be scalar or vector, we shall take only τ and θ as unstable K -mesons. On the hyperons, we shall take Σ and Λ en bloc and represent them by Λ , because Σ is able to change into Λ by the strong interaction ($\Sigma \rightarrow \Lambda + \pi^{**}$). In virtue of the lack of the information about various competing decay processes, we shall exclude Ξ -particle for the moment. Λ is assumed to be spin $1/2$ particle.

In studying the decay processes under the above identifications, we will select the following phenomena. Of course we should take account of the possibility that some of the following facts would be denied by the future development of the experiment.

- (i) $\theta \rightarrow \mu + \nu$ has been found but $\theta \rightarrow e + \nu$ has not. We assume $K_{\mu 2}$ to be identical with θ , but not with τ , tentatively.
- (ii) $\tau \rightarrow e + \nu$ has not been detected also.
- (iii) τ or $\theta \rightarrow \mu + \nu + \pi^0$ and seemingly $\rightarrow e + \nu + \pi^0$ are found.⁽⁸⁾
- (iv) $\Lambda^0 \rightarrow p + e + \nu$ is not yet detected.

Now we shall study the predictions from our theory. Owing to the fact (iii), we cannot forbid (iv) absolutely. To ensure (iv), we may take $(A_0 \gamma_5 P)(e \gamma_5' \nu)$ as the resultant interaction derived through B -field, because γ_5 -interaction reduces the transition probability by a factor $\sim (M_\Lambda - M_p)^2 / M_\Lambda^2$ in comparison with other types of interaction.*** γ_5' is γ_5 or 1 corresponding to the same or opposite parity of Λ^0 to that of proton.**** The B -field to bring this interaction must be of pseudo-scalar type or scalar type, according to that Λ^0 and proton have the same or different parity. Because $\tau \rightarrow e + \nu$ is allowed in the case of pseudo-scalar type B -field, B -field here should be scalar. In this case, vector θ -meson is also forbidden to decay into electron and neutrino. $\tau \rightarrow e + \nu + \pi^0$ is allowed and $\theta \rightarrow e + \nu + \pi^0$ will be expected to be damped from the kinematical consideration. $\theta \rightarrow \mu + \nu$

* n and Λ are assumed to have same parity here.

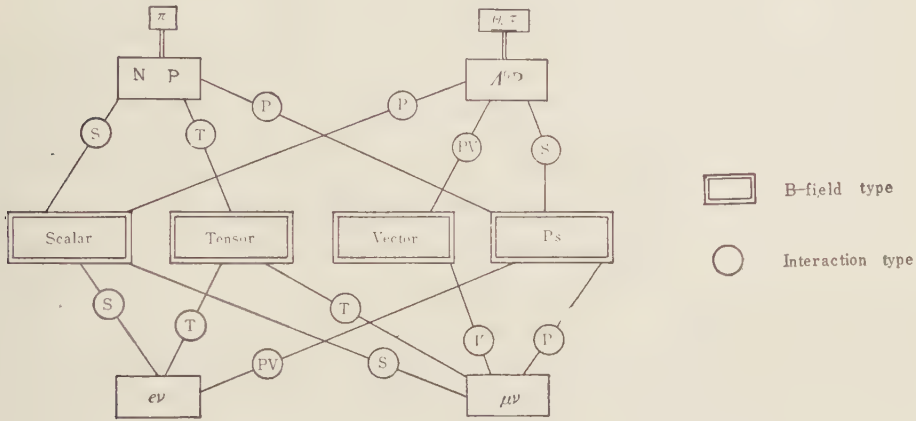
** With the progress of the experiment, more refinement will be necessary.

*** It should be noted that β -decay of Λ is estimated to be less frequent than $\Lambda \rightarrow p + \pi$ decay by a factor $1/200$ in the case of scalar type Fermi interaction.⁽⁵⁾

**** Electron and neutrino are assumed to be of same parity.

becomes comprehensible under the introduction of vector type B -field between $(l^0 P)$ and $(\mu\nu)$. In this case, θ (and τ) $\rightarrow \mu + \nu + \pi^0$ can be realized. In the case of scalar θ , the solution is not able to be found in our theory. The limitation for the particle as this follows from the facts (i) \cdots (iv). When one of these facts, for instance, (iv) is neglected, the possibility extends considerably.

Now, we shall present an example of B -fields and its interactions which are able to explain β -decay and π - μ decay, etc.



Some illustrations will be necessary for this diagram. To give the tensor-type interaction of the ordinary β -decay, B -field with spin 2 is introduced. For a mere derivation of the tensor-type interaction, we may take a vector B -field and assume its tensor interaction with Fermions. In this case, however, we cannot identify this B -field with the vector-type B -field which couples with $(A^0 P)$, because then $\theta \rightarrow e + \nu$ will be allowed, and further to maintain the strength of the resultant tensor interaction comparable with that of scalar interaction, we must enlarge the value of coupling constant ($2'$) by two units only for this case. For this reason, we have excluded the above possibility. This diagram, of course, follows from taking the facts (i) \cdots (iv) obstinately. We should expect the modification of this diagram with the future progress of the experiment. The knowledge of Ξ -particle will also offer another limitation. In this sense, the diagram indicated should not be taken too seriously.*

§ 5. Concluding remarks

In this paper, to understand the various features about weak interactions in a unified manner, we have proposed a possible model. In our view-point, all weak interactions are assumed to originate through the mediation of the charged Bose field B newly introduced. While, in the phenomenological stage, our theory is not so different from the view of primary

* It should be noted that the present diagram does not necessarily forbid $\theta \rightarrow 3\pi$.

Fermi interaction, we have found some disparities in the selection rules. However, in view of the present stage difficult to examine the existence of B -fields, we shall rather intend to take our theory as a possible indication for the origin of Fermi interactions which are to elucidate the weak interaction phenomena. Further, we should note one exceptional case for which our theory alone may not find solution, namely, the prohibition of the process $\mu \rightarrow e + \gamma$. To forbid this process, another restriction, for instance, the conservation of lepton-number will be necessary.⁹⁾

Finally, we shall present a brief consideration on the families and their interactions³⁾. We know two Fermion -, namely, nucleon and lepton-families of which we have nothing to be newly added. For Boson-families, we have now three characteristic families. These are the electromagnetic field, the B -fields and the π -meson-fields, the last of which is assumed to include τ -meson, and θ -meson etc. These families are also characterized by their intrinsic interactions which are, respectively, given by the interactions, (I), (II) and only the *strong* (in contrast with Pais-Nishijima-Gell-Mann theory) interaction

$$G \sum_{\alpha} \sum_{ab} \bar{\psi}_a O_j(a, b, \alpha) \psi_b \cdot K_j(a, b, \alpha) \phi_{\alpha} \quad (\text{III})$$

where ψ is the wave function of the nucleon family and ϕ that of the π -meson family. Thus, from our view-point, we may say that Pais-Nishijima-Gell-Mann theory has given the systematics of what combinations (a, b, α) of the particles are realized in the interaction (III) and that all interactions occurring in nature originate from the existence of three characteristic Boson-families with their three intrinsic interactions (I), (II) and (III).

In conclusion, we should like to express our sincere thanks to Prof. K. Sakuma for his kind interest and continual encouragement throughout this work. We also take great pleasure in thanking Prof. S. Sakata, because this work has received many suggestions from his methodology.

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Interaction between Electrons in Two-Dimensional Free-Electron Model for Conjugated Systems

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An attempt is made to include the effects of electron-electron interactions into the two-dimensional free-electron model of Platt for conjugated systems. The method is explained in the case of the naphthalene molecule as a simple example. The excitation energies of the first four singlets are calculated. The agreement between theory and experiment is moderate. The method may be applied to larger systems with reasonable amount of labor, to which the applications of the usual theories based on atomic orbitals may not be feasible.

§ 1. Introduction

Using a model in which the electrons freely move in a properly shaped box of constant potential, Platt has established amazing results for conjugated systems¹⁾. For example, ten electrons free to move in a rectangle give the total density distribution of π -electrons closely reproducing the geometry of bond skeleton of the naphthalene molecule. This, Platt says, suggests that particular ring configurations are stabilized not merely by the hexagonal framework of the trigonal σ -bond, but also by the self-consistent coincidence of the nuclear positions with the rings of high π -electron density which are formed by the semi-free π -electron motion limited by the gross molecular shape. This interesting situation seems to mean that the wave functions of π -electrons supplied by the Platt model may be considered as useful approximations with realistic meanings to a certain degree. Then there arises a natural temptation to include the effects of electron-electron interactions in this model. This is the subject for which the result of some preliminary calculations will be reported in the present communication. The method of calculation is a straightforward extension of the one devised by Araki and Araki²⁾ for a one-dimensional free-electron model.

§ 2. Interaction potential between two electrons in a rectangle

We shall start with a brief consideration of the essential features of the π -electron approximation. For a molecule containing the so-called π -electrons, one explicitly considers only the π -electrons in place of writing out the complete many-electron Hamiltonian operator. The other electrons are treated merely as supplementing the nuclear field in which the π -electrons move. One writes

$$H = \sum_i A(i) + \sum_{i > j} (1/r_{ij}), \quad (2 \cdot 1)$$

in atomic units, where $A(i)$ includes the kinetic energy of π -electron i , its potential energy of attraction due to all nuclei, and its potential energy of Coulomb repulsion and exchange attraction due to electrons other than π -electrons and the electrostatic repulsion terms $(1/r_{ij})$ pertain to π -electron only. This procedure represents what is known as the π -electron approximation.

It appears reasonable to understand the above form of the Hamiltonian as a special case of the following general expression:

$$H = \sum_i A(i) + \sum_{i>j} B(i, j), \quad (2.2)$$

where the first term means the one-particle part and the second the two-particle part of the Hamiltonian. The manner of separating these two parts in (2.1) is a plausible one but other forms may also be possible. If we assume $H = \sum A(i)$ for the π -electron Hamiltonian, we tacitly understand that the π -electrons repulsions are wholly suppressed into the effective one-electron potential field. The free-electron model falls into this category. There must, however, remain the effects of the π -electron repulsions which cannot be brought into the form of the effective one-electron potential. In this respect, the present communication intends to supply an assumption for the second term of (2.2), $B(i, j)$, the first term being given by the Platt model.

In the Platt model π -electrons are confined to a three-dimensional box of suitable shape with constant potential inside. The z -motion is supposed to be independent of x and y and constant for all conjugated systems and it may be separated, leaving the two-dimensional wave equation,

$$\begin{aligned} H\psi &= E\psi, \\ H &= -(1/2)(p_x^2 + p_y^2) + U(x, y), \end{aligned} \quad (2.3)$$

where $U(x, y)$ may be defined properly according to the box shape adopted. In the following we will treat a rectangle exclusively. In this case

$$\begin{aligned} U(x, y) &= 0, & 0 < x < L_x, & \quad 0 < y < L_y, \\ U(x, y) &= \infty, & \text{otherwise.} \end{aligned} \quad (2.4)$$

The one-electron Hamiltonian in (2.3) is just corresponding to $A(i)$ in (2.2). The equation (2.3) has the well-known solutions,

$$\psi_{n_x, n_y}(x, y) = \frac{2}{(L_x L_y)^{1/2}} \sin(\pi n_x x / L_x) \sin(\pi n_y y / L_y), \quad (2.5)$$

$$E = \pi^2 / 2 \cdot (n_x^2 / L_x^2 + n_y^2 / L_y^2), \quad n_x, n_y \geq 1. \quad (2.6)$$

In order to take into account the electron-electron interaction we must have a two-dimensional version of the second term of the formula (2.2). To meet this need we shall follow closely the procedure devised by Araki and Araki in their treatment of one-dimensional free-electron model¹¹. Coulomb interaction potential between two electrons is approximately expanded in the Fourier series in a parallelepiped with the edge lengths of

$2L_x$, $2L_y$ and $2L_z$ as was done by them. If we take an average of the series over a range $-L_x < z_i - z_j < L_x$, we get $V/2L_x$, where

$$\begin{aligned} & V(x_i - x_j, y_i - y_j) \\ &= V_0 \\ &+ \frac{\pi}{L_x L_y} \left[\sum_{n_x=1}^{\infty} \frac{1}{(\pi n_x / L_x)^2} \cos(\pi n_x / L_x) (x_i - x_j) + \sum_{n_y=1}^{\infty} \frac{1}{(\pi n_y / L_y)^2} \cos(\pi n_y / L_y) (y_i - y_j) \right. \\ &\quad \left. + 2 \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \frac{1}{(\pi n_x / L_x)^2 + (\pi n_y / L_y)^2} \cos(\pi n_x / L_x) (x_i - x_j) \cos(\pi n_y / L_y) (y_i - y_j) \right]. \end{aligned} \quad (2.7)$$

Our assumption is that αV is the two-electron interaction potential in the present calculation where α is a variable parameter with a dimension of $[L^{-1}]$.

The present attempt to include the electron repulsion into the Platt model keeps a very close parallelism with the one-dimensional case treated by Araki and Araki²⁾. It is not a difficult task to expose in a general form our calculational scheme with full expressions of the formulas. But we will not do so here; it will occupy too much spaces. Instead, we shall try to explain the scheme of calculation in an example of preliminary application to the naphthalene molecule. Extensions to other cases will be straightforward.

§ 3. Application to naphthalene

In this section we calculate the excitation energies of four lowest singlet states of the naphthalene molecule. Our first step is to examine the energy levels in the Platt model without electron-electron interactions. According to the Platt's prescription we adopt $L_x = 13.749$ a.u. and $L_y = 10.584$ a.u. for the naphthalene molecule as is shown in Fig. 1. In this respect it is assumed that the carbon-carbon distance is 2.646 a.u. (1.4 Å).

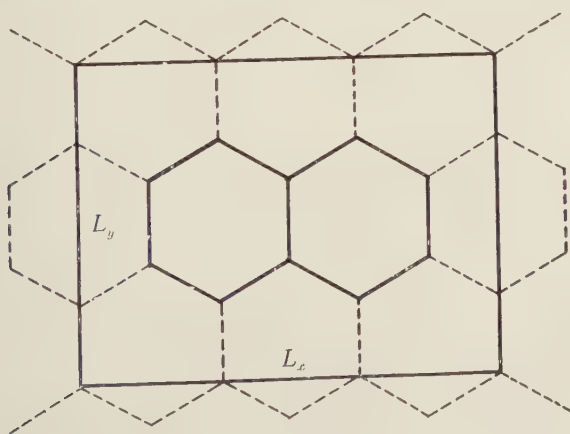


Fig. 1. Dimension of rectangle for naphthalene (according to Platt).

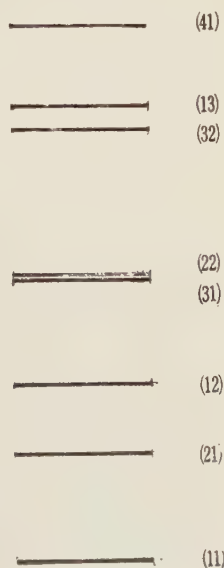


Fig. 2. Energy levels.

If we designate an energy level of the system by a set of quantum numbers (n_x, n_y), the eight lowest levels of naphthalene are (11), (21), (12), (31), (22), (32), (13) and (41). (see Fig. 2.).

Then the ground state configuration may be represented by

$$(0) \quad A_g \quad (11)^2(21)^2(12)^2(31)^2(22)^2.$$

As is seen in Fig. 2 two levels, (31) and (22) are accidentally almost degenerated. Four lowest excitations corresponding to the four observed singlets are described by the electron configurations,

$$\begin{aligned} (1) \quad B_{3u} & \quad (11)^2(21)^2(12)^2(31)^2(22)^1(32)^1, \\ (2) \quad B_{2u} & \quad \text{,,} \quad (31)^2(22)^1(41)^1, \\ (3) \quad B_{2u} & \quad \text{,,} \quad (31)^1(22)^2(32)^1, \\ (4) \quad B_{3u} & \quad \text{,,} \quad (31)^1(22)^2(41)^1. \end{aligned}$$

Numbers 0, 1, 2, 3 and 4 will be used hereafter as the indices of the states. The transition energies of these excitations are listed in Table 1, which are predicted by the Platt model without explicit inclusion of the electron-electron interactions. Platt assigned the same size of the rectangle both for naphthalene and azulene and so we display in Table 1 the experimental data for both molecules after him.

Table 1

ΔE_{calc} : excitation energy predicted by the Platt model (in cm^{-1}).

ΔE_{obs} : first four observed singlets of corresponding polarizations (in cm^{-1}).

Excitation	Polarization	ΔE_{calc}	ΔE_{obs}	
			Naphthalene	Azulene
(22) — (32)	α	$\Delta E_x^{(1)} \ 286 \times 10^3$	32×10^3	29×10^3
(22) — (41)	γ	$\Delta E_y^{(2)} \ 397 \times 10^3$	59×10^3	52×10^3
(31) — (32)	γ	$\Delta E_y^{(3)} \ 290 \times 10^3$	35×10^3	15×10^3
(31) — (41)	α	$\Delta E_x^{(4)} \ 401 \times 10^3$	45×10^3	36×10^3

As a next step, let us see what happens if we take account of the electron-electron interaction terms. The determinantal wave functions for the ground state and first four singlet states can be constructed in the usual manner. Preparing five wave functions, Ψ_0 , Ψ_1 , Ψ_2 , Ψ_3 , and Ψ_4 , we can perform the calculations of the excitation energies in keeping a close parallelism with the calculation of Araki and Araki¹ and we do not reproduce the details of calculations here. Some technical problems encountered in the course of calculation are discussed in the next section. First, we calculate the excitation energy for each configuration separately, which directly corresponds to the one listed in Table 1, and the results are shown in the column (1) of Table 2. However, configurations of the same symmetry character have an interaction between them through the electron-electron interactions. Then our next step is to work with the following two linear combinations:

$$\psi_r=a_1\psi_{r1}+a_4\psi_{r2}$$

$$\psi_{ry}=a_2\psi_{r2}+a_3\psi_{r3},$$

and to solve the corresponding two-dimensional secular equation for each case. The numerical values thus obtained are displayed in the column (2) of Table 2. The assignment of values for α is only tentative.

Table 2. The results of the present calculation (in cm⁻¹).

	(1)		(2)	
	$\alpha=0.2$	$\alpha=0.1$	$\alpha=0.2$	$\alpha=0.1$
$\Delta E_x^{(1)}$	413×10^2	350×10^2	355×10^2	329×10^2
$\Delta E_y^{(2)}$	609×10^2	503×10^2	610×10^2	504×10^2
$\Delta E_y^{(3)}$	162×10^2	226×10^2	162×10^2	226×10^2
$\Delta E_x^{(4)}$	595×10^2	498×10^2	653×10^2	519×10^2

§ 4. Sum of infinite series

In the present method, we have to calculate a number of integrals concerning the electron-electron interaction, which can be integrated term by term. The results thus obtained contain several kinds of infinite series. After Araki and Araki²⁾ we shall resort to what is known as the Fourier and Parseval theorem in order to evaluate the sums of these series.

If a_n and A_n are the Fourier coefficients of the cosine series of $f(x)$ and $F(x)$ respectively, it follows from the Parseval theorem that

$$\int_0^\pi f(x)F(x)dx=(\pi/2)\left\{(1/2)a_0A_0+\sum_{n=1}^\infty a_nA_n\right\}.$$

If the functions are given by

$$f(x)=e^{\lambda x},\qquad F(x)=x,\qquad 0\leq x\leq\pi,$$

their Fourier coefficients are

$$\begin{aligned}a_n&=(2/\pi)(\lambda/\lambda^2+n^2)[(-1)^ne^{\lambda\pi}-1],\\A_0&=\pi,\quad A_n=(2/\pi)(1/n^2)[(-1)^n-1],\quad (n\geq1),\end{aligned}$$

whence

$$\begin{aligned}\frac{1}{\lambda^2}[e^{\lambda\pi}(\lambda\pi-1)+1]&=\int_0^\pi f(x)F(x)dx\\&=\frac{\pi}{2}\left\{\frac{1}{\lambda}[e^{\lambda\pi}-1]+\frac{8}{\pi^2}\lambda[e^{\lambda\pi}+1]\sum_{k=0}^\infty\frac{1}{\lambda^2+(2k+1)^2}\frac{1}{(2k+1)^2}\right\}.\end{aligned}$$

Thus we have the following result :

$$\sum_{k=0}^\infty\frac{1}{\lambda^2+(2k+1)^2}\frac{1}{(2k+1)^2}=\frac{\pi}{8\lambda^3}\left[\lambda\pi-2\frac{e^{\lambda\pi}-1}{e^{\lambda\pi}+1}\right].$$

In the similar manner we can evaluate the sums of several series as follows :

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{1}{\lambda^2 + (2k+1)^2} \frac{1}{(2l)^2 - (2k+1)^2} &= \frac{\pi}{4\lambda} \frac{1}{\lambda^2 + (2l)^2} \frac{e^{\lambda\pi} - 1}{e^{\lambda\pi} + 1}, \\ \sum_{k=1}^{\infty} \frac{1}{\lambda^2 + (2k)^2} \frac{1}{(2l+1)^2 - (2k)^2} &= \frac{\pi}{4\lambda} \frac{1}{\lambda^2 + (2l+1)^2} \frac{e^{\lambda\pi} + 1}{e^{\lambda\pi} - 1} - \frac{1}{2\lambda^2} \frac{1}{(2l+1)^2}, \\ \sum_{k=0}^{\infty} \frac{1}{\lambda^2 + (2k+1)^2} \frac{1}{[(2l)^2 - (2k+1)^2]^2} &= \frac{\pi}{4\lambda} \frac{1}{[\lambda^2 + (2l)^2]^2} \frac{e^{\lambda\pi} - 1}{e^{\lambda\pi} + 1} + \frac{\pi^2}{16(2l)^2} \frac{1}{\lambda^2 + (2l)^2}, \\ \sum_{k=1}^{\infty} \frac{1}{\lambda^2 + (2k)^2} \frac{1}{[(2l+1)^2 - (2k)^2]^2} &= \frac{\pi}{4\lambda} \frac{1}{[\lambda^2 + (2l+1)^2]^2} \frac{e^{\lambda\pi} + 1}{e^{\lambda\pi} - 1} \\ &\quad + \frac{\pi^2}{16(2l+1)^2} \frac{1}{\lambda^2 + (2l+1)^2} - \frac{1}{2\lambda^2} \frac{1}{(2l+1)^4}. \end{aligned}$$

We do not display here the formulas already listed in the paper of Araki and Araki⁽²⁾ and the results which can be obtained by the simple limiting process $\lambda \rightarrow 0$ from the above formulas. Our final comment is that we have to evaluate a double sum corresponding to the last term of $V(x_i - x_j, y_i - y_j)$ in (2.7), which is eventually reduces to a single sum of an infinite series. Fortunately the convergence of the single sum is of sufficient rapidity and usually we need only first few terms of the series.

§ 5. Discussions

The results of the present calculation displayed in Table 2 are not necessarily encouraging. However, the present calculation is remaining in a rather preliminary stage and there opens a possibility of improvement by means of superposition of configurations. This very possibility has been acquired by the introduction of the electron-electron interactions into the Platt model. We have treated ten electron problem of naphthalene as an example. Applications of our method to larger systems may be possible with reasonable amount of labor. An apparent weak point of the present method is the assumption for the electron-electron interaction. We have averaged the three-dimensional Coulomb interaction over z -direction to get a two-dimensional interaction. But the meaning of the procedure is not sufficiently clear. Once, however, it is accepted as established, calculations thereafter can be performed without any doubtful approximations. In the usual theories based on atomic orbitals, we start with the genuine Coulomb interaction but in the later stage of calculation we are forced to resort to several approximations which are not always justified. An example of these is the neglect of multicenter integrals. In the present method, there do not exist the difficulties of multicenter integrals.

In conclusion the author would like to express his sincere thanks to Professor G. Araki for suggesting this problem and for many valuable discussions. The present work has been supported in part by Grant in Aid from Educational Ministry.

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One-Center Expansion of Molecular Wave Function

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The possibility of expanding molecular wave functions into a series of functions with a single center is examined. The calculations are performed for the ground state energies of H_2^+ and H_2 with reasonable results. For example, a simple expansion yields 3.80 e.v. as the binding energy of H_2 . The advantage of this method is, above all, its simplicity in calculations.

§ 1. Introduction

The determination of molecular wave functions is more difficult than that of atomic wave functions because the potential energy of nucleus-electron attraction involves two or more centers in the former case. It is thus a central problem in the molecular theory to overcome many difficulties in calculations caused by this fact and to simplify, wherever possible, the method of calculations. There are two main types of approximation which are most commonly used, known respectively as the valence bond and molecular orbital approximations. The valence bond method is essentially a generalization of the calculation used by Heitler and London for the hydrogen molecule. Here a molecule is regarded as composed of atoms and then the building blocks of the whole wave function are necessarily atomic orbitals. In the molecular orbital theory only the nuclei (or nuclei + inner shells) are first brought into position, and afterwards the valence electrons are allotted to polycentric molecular orbitals. There is no intrinsic connection of this method to atomic orbitals. However it is customary to construct the molecular orbitals as linear combinations of atomic orbitals. There are of course good reasons for the prevailing use of atomic orbitals as building blocks in constructing the whole molecular wave functions. It provides us very effective approximations for many cases.

But the very fact that the electron coordinates have many reference centers creates some obstacles difficult to overcome. Above all, the difficulty in the evaluation of energy integrals concerning the electron-electron interaction is most conspicuous. Two-center integrals are usually evaluated in elliptical coordinates by using the Neumann expansion of the electron-electron interaction. But this procedure is not applicable directly to the multi-center integrals. In all atomic problems we find no grave difficulty like this, where we can use the much simpler Legendre expansion because all electron coordinates are measured from a single center. This leads us very naturally to the idea of expressing the electronic charge cloud in molecules in reference to one appropriate coordinate origin, just as in atoms. Crudely stated, this is nothing but the central idea of the works on multi-center

integrals by Barnett and Coulson,¹⁾ Lundquist and Löwdin.²⁾ They, however, placed their emphasis on the mathematical techniques and they confined themselves to the evaluation of the difficult integrals encountered in the usual theories based on the atomic orbitals. It was a natural consequence that they identified the center of expansion of the wave function with the position of one specially chosen nucleus. We can, however, freely choose the center of the expansion anywhere if it is appropriate physically and mathematically. In this respect we find two very encouraging pioneer works, namely the works of MacDonald³⁾ and Matsen.⁴⁾ MacDonald treated the problem of higher excited states of the hydrogen molecule as a three-center problem. In his treatment the inner unexcited electron occupies the orbit of H_2^+ given by Guillemin and Zener⁵⁾ and the wave function of the outer excited electron is given by the hydrogen-like wave function with the center at the middle point between two nuclei. Of course he had to face a grave difficulty of evaluating three-center integrals, which were performed by means of graphical integrations. The most instructive point in his work is, undoubtedly, that he judged that, in spite of the tedious task of the graphical integrations, the "three-center description" of the whole wave function was superior to the more usual two-center description both with respect to the labors in calculations and the physical interpretations. Matsen tried to calculate the energies of the $1s\sigma$, $2p\sigma$, $2p\pi$, $3d\pi$ and $3d\sigma$ states of H_2^+ , by using the hydrogen-like atomic orbitals with the center at the middle point of two nuclei. For $1s\sigma$ state the result was rather poor but for the higher states the results were amazingly good far beyond our first expectation. Being encouraged by these works, we naturally arrive at an idea of describing all the electrons with reference to one single center appropriately chosen in a molecule. We shall begin with two simple examples, the hydrogen molecule ion H_2^+ and the hydrogen molecule H_2 . These may serve as useful means of test and illustration of our procedure: one-center expansion of molecular wave functions.

§ 2. Hydrogen molecule ion

The one-electron problem of H_2^+ can be solved exactly and the very accurate values of energies and wave functions of various states are now available.⁶⁾ Since the higher states were already treated successfully by Matsen as was mentioned in the introduction we restrict ourselves to the calculation of energy of the ground $1s\sigma$ state. We set up the one-center σ -function in the following form:

$$\phi = c_1 s_1(\zeta; r) + c_2 s_n(\zeta'; r) + c_3 d_n(\zeta'; r, \vartheta), \quad (1)$$

$$s_n(\zeta; r) = N(n, \zeta) r^{n-1} e^{-\zeta r} Y_0,$$

$$d_n(\zeta; r, \vartheta) = N(n, \zeta) r^{n-1} e^{-\zeta r} Y_2,$$

$$N(n, \zeta) = (2\zeta)^{n+1/2} [(2n)!]^{-1/2},$$

where ζ , ζ' , c_1 , c_2 and c_3 are variable parameters which should be determined by the variation method and n is also a variable integral number. $N(n, \zeta)$ is the normalization constant for the radial function and Y_l is the normalized axially symmetric spherical surface

Table 1. Nuclear distance $R=1.4$ a.u.

	(1)	(2)	(3)
c_1	1	1	0.761157
c_2	0	0	0.253422
c_3	0	0	0.110674
n	—	—	4
ζ	2	1.095	1.1
ζ'	—	—	4.3
E_{calc}	-0.44016	-1.168	-1.264020
E_{exact}	-1.284265 a.u.		

Table 2. Nuclear distance $R=2.0$ a.u.

	(1)	(2)	(3)
c_1	1	1	0.661432
c_2	0	0	0.339056
c_3	0	0	0.179056
n	—	—	4
ζ	2	0.912	1
ζ'	—	—	3
E_{calc}	+0.10939	-0.967	-1.074755
E_{exact}	-1.102625 a.u.		

Table 3. Nuclear distance $R=2.6$ a.u.

	(1)	(2)	(3)
c_1	1	1	0.626715
c_2	0	0	0.364435
c_3	0	0	0.227291
n	—	—	4
ζ	2	0.787	0.8
ζ'	—	—	2.5
E_{calc}	+0.49209	-0.826	-0.932967
E_{exact}	-0.975475 a.u.		

harmonic of the l th degree. The essential point is that the origin of coordinates is located at the middle point between two nuclei. The results are listed in Tables 1, 2 and 3.

In each of the above tables the column (1) shows the result obtained by the first order perturbation started from the He atom ground state, the united atom in its strict sense of the word, and (2) the result of Matsen's single-term approximation and (3) the result of the present three-term expansion. The improvement in each stage is quite obvious. The molecule can bind only in the case of (3). Our result seems to be a little bit inferior to the one given by LCAO MO approximation with variable screening but definitely better than LCAO MO without screening. A further comment is that our efforts to find the best values of variational parameters are not exhaustive. Some of them are supplied through rather unconvincing guess works. The situation is the same also in the following section.

§ 3. Hydrogen molecule

In this section we shall calculate the binding energy of the H_2 molecule at the fixed nuclear distance of 1.4 a.u. For the convenience of explanations, let us divide the following description into five stages. We hope this proves to have some heuristic advantage.

Stage (a). As a preliminary step, let us try to calculate the energy of the ground state of the H_2 system by using a wave function, $\Psi = s_1(\zeta; r_1) s_1(\zeta; r_2)$. This may be called a united-atom treatment of H_2 in the sense used by Matsen³⁾ for the H_2^+ case. The origin of electron coordinates is of course located at the middle point between two nuclei as before. With $\zeta = 0.935$ the result is -1.7022 a.u. for the electronic energy which means the binding energy of -0.32934 e.v. Thus the molecule cannot bind at all.

Stage (b). The function in the previous stage has an obvious defect. Two electrons concentrate too much at the middle point. Next we shall try to remedy this by spreading the function as $\Psi \sim \phi(1)\phi(2)$, where ϕ is given by (1) without d term. Although this function is still preserving the complete spherical symmetry around the center of gravity it yields a rather unexpected result that the molecule does bind with the binding energy 0.54289 e.v., the parameter values being $n=4$, $\zeta=1$, $\zeta'=4.3$ and $c_2/c_1=0.2$.

Stage (c). Although we have the stable molecule in the stage (b), the wave function used there cannot be admitted as meaningful because it has a complete spherical symmetry. The hydrogen molecule must have an elongated charge cloud. Then a plausible guess will be to suppose that the inner electron has a wave function similar to the one adopted for H_1 at $R=1.4$ a.u. in the preceding section and the outer electron under the influence of the screened nuclear field has an atomic $1s$ -type wave function. The following function is written down along this idea.

$$\Psi \sim \phi_{in}(1)\phi_{out}(2) + \phi_{out}(1)\phi_{in}(2), \quad (2)$$

$$\phi_{in} = s_1(1.1; r) + 0.5s_4(4.3; r) + 0.24406d_4(4.3; r, \theta),$$

$$\phi_{out} = s_1(0.8; r),$$

where ϕ_{in} is almost of the same form as the wave function ϕ of H_2^+ at $R=1.4$ a.u. ϕ_{out} is simply an s -type function. This function is the so-called "in-out" type function though in a very primitive fashion. The calculation is quite elementary. It gives 2.5766 e.v. as the binding energy. This result tells us that the single-term approximation for the outer electron is too crude for the ground state. However it is probable that it works well for the higher excited states.

Stage (d). As the next step we try to include the so-called "left-right" correlation by adding to the function of (c) stage a term, $p(1)p(2)$, where $p(r, \vartheta) = N(3, 3)r^3e^{-\frac{1}{2}r}Y_1$. This leads to a binding energy 2.8837 e.v.

Stage (e). This is the final stage of our calculation so far performed. Of course this does not mean any final stage of the present approximation. Here we use the following wave function.

$$\Psi = c_1\varphi_1(1, 2) + c_2\varphi_2(1, 2) + c_3\varphi_3(1, 2), \quad (3)$$

$$\varphi_1(1, 2) = [s_1(\zeta; r_1) + as_4(\zeta'; r_1)][s_1(\zeta; r_2) + as_4(\zeta'; r_2)],$$

$$\begin{aligned} \varphi_2(1, 2) = & [s_1(\zeta; r_1) + as_4(\zeta'; r_1)]d_4(\zeta'; r_2, \vartheta_2) \\ & + d_4(\zeta'; r_1, \vartheta_1)[s_1(\zeta; r_2) + as_4(\zeta'; r_2)], \end{aligned}$$

$$\varphi_3(1, 2) = d_4(\zeta'; r_1, \vartheta_1)d_4(\zeta'; r_2, \vartheta_2),$$

where the origin of coordinates is located at the middle point between two nuclei. This may be interpreted as follows. In the stage (c) we allotted a sausage type wave function for the inner electron while a simple s -type function for the outer electron. If, however, we allot a sausage type function also for the outer electron, we may be able to obtain a better result. This is the idea which leads us to the above form of the wave function. Another way of explanation may be supplied through the language of the configuration interaction. In this way the in-out correlation may be included more reasonably than in the stage (c).

With the above wave function the calculation is still straight-forward and not tedious. It gives the following result with the parameter values $\zeta=1$, $\zeta'=4.3$, $a=0.25$.

Binding energy: 3.7999 e.v. ($R=1.4$ a.u.)

Coefficients: $c_1=0.66109$, $c_2=0.07881$, $c_3=0.06788$.

This is better than the value 3.60 e.v. obtained by Coulson's best molecular orbital. It may be expected that if we add to the function a left-right correlation term like $p(1)p(2)$ of the stage (d) we have a binding energy of about 4 e.v.

§ 4. Discussion and conclusion

In view of the results of the exploratory applications of the present method to the hydrogen molecule and ion, it seems that the situation is certainly not hopeless. But it is to be noted that we have treated in the preceding sections only two specially simple

examples. The present method must be examined more carefully from the general point of view.

The first advantage of the present method is the easiness of integral calculations. For the sake of convenience, we shall borrow here the notations of Roothaan⁵⁾ without explanations. In his notations all that we need are the following three integrals:

$$(i) \quad (\chi_a | \chi_b), \quad (\rho=0),$$

$$(ii) \quad [a | Q_b],$$

$$(iii) \quad [Q_a | Q_b], \quad (\rho=0).$$

The nuclear distance, ρ , is entered only in the nuclear attraction integral $[a | Q_b]$ and this is easily evaluated by means of the auxiliary function $A_n(\rho)$,

$$A_n(\rho) = \int_0^\infty \tilde{z}^n e^{-\rho \tilde{z}} d\tilde{z} = n! \rho^{-n-1} e^{-\rho} \sum_{k=0}^n \rho^k / k!,$$

$$A_0(\rho) = e^{-\rho} / \rho, \quad A_n(\rho) = (1/\rho) [e^{-\rho} + n A_{n-1}(\rho)].$$

The most troublesome in the molecular theory is the integral of electron-electron interaction energy,

$$[Q_a | Q_b] = \iint [Q_a(1) Q_b(2) / r_{12}] dv_1 dv_2.$$

In the present scheme, however, charge distribution Q_a , Q_b are co-centric and we can easily evaluate the above integral. This is a very favorable situation for the large scale tabulation of the integrals. There exist no difficulties of hetero-nuclear and multi-center integrals.

The second advantage of the present method is its conceptual simplicity especially for higher excited states of molecules. We shall publish in the near future the analytical version of MacDonald's works on H_2 excited states. While we were not so successful in the single-term approximation for the outer electron of the ground state of H_2 as described in the stage (c) of the preceding section, the higher states will be treated more successfully. This conjecture is supported by the encouraging results obtained by Matsen.¹⁾ This technical possibility may not be limited to the hydrogen molecule only.

Needless to say, the procedure of one-center expansion of molecular wave functions reveals many disadvantages when compared with the prevailing approximations based on atomic orbitals. Expansions of molecular wave functions in one-center functions would be almost impossible for the inner electron still preserving their individual atomic characters in molecules. However it is outer electrons belonging to the whole molecular frame rather than to atom which play the key rôle in determining the physical behavior of molecules. For these outer electrons, the atomic functions form less satisfactory building blocks and conversely the present one-center expansion may increase its suitability both physically and mathematically. Meanwhile the difficulty concerning the inner electrons concentrated on the nuclear positions might be side-stepped by means of certain semi-empirical procedures. In addition the recent advance in computing devices encourages the hope that it may become

possible to expand in good accuracies the molecular wave function into a series of one-center functions by including a large number of terms.

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Letters to the Editor

On the Feynman's Theory of Polarons

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Recently Feynman¹⁾ developed a powerful approach to the polaron problem, namely he has introduced the variational method to the path integral formulation. His results cover fairly well all values of the coupling constant, and the detailed comparison with the more usual method is given by Höhler.²⁾ Although the method could have much wider applicability, as it is based on the path integral, the choice of the trial functional is limited to quadratic forms; and thus it is desirable to find out how his method may be expressed in conventional notation, for a wider class of trial functional might thereby become available, as Feynman himself suggested in his paper. Even if we limit our attention to the polaron problems, it cannot supply any knowledge about the wave functions and it is difficult to estimate the other physical quantities, although Feynman's method is very useful for the ground state energy problems.

In this note as a preliminary to find out the trial wave function which just gives the Feynman's result, we develop a way to express his method in conventional notation. Our method consists in the transcription of the path integral to the operator calculus of Feynman,³⁾ and to the Hamiltonian

formulation.

We use the same symbols and notations as those used by Feynman and cite his equations as, for example, (F.1), etc. Let us write

$$H = H_0 + H_I \quad (1)$$

where

$$\left. \begin{aligned} H_0 &= P^2/2 + \int dk a_k^* a_k, \\ H_I &= i \sqrt{\frac{\sqrt{2} \alpha}{4\pi^2}} \int \frac{dk}{K} (a_k^* e^{-ikX} - a_k e^{ikX}) \end{aligned} \right\} \quad (2)^*$$

First we introduce a fictitious field operator A , which should describe the p -wave phonon cloud and has the following properties,

$$\begin{aligned} [A_i, A_j^*] &= \delta_{ij}, \quad [A_i, a_k^*] = [A_i, a_k] = 0, \\ &\text{etc.} \end{aligned} \quad (3)$$

Roughly speaking, in Feynman's method the phonon cloud is taken to be the p -wave, which is governed by the Hamiltonian H_I (see below) and is treated exactly, and as the parameter of variation the best value of their frequency ω and coupling strength C are determined.

Let us consider the Hamiltonian

$$H_I = H_0 + H_I,$$

where

$$\begin{aligned} H_{II} &= \omega \left(A^* + \frac{\sqrt{2}C}{\omega} X \right) \left(A + \frac{\sqrt{2}C}{\omega} X \right) \\ &= \frac{\omega}{2} (\pi^2 + \phi^2 - 3) + \end{aligned}$$

* (F.1) contains a misprint by a factor $\sqrt{2}$.

$$+2\sqrt{C}(X\phi) + \frac{2C}{\omega}X^2, \quad (4)$$

ϕ and π are defined as usual: $\phi = (1/\sqrt{2}) \times (A + A^*)$, $\pi = (i/\sqrt{2}) \cdot (A^* - A)$. H_1 is so chosen as to become

$$\langle \phi_{0A} | e^{-H_1 T} | \phi_{0A} \rangle = e^{\delta_1}, \quad (5)$$

where $|\phi_{0A}\rangle$ is the vacuum state belonging to a_k and A , namely it is defined by $a_k |\phi_{0A}\rangle = A |\phi_{0A}\rangle = 0$. Note that S_1 is the operator concerning the electron variables X, P .

We can diagonalize the Hamiltonian H_1 by a unitary transformation U ,

$$U^{-1}H_1U = H_0 + vB^*B + 3/2(v - \omega), \quad (6)$$

where $\phi = \sqrt{\omega/2v}(B + B^*)$, $\pi = i\sqrt{v/2\omega} \times (B^* - B)$ and $v^2 = \omega^2 + 4C/\omega$. U is given by

$$U = \exp \{ i \sin^{-1} (\sqrt{4C/\omega^3}) \times (\sqrt{\omega}X\pi - (1/\sqrt{\omega})P\phi) \}. \quad (7)$$

Thus we can find the ground state of H_1 with the eigenvalue E_1 as

$$H_1|\Psi\rangle = E_1|\Psi\rangle, \quad E_1 = (3/2)(v - \omega), \\ |\Psi\rangle = U|\phi_0\rangle, \quad (8)$$

where $B|\phi_0\rangle = a_k|\phi_0\rangle = P|\phi_0\rangle = 0$. We thus see that the Feynman's lowest energy value E_1 belonging to the action S_1 is the lowest eigenvalue of our Hamiltonian H_1 and represents the increase of zero point energy due to the coupling C .

With the aid of this supplementary Hamiltonian H_1 and using the operator calculus of Feynman,³⁾ we can find without much difficulty that the expression A of (F·31) can be obtained from

$$\langle \Psi | H_I (H_1 - E_1)^{-1} H_I | \Psi \rangle. \quad (9)$$

It just has the form of the second order

perturbation formula, H_1 being the unperturbed and H_I the perturbation. This is our main result obtained in this note. Note that, as H_I has no diagonal element, $(H_1 - E_1)^{-1}$ has no singularity in eq. (9).

Finally the B of (F·32) can be expressed as

$$B = \langle \Psi | (2C/\omega)X^2 - \sqrt{2C}X(H_1 - E_1 + \omega)^{-1}\sqrt{2C}X | \Psi \rangle, \quad (10)$$

if we directly transcribe the Feynman's method using the operator calculus. It can be expressed also by

$$B = \langle \Psi | (2C/\omega)X^2 + \sqrt{C}(X\phi) | \Psi \rangle \\ = \langle \Psi | (1/2)P^2 | \Psi \rangle = \langle \Psi | H_0 | \Psi \rangle. \quad (11)$$

Thus we can express the Feynman's results in our notation

$$E = E_1 - B - A \\ = \langle \Psi | H_1 - H_0 - H_I(H_1 - E_1)^{-1}H_I | \Psi \rangle. \quad (12)$$

This can be interpreted as follows. Let us consider the Hamiltonian $H_0 + H_I + H_{I'} = H_1 + H_{I'}$, the minimum eigenvalue of which is certainly larger than that of H_1 as $H_{I'}$ is positive definite, and if we apply the second order perturbation formula, H_I being perturbation, we obtain $E_1 - A$. Now we must subtract the contribution due to the extra interaction $H_{I'}$, which is given by the increase of the kinetic energy of the electron i.e., $\langle \Psi | P^2/2 | \Psi \rangle (=B)$. Thus we can obtain the simple interpretation of Feynman's theory in terms of the usual notation, but it is regrettable that owing to the lack of the powerful inequality corresponding to (F·10), we cannot yet complete our program, as the sign of the higher order terms

cannot so easily be estimated.

If we express H to be $H_1 + (H - H_1)$ and treat $H - H_1$ as perturbation we obtain

$$B - A - (1/3v)B^2 \quad (13)$$

but as the higher order contribution has no definite sign, it cannot be used.

We can generalize the Feynman's method by taking the electron mass in the action S_1 as the variational parameter, namely we consider

$$H'_1 = (1/2m)\mathbf{P}^2 + \int dk a_k^* a_k + H_{1I}. \quad (14)$$

We can proceed in the same way as the above and the final answer is

$$E' = E_1 - B - (3/4)(1 - m) \times \{(v^2 - \omega^2)/v\} - \sqrt{m}A, \quad (15)$$

which gives somewhat better results than those of Feynman in both strong and weak coupling regions, although the main term never changes.

I should like to express my heartfelt thanks to Prof. S. Hayakawa for his kind interest shown to this work.

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Hydrodynamical Theory of the Multiple Production of Particles in High Energy Nucleon-Nucleus Collision

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In the cosmic ray experiments, we have

more chance to observe the multiple production of particles in the nucleon-nucleus collision than in the nucleon-nucleon one. It is therefore desirable to apply the theories of the elementary process of the high energy nuclear interaction to the nucleon-nucleus collision and compare the characteristic results with experimental data. The application of the Fermi theory is well-known. So we shall consider here Landau's hydrodynamical theory.¹⁾ It may be interesting to note that the continuous medium model of Landau, when applied to the collision of a nucleon (or light nucleus) with a nucleus, bears some resemblance to the so-called "individual" collision model.

The nuclear mass number dependence of the multiplicity of secondary particles was calculated by Belen'kij and Milenkhin²⁾ in the hydrodynamical theory, but the result was much similar to the one which had been obtained from Fermi's theory. So we have treated the motion of the nuclear fluid to get the energy and angular distribution of the emitted particles, by solving the equations of the relativistic hydrodynamics under suitable boundary conditions. We here report only the results and the detailed calculations and discussions will be published later in this journal.

Consider a nucleon, with Lorentz contracted thickness d and momentum and energy (P, E) , which collides with a tube of nuclear matter, with thickness $n \times d$ and momentum and energy $(-nP, nE)$. We are considering the problem in the equi-velocity system of reference (e. v. s.), i. e., the coordinate system in which the incident nucleon and the target nucleus have the same magnitude and the opposite directions of velocity. n is approximately equal to the number of nucleons in the tube. If we

confine ourselves to the case $n \lesssim 3.7$, the potential of the one-dimensional motion of the fluid, introduced by Khalatnikov,¹⁰ can be obtained as follows:

$$\begin{aligned} \bar{\chi}(\alpha, \beta) = & \int_0^\alpha da I_0(\sqrt{A(\alpha-a)}\beta) \\ & \times F(a; t_2, x_2) \\ & + \int_0^\beta db I_0(\sqrt{A\alpha(\beta-b)}) \\ & \times F(b; t_1, -x_1), \end{aligned} \quad (1)$$

where, I_0 is the zeroth order imaginary Bessel function and

$$\begin{aligned} F(a; t_2, x_2) = & -(1/2) [(c_0 - 1/2 - B) \\ & \times (t_2 - x_2) \exp\{-(c_0 - 1/2)a\} \\ & + (c_0 + 1/2 - B)(t_2 + x_2) \\ & \times \exp\{-(c_0 + 1/2)a\} - (c_0 - 2B)t_2 \\ & \times \exp(-c_0 a/2)]. \end{aligned}$$

This was obtained under the boundary condition that, at the both wave fronts, the solution should be connected with the simple wave solution. In eq. (1), the variables (α, β) are defined by

$$\alpha = \eta - (\gamma/c_0), \quad \beta = -\eta - (\gamma/c_0)$$

with $T/T_0 = \exp(\gamma)$: the temperature divided by its initial value, and $v = \tanh \eta$: the velocity of the fluid element in the e. v. system. A and B are related to the sound velocity c_0 , which we have assumed to be constant, by $A = 1 - 2c_0^2 + (1 - 3c_0^2)^2/(4c_0^2)$ and $B = (1 - 3c_0^2)/(4c_0)$. (In Landau's theory, $c_0 = (1/3)^{1/2}$, $A = 1/3$ and $B = 0$.) t_1, x_1 and t_2, x_2 are the time and positions at which the out-flow of the matter begins in backward and forward directions, respectively. The coordinate variables t and x are related to the γ and η by the relation¹¹

$$\begin{aligned} t = & \exp(-\gamma) \{(\partial \chi / \partial \gamma) \cosh \eta - (\partial \chi / \partial \eta) \sinh \eta\} \\ x = & \exp(-\gamma) \{(\partial \chi / \partial \gamma) \sinh \eta - (\partial \chi / \partial \eta) \cosh \eta\}. \end{aligned} \quad (2)$$

where $\chi = \bar{\chi} \exp(-\gamma)$.

From the exact solution (1) and (2) of the one-dimensional fluid motion, we have obtained the approximate expressions for the temperature and velocity. The energy and angular distribution of the secondary particles can be estimated from them roughly. Some characteristic results from our calculation are summarized as follows:

1) The ratio of the amount of energy carried in forward and backward directions (in the e. v. s.) is

$$E_{\text{forward}} : E_{\text{backward}} = 1 : n.$$

(In Fermi's model, this ratio is 1:1 in the c. m. system.)

2) Most of the emitted particles are slow with energies not so much different from the nucleon rest mass, and about equal numbers of particles are emitted in forward and backward directions. While as for particles with very high energies (totally several ten percents of the available energy), they are emitted in the forward and backward direction with the ratio 1:n. (In Fermi's model with zero impact parameter, every particle has the same energy and emitted isotropically in the c. m. system.) The angular distribution of particles, in the laboratory system, will become more sharp than in Fermi's model.

We finally remark that the angular distribution of particles in the nucleon-nucleon collision, which we can estimate from the solution (1) and (2) with $n=1$, is somewhat more sharp than the approximate formula given in Landau's paper,¹¹

if we disregard the uncertainties in the rough treatment of the three dimensional motion.

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Orbits of an Electron in Static Electromagnetic Fields, I

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Static electromagnetic fields in which the Hamilton-Jacobi equation of an electron is integrable by separation of variables are determined. The field for a perfect imaging system is rediscovered.

§ 1. Introduction and outline

Recent progress in the field of electron microscope, mass spectrometer, accelerator, etc., encourages one to study the general properties of orbits of an electron in static electromagnetic fields. Since the close of last century an ingenious approach to the study of orbits has been opened by Poincaré, succeeded by Birkhoff, Morse and others. The approach is, however, not a paved one.

When the equation of motion is integrable, not merely the study of orbits is more tractable, but some features of orbits otherwise to be overlooked may be revealed.

The determination of the Hamilton-Jacobi equations, that are integrable by separation of variables, has a paramount importance for the integration of the equation of motion and has a long history.

In 1893, generalizing the Liouville's theorem, Staeckel¹⁾ showed that a dynamical system with its kinetic energy

$$1/2 \cdot \sum h_i \dot{q}_i^2$$

is integrable when the h_i 's have the expression

$$1/h_i = \varphi^{1i}, \tag{S}$$

the matrix $\|\varphi^{ij}\|$ being inverse to the matrix $\|\varphi_{ij}\| \equiv \Phi$, φ_{ij} being a function of q_i alone. He showed further that there exists not only the energy integral

$$\sum 1/h_i \cdot p_i^2 = \sum \varphi^{1i} p_i^2 = 2\alpha_1$$

but also $n-1$ other integrals

$$\sum \varphi^{ki} p_i^2 = 2\alpha_k \quad (k=2, 3, \dots, n)$$

p_i being the momentum conjugate to q_i .

In 1893 Goursat²⁾ showed that when the potential function B of the dynamical system has the expression

$$B = \sum \varphi^{i_1} \phi_{i_1}, \quad \phi_i = \phi_i(q_i), \quad (G)$$

the Hamilton-Jacobi equation is integrable and has a complete integral

$$-\alpha_1 t + \sum_i \int \sqrt{2} (\alpha_1 \varphi_{i1} + \alpha_2 \varphi_{i2} + \cdots + \alpha_n \varphi_{in} - \phi_i)^{1/2} dq_i,$$

$\alpha_1, \alpha_2, \dots, \alpha_n$ being arbitrary constants.

In 1904 Levi-Civita⁵⁾ showed that the Hamilton-Jacobi equation is integrable by separation of variables when and only when the Hamiltonian H satisfies the $n(n-1)/2$ equations of the second order

$$\begin{aligned} \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial q_i \partial q_j} - \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_j} \frac{\partial^2 H}{\partial q_i \partial p_j} - \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_j} \frac{\partial^2 H}{\partial p_i \partial q_j} + \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial q_j} \frac{\partial^2 H}{\partial p_i \partial p_j} \\ = 0, \quad i \neq j \quad (L) \end{aligned}$$

Levi-Civita determined the H of two degrees of freedom, leaving the determination of the H of three degrees of freedom to Dall'Aqua³⁾, who solved the problem completely.

In 1924 Weinacht⁵⁾ determined the h_i 's in the euclidian space of two and three dimensions together with respective potential functions. His result is, however, cursory and not complete.

In 1928 Robertson⁶⁾ remarked that the Schroedinger equation

$$\sum \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \frac{\sqrt{g}}{h_i} \frac{\partial \psi}{\partial q_i} + k^2 (E - V) \psi = 0, \quad g = h_1 \cdots h_n$$

is separable when the Hamilton-Jacobi equation is of the Staeckel form and when $\det \psi = \sqrt{g} f_1(q_1) \cdots f_n(q_n)$, f_i being a function of q_i alone.

In 1934 Eisenhart⁷⁾ showed that the latter condition of Robertson is equivalent to the vanishing of Einstein tensor

$$R_{ik} = 0$$

so that the Schroedinger equation is separable simultaneously with the Hamilton-Jacobi equation when the space is euclidian. Eisenhart determined further the h_i 's in the euclidian space of three dimensions and respective potential functions.

In this paper are determined static electromagnetic fields in which the Hamilton-Jacobi equation of an electron is separable.

Separable coordinate systems and respective Staeckel matrices are listed in § 2. Electric fields, magnetic fields, electromagnetic fields are listed in §§ 3, 4, 5 respectively.

Let the line element be denoted by

$$ds^2 = h_1 dq_1^2 + h_2 dq_2^2 + h_3 dq_3^2$$

in a coordinate system (q_1, q_2, q_3) , then the Hamiltonian of an electron in a static field is written

$$H = (1/2h_1) (p_1 - b_1)^2 + (1/2h_2) (p_2 - b_2)^2 + (1/2h_3) (p_3 - b_3)^2 + B$$

where

$$b_i = e/m \cdot A_i(q_1, q_2, q_3),$$

$$B = e/m \cdot A(q_1, q_2, q_3)$$

are respectively vector potentials and scalar potential multiplied by the specific charge of electron.

The b_i 's are determined by the Levi-Civita's condition (L) combined with the Lamé's equations. The b_i 's thus determined are identical with that of Eisenhart except for some notations.

When the field is purely electric, the scalar potential is determined by the Goursat's condition (G) and the Maxwell's equation

$$\frac{\partial}{\partial q_1} \sqrt{\left(\frac{b_2 b_3}{h_1}\right)} \frac{\partial B}{\partial q_1} + \frac{\partial}{\partial q_2} \sqrt{\left(\frac{b_3 b_1}{h_2}\right)} \frac{\partial B}{\partial q_2} + \frac{\partial}{\partial q_3} \sqrt{\left(\frac{b_1 b_2}{h_3}\right)} \frac{\partial B}{\partial q_3} = 0.$$

When the magnetic field is present, the determination of b_i and B is difficult. The Levi-Civita's method is cumbersome. The Staeckel's method of special values⁸⁾ combined with the method of Caratheodory⁹⁾ concerning radicals is employed there. The equations for b_i are

$$\sum_k (\partial/\partial q_k) \cdot (\sqrt{g}/h_i b_k) f_{ik} = 0,$$

$$f_{ik} = (\partial b_i / \partial q_k) - (\partial b_k / \partial q_i), \quad g = h_1 h_2 h_3.$$

The magnetic field strength is represented by three orthogonal components

$$H_1 = (1/\sqrt{h_2 h_3}) \cdot f_{23}, \quad H_2 = (1/\sqrt{h_3 h_1}) \cdot f_{31}, \quad H_3 = (1/\sqrt{h_1 h_2}) \cdot f_{12}$$

that contain the factor e/m .

An objection is raised that to exhaust all integrable systems might be nothing more than a mathematical exercise that is out of use in physics. A brief inspection into the list of § 5, however, shows us that there appears repeatedly the field for a perfect imaging system,¹⁰⁾ which was discovered by another approach. It is shocking. The study of all systems integrable by separation of variables seems to be a task worth working for.

The writer is much obliged to Prof. Miyamoto for his criticism and to the members of Miyamoto laboratory for their encouragement.

§ 2. Coordinate systems of Staeckel

1) Cartesian coord.

$$\begin{aligned} ds^2 &= dq_1^2 + dq_2^2 + dq_3^2 \\ &= dx^2 + dy^2 + dz^2, \end{aligned}$$

$$\phi = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{vmatrix}.$$

2) Cylindrical coord.

$$\begin{aligned}
 ds^2 &= dr^2 + r^2 d\theta^2 + dz^2 \\
 x &= r \cos \theta \\
 y &= r \sin \theta \\
 z &= z,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} 1 & -1/r^2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

3) Spherical coord.

$$\begin{aligned}
 ds^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \\
 x &= r \sin \theta \cos \varphi \\
 y &= r \sin \theta \sin \varphi \\
 z &= r \cos \theta,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} 1 & -1/r^2 & 1 \\ 0 & 1 & -1/\sin^2 \theta \\ 0 & 0 & 1 \end{vmatrix}.$$

4) Elliptic coord.

$$\begin{aligned}
 ds^2 &= c^2 (\operatorname{ch}^2 u - \cos^2 v) (du^2 + dv^2) + dz^2 \\
 x &= c \operatorname{ch} u \cos v \\
 y &= c \operatorname{sh} u \sin v \\
 z &= z,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} 1 & c^2 \operatorname{ch}^2 u & 0 \\ -1 & -c^2 \cos^2 v & 0 \\ 0 & -1 & 1 \end{vmatrix}.$$

5) Parabolic coord.

$$\begin{aligned}
 ds^2 &= (u^2 + v^2) (du^2 + dv^2) + dz^2 \\
 x &= 1/2 \cdot (u^2 - v^2) \\
 y &= uv \\
 z &= z,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} 1 & u^2 & 0 \\ -1 & v^2 & 0 \\ 0 & -1 & 1 \end{vmatrix}.$$

6) $ds^2 = dq_1^2 + q_1^2 (f_2(q_2) + f_3(q_3)) (dq_2^2 + dq_3^2)$

$$\begin{aligned}
 f_i &= f(q_i) \\
 f'^2 &= -4f^3 - \lambda f^2 - \mu f - \nu,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} 1 & -1/q_1^2 & 0 \\ 0 & f_2' & -1 \\ 0 & f_3' & 1 \end{vmatrix}.$$

7) Oblate spheroidal coord.

$$\begin{aligned}
 ds^2 &= c^2 (\operatorname{ch}^2 u - \cos^2 v) (du^2 + dv^2) + c^2 \operatorname{ch}^2 u \cos^2 v d\varphi^2 \\
 x &= c \operatorname{ch} u \cos v \cos \varphi \\
 y &= c \operatorname{ch} u \cos v \sin \varphi \\
 z &= c \operatorname{sh} u \sin v,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} c^2 \operatorname{ch}^2 u & -1 & 1/\operatorname{ch}^2 u \\ -c^2 \cos^2 v & 1 & -1/\cos^2 v \\ 0 & 0 & 1 \end{vmatrix}.$$

8) Prolate spheroidal coord.

$$\begin{aligned}
 ds^2 &= c^2 (\operatorname{ch}^2 u + \sin^2 v) (du^2 + dv^2) + c^2 \operatorname{sh}^2 u \sin^2 v d\varphi^2 \\
 x &= c \operatorname{sh} u \sin v \cos \varphi \\
 y &= c \operatorname{sh} u \sin v \sin \varphi \\
 z &= c \operatorname{ch} u \cos v,
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} c^2 \operatorname{sh}^2 u & -1 & -1/\operatorname{sh}^2 u \\ c^2 \sin^2 v & 1 & -1/\sin^2 v \\ 0 & 0 & 1 \end{vmatrix}.$$

9) Paraboloidal coord.

$$\begin{aligned}
 ds^2 &= (u^2 + v^2) (du^2 + dv^2) + u^2 v^2 d\varphi^2 \\
 x &= uv \cos \varphi \\
 y &= uv \sin \varphi \\
 z &= (1/2) \cdot (u^2 - v^2),
 \end{aligned}
 \quad
 \Phi = \begin{vmatrix} u^2 & -1 & -1/u^2 \\ v^2 & 1 & -1/v^2 \\ 0 & 0 & 1 \end{vmatrix}.$$

10) Ellipsoidal coord.

$$ds^2 = (f_1 - f_2)(f_1 - f_3)dq_1^2 + (f_2 - f_1)(f_2 - f_3)dq_2^2 + (f_3 - f_1)(f_3 - f_2)dq_3^2$$

$$f_i = f(q_i) \quad \Phi = \begin{vmatrix} f_1^2 & f_1 & 1 \\ f_2^2 & f_2 & 1 \\ f_3^2 & f_3 & 1 \end{vmatrix}$$

$$f'^2 = 4f^3 + \lambda f^2 + \mu f + \nu$$

11) Confocal-paraboloidal coord.

The same form as the preceding 10), except that

$$f'^2 = \lambda f^2 + \mu f + \nu.$$

§ 3. Electric fields

1) Cartesian coord.

$$B = \phi_1(x) + \phi_2(y) + \phi_3(z)$$

$$\phi_1 = \alpha x^2 + \alpha_1 x$$

$$\phi_2 = \beta y^2 + \beta_1 y$$

$$\phi_3 = -(\alpha + \beta)z^2 + \gamma z + \gamma_1,$$

(E 1)

α, \dots, γ_1 being arbitrary constants.

2) Cylindrical coord.

$$B = \phi_1(r) + \frac{\phi_2(\theta)}{r^2} + \phi_3(z)$$

$$= -\frac{1}{2}\alpha r^2 - \alpha_1 \log r + \frac{\beta \cos 2\theta + \beta_1 \sin 2\theta}{r^2} + \alpha z^2 + \gamma z + \gamma_1. \quad (\text{E } 2)$$

3) Spherical coord.

$$B = \phi_1(r) + \frac{\phi_2(\theta)}{r^2} + \frac{\phi_3(\varphi)}{r^2 \sin^2 \theta}$$

$$= \frac{\alpha}{r} + \alpha_1 + \frac{\beta P_1(\cos \theta) + \beta_1 Q_1(\cos \theta)}{r^2} + \frac{\gamma \cos 2\varphi + \gamma_1 \sin 2\varphi}{r^2 \sin^2 \theta}. \quad (\text{E } 3)$$

P_1, Q_1 being Legendre's functions.

4) Elliptic coord.

$$B = \frac{\phi_1(u) + \phi_2(v)}{c^2(\text{ch}^2 u - \cos^2 v)} + \phi_3(z)$$

$$= \frac{(\beta_1 + \beta u) \text{sh } 2u + (\beta_2 + \beta v) \sin 2v}{c^2(\text{ch}^2 u - \cos^2 v)} - \frac{\alpha c^2}{8} (\text{ch } 2u + \cos 2v)$$

$$+ 1/2 \cdot \alpha z^2 + \alpha_1 z + \gamma.$$

(E 4)

5) Parabolic coord.

$$\begin{aligned}
 B &= \frac{\psi_1(u) + \psi_2(v)}{u^2 + v^2} + \psi_3(z) \\
 &= \frac{\alpha(u^6 + v^6) + \beta(u^4 - v^4) + \beta_1 u + \beta_2 v}{u^2 + v^2} \\
 &\quad - 5\alpha z^2 + \gamma z + \gamma_1.
 \end{aligned} \tag{E 5}$$

$$\begin{aligned}
 6) \quad B &= \psi_1(q_1) + \frac{\psi_2(q_2) + \psi_3(q_3)}{q_1^2 (f_1(q_2) + f_2(q_3))} \\
 &= \alpha/q_1 + \alpha_1.
 \end{aligned} \tag{E 6}$$

7) Oblate spheroidal coord.

$$\begin{aligned}
 B &= \frac{\psi_1(u) + \psi_2(v)}{c^2 (\text{ch}^2 u - \cos^2 v)} + \frac{\psi_3(\varphi)}{c^2 \text{ch}^2 u \cos^2 v} \\
 &= \frac{1}{c^2 (\text{ch}^2 u - \cos^2 v)} \left\{ \alpha (\text{sh} \tan^{-1} \text{sh} u + \sin v \tanh^{-1} \sin v) \right. \\
 &\quad \left. + \alpha_1 \text{sh} u + \alpha_2 \sin v \right\} + \frac{\beta \cos 2\varphi + \beta_1 \sin 2\varphi}{c^2 \text{ch}^2 u \cos^2 v} + \gamma.
 \end{aligned} \tag{E 7}$$

8) Prolate spheroidal coord.

$$\begin{aligned}
 B &= \frac{\psi_1(u) + \psi_2(v)}{c^2 (\text{sh}^2 u + \sin^2 v)} + \frac{\psi_3(\varphi)}{c^2 \text{sh}^2 u \sin^2 v} \\
 &= \frac{1}{c^2 (\text{sh}^2 u + \sin^2 v)} \left\{ \alpha (\text{ch} u \coth^{-1} \text{ch} u + \cos v \tanh^{-1} \cos v) \right. \\
 &\quad \left. + \alpha_1 \text{ch} u + \alpha_2 \cos v \right\} + \frac{\beta \cos 2\varphi + \beta_1 \sin 2\varphi}{c^2 \text{sh}^2 u \sin^2 v} + \gamma.
 \end{aligned} \tag{E 8}$$

9) Paraboloidal coord.

$$\begin{aligned}
 B &= \frac{\psi_1(u) + \psi_2(v)}{u^2 + v^2} + \frac{\psi_3(\varphi)}{u^2 v^2} \\
 &= \frac{\alpha(u^4 - v^4) + \alpha_1 (\log u - \log v) + \alpha_2}{u^2 + v^2} \\
 &\quad + \frac{\beta \cos 2\varphi + \beta_1 \sin 2\varphi}{u^2 v^2} + \gamma.
 \end{aligned} \tag{E 9}$$

10) Ellipsoidal coord.

$$\begin{aligned}
 B &= \frac{\psi_1(q_1)}{(f_1 - f_2)(f_1 - f_3)} + \frac{\psi_2(q_2)}{(f_2 - f_1)(f_2 - f_3)} + \frac{\psi_3(q_3)}{(f_3 - f_1)(f_3 - f_2)}, \\
 \psi_i(q_i) &= f_i'(q_i) \left\{ \int \frac{F(f_i)}{f_i'^2} dq_i + \alpha_i \right\},
 \end{aligned} \tag{E 10}$$

$$F(f) = \gamma f^2 + \gamma_1 f + \gamma_2.$$

11) Confocal paraboloidal coord.

the same as 10), except that

$$F(f) = \gamma f^4 + \gamma_1 f^3 + \gamma_2 f^2 + \gamma_3 f + \gamma_4. \quad (\text{E } 11)$$

§ 4. Magnetic fields

1) Cartesian coord.

$$\begin{aligned} 2H &= (p_x - b_x)^2 + (p_y - b_y)^2 + p_z^2 \\ b_x &= \beta_1 z, \quad b_y = \beta_2 z \\ H_x &= \beta_2, \quad H_y = -\beta_1, \quad H_z = 0. \end{aligned} \quad (\text{M } 1)$$

2) Cylindrical coord.

$$\begin{aligned} 2H &= p_r^2 + \frac{(p_\theta - b_\theta)^2}{r^2} + (p_z - b_z)^2 \\ b_\theta &= 1/2 \cdot \alpha \cdot r^2, \quad b_z = \beta \log r \\ H_r &= 0, \quad H_\theta = \beta/r, \quad H_z = -\alpha. \end{aligned} \quad (\text{M } 2)$$

The field $H_\theta = \beta/r$ is produced by a rectilinear current.

3) Spherical coord.

$$\begin{aligned} 2H &= p_r^2 + p_\theta^2/r^2 + (p_\varphi - b_\varphi)^2/r^2 \sin^2 \theta \\ b_\varphi &= \beta \cos \theta \\ H_r &= \beta/r^2, \quad H_\theta = H_\varphi = 0. \end{aligned} \quad (\text{M } 3)$$

The field $H_r = \beta/r^2$ is produced by a magnetic monopole.

§ 5. Electromagnetic fields

1) Cartesian coord.

$$\begin{aligned} \alpha) \quad 2H &= (p_x - b_x)^2 + (p_y - b_y)^2 + p_z^2 + 2B \\ b_x &= \beta_1 z, \quad b_y = \beta_2 z, \quad B = \gamma z + \gamma_1, \end{aligned} \quad (\text{EM } 1)$$

$$\begin{aligned} \beta) \quad 2H &= p_x^2 + p_y^2 + (p_z - b_z)^2 + 2B \\ b_z &= 1/2 \cdot \beta (x^2 - y^2) + \beta_1 x + \beta_2 y \end{aligned} \quad (\text{EM } 2)$$

$$\begin{aligned} B &= \left(\frac{1}{2} \beta x^2 + \beta_1 x \right) \left(\frac{1}{2} \beta y^2 - \beta_2 y \right) - \frac{\beta^2}{24} (x^4 + y^4) \\ &\quad - \frac{1}{6} (\beta_1 x^3 - \beta_2 y^3) + \frac{\gamma}{2} (x^2 - y^2) + \gamma_1 x + \gamma_2 y + \gamma_3. \end{aligned}$$

2) Cylindrical coord.

$$\alpha) \quad 2H = p_r^2 + \frac{(p_\theta - b_\theta)^2}{r^2} + (p_z - b_z)^2 + 2B, \\ b_\theta = (1/2) \alpha r^2, \quad b_z = \beta \log r, \quad B = \gamma \log r + \gamma_1. \quad (\text{EM } 3)$$

$$\beta) \quad 2H = p_r^2 + \frac{(p_\theta - b_\theta)^2}{r^2} + p_z^2 + 2B, \\ b_\theta = r^2(\beta_1 z + \beta_2), \\ B = \frac{1}{16} \beta_1^2 r^4 - \frac{\alpha}{4} r^2 + \alpha_1 \log r + \frac{1}{6\beta_1^2} (\beta_1 z + \beta_2)^4 \\ + \frac{\alpha}{2} z^2 + \gamma_1 z + \gamma_2 - (1/2) r^2 (\beta_1 z + \beta_2)^2. \quad (\text{EM } 4)$$

$$\gamma) \quad 2H = p_r^2 + \frac{p_\theta^2}{r^2} + (p_z - b_z)^2 + 2B, \\ b_z = \alpha \log r + \frac{\beta \cos 2(\theta - \varepsilon)}{r^2}, \\ B = -\frac{\beta^2}{4r^4} \cos 4(\theta - \varepsilon) - \frac{\alpha\beta}{r^2} \theta \sin 2(\theta - \varepsilon) \\ - (\alpha \log r + \alpha_1)/r^2 \cdot \beta \cos 2(\theta - \varepsilon) - \beta_1 \log r - \beta_2. \quad (\text{EM } 5)$$

3) Spherical coord

$$2H = p_r^2 + \frac{p_\theta^2}{r^2} + \frac{(p_\phi - b_\phi)^2}{r^2 \sin^2 \theta} + 2B.$$

$$\alpha) \quad b_\phi = \beta \cos \theta, \quad 2B = \frac{\alpha}{r} + \alpha_1 + \frac{\gamma P_1(\cos \theta) + \gamma_1 Q_1(\cos \theta)}{r^2}. \quad (\text{EM } 6)$$

$$\beta) \quad b_\phi = \beta r^2 \sin^2 \theta, \quad 2B = \frac{\alpha}{r} + \alpha_1 + \frac{2}{3} \beta^2 r^2 + \frac{\gamma P_1(\cos \theta) + \gamma_1 Q_1(\cos \theta)}{r^2} \\ - \beta^2 r^2 \sin^2 \theta. \quad (\text{EM } 7)$$

4) Elliptic coord.

$$2H = \frac{p_u^2 + p_v^2}{c^2(\text{ch}^2 u - \cos^2 v)} + (p_z - b_z)^2 + 2B,$$

$$b_z = \frac{\beta_1 \text{sh } 2u + \beta_2 \sin 2v}{\text{ch}^2 u - \cos^2 v},$$

$$2B = \frac{\alpha_1 \text{ch } 2u + \alpha_2 \cos 2v}{\text{ch}^2 u - \cos^2 v} + \frac{(\beta_3 - \beta_1 u) \text{sh } 2u + (\beta_4 - \beta_2 v) \sin 2v}{\text{ch}^2 u - \cos^2 v} - b_z^2,$$

$$\alpha_1 + \alpha_2 = 2(\beta_1^2 + \beta_2^2). \quad (\text{EM } 8)$$

5) Parabolic coord.

$$2H = \frac{p_u^2 + p_v^2}{u^2 + v^2} + (p_z - b_z)^2 + 2B,$$

$$b_z = \alpha(u^2 - v^2) + \frac{\alpha_1 u + \alpha_2 v}{u^2 + v^2},$$

$$2B = \frac{1}{u^2 + v^2} \left\{ \frac{\alpha_1^2 + \alpha_2^2}{2} + 4\alpha(\alpha_1 u^3 - \alpha_2 v^3) - \frac{4\alpha^2}{5}(u^6 + v^6) \right\} \quad (\text{EM } 9)$$

$$+ \frac{\beta_1 u + \beta_2 v}{u^2 + v^2} + \beta(u^2 - v^2) + \gamma - b_z^2.$$

7) Oblate spheroidal coord.

$$2H = \frac{p_u^2 + p_v^2}{c^2(\text{ch}^2 u - \cos^2 v)} + \frac{(p_\varphi - b_\varphi)^2}{c^2 \text{ch}^2 u \cos^2 v} + 2B,$$

$$b_\varphi = \beta c^2 \text{ch}^2 u \cos^2 v, \quad (\text{EM } 10)$$

$$2B + \beta^2 c^2 \text{ch}^2 u \cos^2 v = \frac{2}{3} \beta^2 c^2 \frac{\text{sh}^4 u - \sin^4 v}{\text{ch}^2 u - \cos^2 v} + \frac{1}{\text{ch}^2 u - \cos^2 v} \left\{ \alpha \text{sh } u + \alpha_1 \sin v \right.$$

$$\left. + \alpha_2 (\text{sh } u \tan^{-1} \text{sh } u + \sin v \tanh^{-1} \sin v) \right\} + \gamma.$$

8) Prolate spheroidal coord.

$$2H = \frac{p_u^2 + p_v^2}{c^2(\text{sh}^2 u + \sin^2 v)} + \frac{(p_\varphi - b_\varphi)^2}{c^2 \text{sh}^2 u \sin^2 v} + 2B,$$

$$b_\varphi = \beta c^2 \text{sh}^2 u \sin^2 v, \quad (\text{EM } 11)$$

$$2B + \beta^2 c^2 \text{sh}^2 u \sin^2 v = \frac{2}{3} \beta^2 c^2 \frac{\text{ch}^4 u - \cos^4 v}{\text{sh}^2 u + \sin^2 v} + \frac{1}{\text{sh}^2 u + \sin^2 v} \left\{ \alpha \text{ch } u + \alpha_1 \cos v \right.$$

$$\left. + \alpha_2 (\text{ch } u \coth^{-1} \text{ch } u + \cos v \tanh^{-1} \cos v) \right\} + \gamma.$$

9) Paraboloidal coord.

$$2H = \frac{p_u^2 + p_v^2}{u^2 + v^2} + \frac{(p_\varphi - b_\varphi)^2}{u^2 v^2} + 2B,$$

$$b_\varphi = \beta u^2 v^2,$$

$$2B + \beta^2 u^2 v^2 = \frac{\beta^2}{3} \frac{u^6 + v^6}{u^2 + v^2} + \frac{1}{u^2 + v^2} \left\{ \alpha(u^4 - v^4) \right.$$

$$\left. + \alpha_1 (\log u - \log v) + \alpha_2 \right\} + \gamma. \quad (\text{EM } 12)$$

If one puts all constants = 0 other than β in (EM 7), (EM 10) and (EM 11), one gets the field for a perfect imaging system

$$H_x = H_y = 0, \quad H_z = -2\beta^2$$

$$B = \beta^2/6 \cdot (-x^2 - y^2 + 2z^2).$$

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Symmetry in Time and Tanikawa's Method of Superquantization in Regard to Negative Energy Fields

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It will be shown (1) that the Dirac equation of electron fits exactly in the pattern of theory required by the method of double inferential state-vector (hereinafter referred to as DIV-method) which was introduced to establish a perfect symmetry in time¹⁾ and (2) that application of the DIV-method to fields leads to a third quantization similar to the formalism proposed by Tanikawa²⁾. This paper is intended to shed a further light on the relationship between the electric charge and the two directions of time, and to suggest a consistent method of handling negative energy fields. A new concept of "supercharge" will be introduced.

§ 1. Introduction

In classical mechanics, we deal, at different stages of theory, with a single particle, an ensemble of particles and then a Gibbs ensemble, or an ensemble of ensembles of particles. If we pass from the quantum mechanics of a single particle to that of an ensemble of particles by a procedure called second quantization, it may not be an idle suggestion to construct a quantum mechanics of an ensemble of ensembles by a third quantization or a superquantization. There should not be any justifiable objection to the proposition that the classical concept of Gibbs ensemble may have two quantum-mechanical counterparts, one in the density matrix and the other in the superquantized ensembles, offering probably two different viewpoints in statistical handling of empirical data. One of the objections to a superquantization may be that one cannot actually prepare an ensemble of ensembles at hand. However, the same objection can obviously be directed to the classical Gibbs ensemble. This actual unrealizability of a Gibbs ensemble really does not impair the usefulness of this concept. Since it is unconceivable, or at least useless, to allow two identical ensembles (occupying the same space-time region) to coexist, it seems more natural to follow the pattern of Fermi-Dirac statistics in the process of superquantization.

These considerations may add an argument in favor of Tanikawa's idea of superquantization of quantized fields,²⁾ which he introduced to cope with the negative energy fields, in particular, in connection with Gupta-Bleuler's treatment of scalar and longitudinal photons³⁾ and with Pauli-Villars's regulators.⁴⁾ The present paper is intended to show, in addition to the above considerations, that an entirely different line of thought pertaining to a purely formal treatment of time-symmetry leads also to a formalism fundamentally equivalent to Tanikawa's theory.

The method of double inferential state-vector (hereinafter referred to as DIV-method) was introduced by the author¹⁾ in order (a) to linearize the operation of time-reversal, (b) to make the double time-reversal an identity transformation and (c) to treat "prediction" and "retrodiction" on the same footing. This DIV-method can be applied either to single-particle quantum mechanics or to field quantum mechanics. It will first be demonstrated that in the former application the DIV-method perfectly agrees with the Dirac equation of the electron. More specifically, the relation of the Dirac equation to the two-component theory of spinning electron turns out to be exactly the relation of the DIV-formalism to the ordinary quantum mechanics. This analysis will offer a new point of view in interpreting the structure of Dirac's four-component spinor theory.

The positron theory that solved the problems pertaining to the negative energy states in the Dirac theory then serves as the model to reinterpret the DIV-method in which also negative energy states appear. This reinterpreted DIV-method, when applied to the quantized fields, turns out to be essentially equivalent to the Tanikawa formulation. In other words, the relation of the ordinary field quantum theory to the superquantized field theory is precisely the relation of the two-component spin theory to the Dirac theory.

The DIV-method involves a doubling of state-vectors corresponding to the two directions of time. As a result, there appears a new degree of freedom, which in the case of single particle quantum mechanics is nothing but the electric charge. In the case of quantized fields, this new degree of freedom may be called "supercharge." Just in the same way as Dirac's positron theory could give a physical meaning to the negative energy levels in terms of opposite electric charge, the DIV-method provides a way to make negative energy fields physically meaningful by the use of opposite supercharge. It is, however, intentionally refrained in the present paper from elaborating this idea into a fixed formalism with detailed prescriptions, since there seem to be various possible ways to implement this programme, of which Tanikawa's formulation is certainly one. Although it will not be emphasized in the following, the reader will also perceive all through this paper a close connection between the DIV-method and the indefinite metric method.

§ 2. The DIV-method

When $\psi_1(t)$ represents the state of a system, we introduce a second state-vector $\psi_2(t)$ by*

$$\psi_2(t) = R^T \psi_1^*(t), \quad (2.1)$$

where R is a time-independent, unitary operator called reversion operator, the asterisk means the complex conjugate, and the superscript T means the transpose. If a sign function ν_R is defined as being -1 or $+1$ according as the quantity $Q(t)$ to which it refers changes or does not change its sign by the operation of time-reversal, R should satisfy**

*) See eq. (3.11) of reference 5. Note $\psi_R(-t) = \psi_2(t)$.

**) See eq. (3.19) of reference 5.

$$Q(-t) = \rho_R(R^{-1}Q(t)R)^T, \quad (2.2)$$

for all Q 's. If φ represents a state, $R^T \varphi^*$ represents in the ordinary theory its reversed state, i.e., the state in which the expectation value of each quantity is ρ_R times that in the state φ , provided the time-dependence of the operator Q is appropriately taken into account. Thus, $\psi_1(t)$ and $\psi_2(t)$ represent the reversed state of each other.* The DIV-method consists in using $\psi_1(t)$ and $\psi_2(t)$ as equally qualified representatives of the state usually represented only by $\psi_1(t)$.

Let us assume that $\psi_1(t)$ develops with time by the Schrödinger equation:

$$\partial\psi_1(t)/\partial t = -iH(t)\psi_1(t), \quad (2.3)$$

$$\text{or} \quad \psi_1(t_2) = U(t_2, t_1)\psi_1(t_1), \quad (2.4)$$

$$\text{where} \quad \partial U(t_2, t_1)/\partial t_2 = -iH(t_2)U(t_2, t_1), \quad U(t_1, t_1) = 1. \quad (2.5)$$

It is obvious that for a closed system $U(t_2, t_1)$ is really a function only of the difference $t_2 - t_1$.

It follows from eqs. (2.1) and (2.3) that $\psi_2(t)$ develops with time according to

$$\partial\psi_2(t)/\partial t = -iK(t)\psi_2(t) = +iH(-t)\psi_2(t), \quad (2.6)$$

$$\text{or} \quad \psi_2(t_2) = V(t_2, t_1)\psi_2(t_1) = U(-t_2, -t_1)\psi_2(t_1), \quad (2.7)$$

for ρ_R of the hamiltonian is $+1$. $K(t)$ and $V(t_2, t_1)$ are respectively the hamiltonian and the transition matrix for $\psi_2(t)$.

As a simple consequence of the properties of the transition matrix:

$$U^{-1}(t_2, t_1) = \bar{U}(t_2, t_1) = U(t_1, t_2), \quad (2.8)$$

the following theorem in the ψ_1 -representation, i.e., in the ordinary theory, was proven in a previous paper.¹⁾

Theorem: The predictive probability that a system found in state ξ will be found τ seconds later in state η can be calculated, in ψ_1 -representation, either by

$$P_1 = |(\eta, U(\tau, 0)\xi)|^2, \quad (2.9)$$

$$\text{or} \quad P_2 = |(\xi, U(-\tau, 0)\eta)|^2. \quad (2.10)$$

It was pointed out that considering in the second expression eq. (2.10) η as a given state and ξ as a state to be arbitrarily chosen, the expression (2.10) suits better an alternative interpretation, viz., P_2 in eq. (2.10) can be considered as representing the "retrodictive" probability (without other preliminary knowledge) that a system which is found in state η at $t=0$ was found at $t=-\tau$ in state ξ .¹⁾ The "retrodictive" state-vector $U(-\tau, 0)\eta$ which appears in eq. (2.10) is a solution of the Schrödinger equation with the final condition that it should become η at $t=0$. Partiality, or rather inconsistency, of those metaphysically inclined physicists was pointed out,¹⁾ who attribute "physical reality"

*) This statement is true without qualification in the Schrödinger picture. A more general statement will presently be presented.

only to the predictive state-vector between two observations. If coexistence of two different "realities" (predictive and retrodictive state-vectors) is irreconcilable with their philosophy, they should rather abandon their metaphysical principle.

It should be noted that the backward development of the state-vector in eq. (2.10) as given by

$$U(-\tau, 0)\eta \quad (2.11)$$

is the same as the forward development of ψ_2 -vector which starts from η at $t=0$, for

$$\varphi_2(\tau) = V(\tau, 0)\eta = U(-\tau, 0)\eta. \quad (2.12)$$

In other words, $\varphi_2(\tau)$ represents a retrodictive state-vector at $t=-\tau$.

Now, reversibility of physical phenomena means that the probability of a system at present in $\hat{\xi}$ being found τ seconds later in η is equal to its reversed probability, i.e., the probability of a similar system at present in the reversed state $R^T \eta^*$ of η being found τ seconds later in the reversed state $R^T \hat{\xi}^*$ of $\hat{\xi}$. The first probability is given by eq. (2.9) while the second is given by

$$P_R = |(R^T \hat{\xi}^*, U(\tau, 0) R^T \eta^*)|^2. \quad (2.13)$$

This probability P_R is equal to P_1 of eq. (2.9), for it follows from eq. (2.2) with $Q=H$ that

$$U(t, 0) = R U^T(t, 0) R^{-1}. \quad (2.14)$$

The reverse probability P_R of eq. (2.13) can be expressed in a much simpler form in terms of ψ_2 -vectors. If we apply the aforementioned theorem which states equality of P_1 and P_2 to the probability in eq. (2.13), we obtain

$$P_R = |(R^T \eta^*, U(-\tau, 0) R^T \hat{\xi}^*)|^2. \quad (2.15)$$

Denoting by $\hat{\xi}_2$ and η_2 the ψ_2 -vectors corresponding to the ψ_1 -vectors $\hat{\xi}$ and η , we can rewrite eq. (2.15) as

$$P_R = |(\eta_2, V(\tau, 0) \hat{\xi}_2)|^2, \quad (2.16)$$

where V , as introduced in eq. (2.7), is the transition matrix of the ψ_2 -vectors. Eq. (2.16) shows an exact parallelism with eq. (2.9), and both probabilities are equal provided reversibility exists. Thus we are led to use either the ψ_1 -vector or the ψ_2 -vector as an equally useful instrument to calculate the transition probability which is the ultimate measurable quantity testing a theory.

If $\psi_1(t)$ and $\psi_2(t)$ are connected by eq. (2.1), then we have

$$\begin{aligned} \langle Q(t) \rangle &= (\psi_1(t), Q(t) \psi_1(t)) \\ &= \rho_R(\psi_1(t), R Q^T(-t) R^{-1} \psi_1(t)) = \rho_R(\psi_2(t), Q(-t) \psi_2(t)), \end{aligned} \quad (2.17)$$

in virtue of eq. (2.2). This would mean that we should use in the ψ_2 -representation the operator:

$$Q_2(t) = \rho_R Q_1(-t) = (R^{-1} Q_1(t) R)^T, \quad (2.18)$$

wherever we use $Q_1(t)$ in the ψ_1 -representation.

This situation at first glance seems to cause a difficulty, for the hamiltonian for $\psi_2(t)$ in eq. (2.6) is

$$K(t) = -H(-t) = -\rho_R H(-t), \quad (2.19)$$

which contradicts eq. (2.18). To circumvent this difficulty, we can devise two alternative prescriptions. The one possibility is to divorce the concept of hamiltonian as the generating function of the Schrödinger equation from the concept of hamiltonian as the energy operator, the former for ψ_2 being $-\rho_R H(-t)$ while the latter for ψ_2 being $+\rho_R H(-t)$. The other alternative is to use $Q_2(t) = -\rho_R Q_1(-t)$ consistently and to say that the actual expectation value in ψ_2 is equal to the negative of $\langle \psi_2(t), Q_2(t) \psi_2(t) \rangle$.

The next step in the DIV-method is to incorporate ψ_1 and ψ_2 in the same system of equations, i.e., we introduce the double state-vector ϕ defined by

$$\phi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \quad (2.20)$$

whose time-development is governed by

$$H(t) = \begin{pmatrix} H(t) & 0 \\ 0 & -H(-t) \end{pmatrix}. \quad (2.21)$$

In the first prescription, the physical quantity must then be

$$Q(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & \rho_R Q(-t) \end{pmatrix} = \begin{pmatrix} Q(t) & 0 \\ 0 & (R^{-1} Q(t) R)^T \end{pmatrix}. \quad (2.22)$$

The expectation value in $\phi(t)$ will be

$$\langle Q(t) \rangle = (1/2) (\phi(t), Q(t) \phi(t)) \quad (2.23)$$

if the same state is expressed simultaneously by the ψ_1 -part and ψ_2 -part. It will be

$$\langle Q(t) \rangle = (\phi(t), Q(t) \phi(t)) \quad (2.24)$$

if a state should be expressed either by ψ_1 -part or ψ_2 -part. The fact that $H(t)$ in eq. (2.21) as the generator of the Schrödinger equation is different from the energy operator as defined by eq. (2.22) cannot be considered as any internal contradiction, since the entire scheme, in the form so far developed, is nothing more, or nothing less, than the ordinary quantum mechanics. If, however, we start to consider matrices that mix (not merely interchange) ψ_1 and ψ_2 , the situation will become different.

It is easy to see that if $\phi(t)$ is a solution of the Schrödinger equation with hamiltonian given in eq. (2.21), then

$$\phi'(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \phi(-t) \quad (2.25)$$

is also a solution, and that

$$(\psi'(-t), Q(-t)\psi'(-t)) = \rho_R(\psi(t), Q(t)\psi(t)), \quad (2.26)$$

showing that $\psi'(t)$ represents the time-reversed process of $\psi(t)$. The operation of time-reversal is thus a linear transformation and its reiteration is the identity transformation.

According to the second prescription, the physical quantity will be expressed by

$$Q(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & -\rho_R Q(-t) \end{pmatrix} = \begin{pmatrix} Q(t) & 0 \\ 0 & -(R^{-1}Q(t)R)^T \end{pmatrix}. \quad (2.27)$$

In this case, the expectation value can be given by

$$\langle Q(t) \rangle = (1/2) (\psi(t), A Q(t) \psi(t)) \quad (2.28)$$

if both ψ_1 -part and ψ_2 -part are used, and

$$\langle Q(t) \rangle = (\psi(t), A Q(t) \psi(t)) \quad (2.29)$$

if either ψ_1 -part or ψ_2 -part is used. The matrix A defined by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.30)$$

can be interpreted as the indefinite metric matrix. The time-reversal is, here too, represented by eq. (2.25).

In the interaction picture, $H(t)$ is the interaction hamiltonian, and the physical quantities $Q(t)$, whether we use eq. (2.22) or eq. (2.27), are governed by the Heisenberg equation with the free hamiltonian:

$$H_0 = \begin{pmatrix} H_0 & 0 \\ 0 & -H_0 \end{pmatrix}. \quad (2.31)$$

The DIV-method, however, can be used in any picture.

§ 3. Structure of Dirac equation

It will be shown in this section that the relation that Dirac's 4-component spinor theory has to the 2-component theory of spinning electron is precisely the relation that the DIV-method has to the ordinary quantum theory.

It should first be noticed that in the 2-component theory with spin matrices σ_1 , σ_2 , and σ_3 the reversion operator R as defined in eq. (2.2) should be such that

$$R_2^{-1} \sigma_i R_2 = -\sigma_i^T, \quad (i=1, 2, 3), \quad (3.1)$$

since the ρ_R of spin is -1 . If σ_1 and σ_3 have real matrix elements and σ_2 has imaginary elements, as is usually assumed, R_2 becomes, in agreement with Wigner's early paper,⁽¹⁾ simply

$$R_2 = \sigma_{2*} \quad (3.2)$$

The linear momentum, for instance, also satisfies eq. (2.2) with R_2 given in eq. (3.2), since the operator $p_i = -i\partial/\partial x_i$ is an antisymmetric operator in the x -representation⁽⁵⁾ and ρ_R of the linear momentum is -1 .

Returning to the Dirac theory, the hamiltonian is given by

$$H = \alpha_i p_i + m\beta, \quad (3.3)$$

or using the customary representation,

$$H = \begin{pmatrix} m & P \\ P & -m \end{pmatrix}, \quad (3.4)$$

with

$$P = p_i \sigma_i. \quad (3.5)$$

Each matrix element in eq. (3.4) is again a two-two-matrix. To use the linear momentum representation, the linear momentum π_i and the spin κ in the direction of propagation vector are respectively given by

$$\pi_i = \begin{pmatrix} p_i & 0 \\ 0 & p_i \end{pmatrix}, \quad (3.6)$$

and

$$\kappa = (1/2p) \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}, \quad (3.7)$$

with

$$p = \sqrt{P^2} = \sqrt{\pi_i \pi_i} = \sqrt{p_i p_i}. \quad (3.8)$$

The hamiltonian given in eq. (3.4) can be diagonalized by the transformation matrix T given by

$$T = \bar{T} = T^{-1} = \frac{1}{[2E(E-m)]^{1/2}} \begin{pmatrix} P & E-m \\ E-m & -P \end{pmatrix} \quad (3.9)$$

with

$$E = [(p^2 + m^2)^{1/2}]. \quad (3.10)$$

The hamiltonian becomes

$$H' = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad (3.11)$$

while the transformed π'_i and κ' are the same as their original forms π_i and κ given in eqs. (3.6) and (3.7).

As we can write $\kappa' = (1/p) (\pi'_i s'_i)$ with

$$s'_i = (1/2) \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad (3.12)$$

we may interpret this latter quantity as the spin vector. It is true that if we decompose κ before the transformation by T into p_i and s_i and then transform this s_i by T , we shall obtain an operator different from eq. (3.12) and uncommutable with H' of eq. (3.11). This situation, however, will not interfere with the following argument.

Now we see that H' in eq. (3.11), π'_i in eq. (3.6) and s'_i in eq. (3.12) all satisfy

$$Q = \begin{pmatrix} Q & 0 \\ 0 & -(R_0^{-1} Q R_0)^T \end{pmatrix}, \quad (3.13)$$

with R_0 given in eq. (3.2), in agreement with eq. (2.27) as required by the DIV-

method in the second "prescription." All the physical quantities here considered are in the Schrödinger picture.

It should, however, be noted that the quantities thus far considered, viz., H , π_i , and s_i , are all "mechanical" in nature, i.e., quantities which do not change their sign by charge conjugation, or symbolically $\rho_c = +1$.⁵⁾ There are at least two "electric" quantities (i.e., $\rho_c = -1$) which commute with the hamiltonian and the linear momentum. One is the magnetic moment μ in a direction perpendicular to the propagation vector, and the other is the charge density d . The former can be written as $\rho_i \sigma_i l_i$, where l_i is a unit vector perpendicular to p_i , i.e., $p_i l_i = 0$ and $l_i l_i = 1$. These quantities in matrix form are

$$\mu = \begin{pmatrix} \sigma_i l_i & 0 \\ 0 & -\sigma_i l_i \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.14)$$

(The matrix μ changes its sign by the transformation by T). These quantities fall in the type:*

$$Q = \begin{pmatrix} Q & 0 \\ 0 & + (R_2^{-1} Q R_2)^T \end{pmatrix} \quad (3.15)$$

since ρ_R of μ and d are respectively -1 and $+1$. The scalar spin κ considered in eq. (3.7) is, in this sense, an electric quantity. Unifying eqs. (3.13) and (3.15), we can write

$$Q = \begin{pmatrix} Q & 0 \\ 0 & -\rho_c (R_2^{-1} Q R_2)^T \end{pmatrix}. \quad (3.16)$$

A complete set of eigenfunctions can now be determined by the eigenvalues of H , π_i and κ . Since the operators are now diagonalized at least in a two-two-matrix form, their eigenfunctions can be classified into those which have only the first two components and those which have only the last two components, such as

$$\phi_\lambda = \begin{pmatrix} \phi_\lambda \\ 0 \end{pmatrix}, \quad \varphi_\lambda = \begin{pmatrix} 0 \\ \varphi_\lambda \end{pmatrix}. \quad (\lambda = 1, 2, 3, \dots) \quad (3.17)$$

Furthermore, we see that if a ϕ_λ of the form given in eq. (3.17) is an eigenfunction, then

$$\varphi_\lambda = \begin{pmatrix} 0 \\ e^{i\alpha} R_2^T \phi_{\lambda*} \end{pmatrix} \quad (3.18)$$

is also an eigenfunction in virtue of eq. (3.13) or eq. (3.16), where α is an arbitrary real constant. More precisely, the eigenvalues of H and π_i for φ_λ given in eq. (3.18) are the negative of those for ϕ_λ . The scalar spin κ has the same value for φ_λ and ϕ_λ . This of course implies that the expectation value of the vector spin s_i for φ_λ is opposite to that for ϕ_λ . Summarizing we can say that the mechanical quantities have the opposite values for φ_λ and ϕ_λ , and the electric quantities have the same values for φ_λ and ϕ_λ . The

*) The direction of l_i is supposed to be kept fixed when time-reversal is applied.

scalar spin is here considered as an electric quantity.*

This situation is ingeniously exploited in the positron theory by a parallel treatment of emission operators of one charge with absorption operators of the other charge. Namely, in Schrödinger picture, we expand a general $\phi(t)$ as

$$\phi(t) = \sum (a_\lambda \alpha_\lambda e^{-iE_\lambda t} + \bar{b}_\lambda \beta_\lambda e^{+iE_\lambda t}). \quad (3.19)$$

According to the positron theory, the α 's and the β 's should be connected by**

$$\beta_\lambda = -K\alpha_\lambda^* = -iE_0 K\alpha_\lambda^* \quad (3.20)$$

where K is the charge conjugation matrix in the E -system. In the representation in which σ_2 (consequently α_0) has imaginary elements and σ_1 and σ_3 (consequently α_1 , α_3 and β) have real elements, K and E_0 are respectively given by***

$$K = E_1 E_3 = \alpha_1 \alpha_3 = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad E_0 = -i\rho_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.21)$$

As a result, eq. (3.20) becomes

$$\beta_\lambda = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \alpha_\lambda^*, \quad (3.22)$$

which is exactly the relationship we have already established in eq. (3.18) if we identify $\alpha_\lambda \exp(-iE_\lambda t)$ with ψ_λ and $\beta_\lambda \exp(+iE_\lambda t)$ with φ_λ . In the two-component way of writing, eq. (3.22) becomes

$$\varphi_\lambda = \sigma_2^T \psi_\lambda^*. \quad (3.23)$$

If we build the expectation value of a quantity, eq. (3.16), diagonal in the ψ_λ - φ_λ -representation, we get

$$(\psi, Q\psi) = \sum (\bar{a}_\lambda a_\lambda - \rho_0 b_\lambda \bar{b}_\lambda) Q_\lambda, \quad (3.24)$$

where Q_λ is the eigenvalue of Q in eigenfunction ψ_λ . This allows one to interpret $N_\lambda^{(+)} = \bar{a}_\lambda a_\lambda$ and $N_\lambda^{(-)} = \bar{b}_\lambda b_\lambda$ as occupation numbers of states having the same mechanical properties and opposite electric properties, provided of course the anticommutation rules are assumed for emission and absorption operators.

It is of importance, particularly in connection with the superselection rule,^{b)5)} to note that a double application of time-reversal:

$$\psi \rightarrow R^T \psi^* \quad (3.25)$$

results in

$$\psi \rightarrow R^{-1} R^T \psi. \quad (3.26)$$

*) It is of some interest to recall that in the indefinite metric method there sometimes arises a certain ambiguity regarding the metric to be used for a product of two quantities.

**) See eq. (7.22) of reference 5.

***) See eqs. (2.25) and (2.50) of reference 5.

It should be noted that if we use $e^{i\alpha}R$ instead of R the combination $R^{-1}R^T$ is not affected.

If we take the point of view of quantized fields, we obtain*

$$R_f^{-1} R_f^T = \mathcal{A}. \quad (3 \cdot 27)$$

If we take the one-particle viewpoint in the two-component theory, we get⁸⁾

$$R_2^{-1} R_2^T = \sigma_2^{-1} \sigma_2^T = -1, \quad (3 \cdot 28)$$

which is obviously in agreement with eq. (3·27), for eq. (3·28) refers to one particle.

If we take the one-particle viewpoint in the four-component theory, we have**

$$\psi \rightarrow E_0 K \psi^* = -iK\psi^* = -i\alpha_1 \alpha_3 \psi^* = - \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \psi^* \equiv R_4^T \psi^*. \quad (3 \cdot 29)$$

This is the transformation used by the author in an early paper.⁷⁾ The field theoretical R_f applied to ψ results in the same transformation, eq. (3·29), because, as was often emphasized by the author, the time-reversal without charge-conjugation is independent of commutation rules.⁵⁾⁹⁾ If we reiterate eq. (3·29), we obtain again

$$R_4^{-1} R_4^T = -1. \quad (3 \cdot 30)$$

If we combine the charge conjugation: $\psi \rightarrow e^{i\alpha} K \psi^* = ie^{i\alpha} E_0 K \psi^*$ with the time-reversal, eq. (3·29), we obtain***

$$\psi \rightarrow \begin{pmatrix} 0 & -e^{i\alpha} \\ e^{i\alpha} & 0 \end{pmatrix} \psi. \quad (3 \cdot 31)$$

Except for a phase-factor, this transformation amounts to interchanging the first part and the second part of ψ . A double application of this transformation results in a multiplication by any arbitrary phase-factor, including $+1$ and -1 . Thus, the double time-reversal with charge-conjugation and the double time-reversal without charge-conjugation behave entirely differently.***

§ 4. Superquantization and supercharge

The fact being established that the Dirac equation has the same structure as the dynamical equation of the DIV-method of a single particle, it is now an easy matter to give a definite meaning to the DIV-formalism of fields following the model of the positron theory. Just as we have in the quantized Dirac field the electron wave-function (which is second-quantized) and the field state-vector, we shall here have to introduce the super-quantized field state-vector ψ and the state-vector Ψ of an ensemble of field-states. The expectation value of a physical quantity Q should now be calculated by

$$\langle\langle Q(t) \rangle\rangle = (\Psi(t), \langle Q(t) \rangle \Psi(t)), \quad (4 \cdot 1)$$

*) See eq. (8·15) of reference 9 and eq. (7·5) of reference 5.

**) See eq. (8·11) of reference 9 and eq. (7·3) of reference 5.

***) See eq. (6·5) of reference 9 and eq. (7·8) of reference 5.

****) See eq. (6·5) of reference 9 and eq. (7·10) of reference 5.

with
$$\langle Q(t) \rangle = (\phi(t), Q(t)\phi(t)), \quad (4.2)$$

where $Q(t)$ must be taken from eq. (2.27), in which $Q(t)$ now is a field theoretical one, i.e. something of the nature of eq. (3.24).

The super-state vector $\Psi(t)$ will develop in time with the Schrödinger equation with the hamiltonian :

$$\langle H(t) \rangle = (\phi(t), H(t)\phi(t)), \quad (4.3)$$

where $H(t)$ should be taken from eq. (2.21), in which of course $H(t)$ is the field theoretical hamiltonian.

The double state-vector $\phi(t)$ of field is now an operator and should be expanded as

$$\phi(t) = \sum a_\lambda A_\lambda e^{-iE_\lambda t} + \sum \bar{b}_\lambda C A_\lambda^* e^{+iE_\lambda t} \quad (4.4)$$

with
$$C = \begin{pmatrix} 0 & R_f^T \\ R_f & 0 \end{pmatrix}, \quad E_\lambda > 0, \quad (4.5)$$

where A_λ is an eigenfunction of the free hamiltonian H_0 , eq. (2.31), in the interaction picture. We could as well adopt the Heisenberg picture, as Tanikawa did,²⁾ or the Schrödinger picture. a_λ and \bar{b}_λ are respectively an annihilation operator and a creation operator of an ensemble of particles, or an entire field-state, and should obey the Fermi-Dirac-type commutation relation. The difference between the positive energy terms and the negative energy terms in eq. (4.4) should be attributed to a difference in "supercharge". $N_\lambda^{(+)} = \bar{a}_\lambda a_\lambda$ and $N_\lambda^{(-)} = \bar{b}_\lambda b_\lambda$ will be then the occupation number of a positive supercharge field-state and that of a negative supercharge field-state. If H_0 is positive definite, A_λ will occupy only the first half of ϕ , and $C A_\lambda^*$ will occupy only the last half of ϕ . If H_0 is not positive definite, A_λ will spread over both parts of ϕ , and so will $C A_\lambda^*$. However, eqs. (4.4) and (4.5) are general enough to cope with such a case. The expectation value of H_0 in the sense of eq. (4.1), is always positive definite.

Analogously to the fact that in the positron theory there occurs no electron-pair-creation in the absence of photons, there is no danger of a pair-creation of ensembles in the superquantized theory, for there is no entity here corresponding to the photon, or any other interacting particle, which provides the necessary energy.

The quantities considered in eq. (2.27) are all "mechanical" in respect to the supercharge, or for brevity, supermechanical. A more general definition of physical quantities will be

$$Q(t) = \begin{pmatrix} Q(t) & 0 \\ 0 & -\rho_{sc}(R_f^{-1}Q(t)R_f)^T \end{pmatrix}, \quad (4.6)$$

where $\rho_{sc} = +1$ for supermechanical quantities and $\rho_{sc} = -1$ for superelectric quantities. The simplest example of a superelectric quantities is

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.7)$$

measuring the total supercharge, which is a constant of motion.

From the non-existence of field-pair-creation and from the conservation of total supercharge, we have

$$\sum (N_{\lambda}^{(+)} + N_{\lambda}^{(-)}) = \text{constant}, \quad (4.8)$$

$$\sum (N_{\lambda}^{(+)} - N_{\lambda}^{(-)}) = \text{constant}. \quad (4.9)$$

Thus, for instance, if we start from an initial state with $\sum N_{\lambda}^{(+)} = 1$, this condition will persist throughout the course of time.

The operation of time-reversal in the superquantized theory is

$$\Psi(t) \rightarrow R^T \Psi^*(-t), \quad (4.10)$$

with

$$(R^{-1} \phi(t) R)^T = \begin{pmatrix} R_f^T & 0 \\ 0 & R_f \end{pmatrix} \bar{\phi}(-t). \quad (4.11)$$

The double time-reversal is here

$$\Psi(t) \rightarrow \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \Psi(t). \quad (4.12)$$

The supercharge-conjugation is given by

$$\Psi(t) \rightarrow S \Psi(t), \quad (4.13)$$

with

$$S \phi(t) S^{-1} = \begin{pmatrix} 0 & R_f^T \\ R_f & 0 \end{pmatrix} \bar{\phi}(t) = C \bar{\phi}(t). \quad (4.14)$$

In order that this transformation may have the desired effect, Ψ has to obey anticommutation rules.

Finally, the combination of time-reversal and supercharge-conjugation results in

$$\Psi(t) \rightarrow R'^T \Psi^*(-t), \quad (4.15)$$

with

$$(R'^{-1} \phi(t) R')^T = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \phi(-t). \quad (4.16)$$

This corresponds to the time-reversal considered in the original DIV-method. The interchange of the first and the second parts of ϕ causes the sign-change of superelectric quantities, but it serves the purpose of time-reversal for the ordinary (supermechanical) quantities. If the supercharge is in principle unobservable, then this transformation, which is linear and whose reiteration is an identity transformation, is a suitable representation of time-reversal.

By the DIV-method in its superquantized version, as sketched in this section, the cases in which the original hamiltonian has negative energy levels of fields can be treated without giving rise to negative expectation values of energy and without causing any new inconsistency. Besides this positive advantage, the method may be credited for its esthetical merit of having better symmetry. A novel situation, which does not exist in the ordinary theory, will appear if one introduces operators which cannot be simultaneously diagonalized with H_0 in a two-two-matrix form. For a successful application of the method developed in the present paper to involved problems of the field theory, a further refinement of

the details may be necessary, but it is hoped that the idea presented here may suggest a new path of approach in this domain of quantum theory.

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Deuteron Photodisintegration at High Energies

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The problem of deuteron photodisintegration at high energies has something to do with meson scattering in the intermediate states. In this paper we have investigated this process by making use of the Tamm-Dancoff method which involves the p -wave interaction of virtual mesons.

§ 1. Introduction

The disintegration cross section of the deuteron for various photon energies has been calculated on various assumptions, and it has been obvious that the deuteron photodisintegration at low energies is independent of the shape of potential and depends on the effective range of the proton-neutron interaction. However, at high energies the situation is not so simple; that is to say, the absorption of a photon is, as is expected, accompanied with emission and reabsorption of virtual meson in the energy region above the meson threshold. On the other hand the recent experiments¹⁾ show that total cross section has a peak maximum at 250 Mev and, further, the angular distribution is flattened with the increasing photon energy.

In order to analyse these results, we investigate in this paper the deuteron photodisintegration at high energies on the assumption that the occurrence of the peak maximum in the cross section is due to the resonant term resulting from a state of the system with the total angular momentum $3/2$ and isotopic spin $3/2$ which involves p -wave meson in the intermediate states. In the present calculation we shall consider the pseudoscalar form of the coupling in the pseudoscalar theory.

In Sec. 2 we shall give the formal solution of deuteron photodisintegration when the meson interaction is taken into account; in Sec. 3 the cross section will be evaluated; and finally in Sec. 4 we shall summarize the results and make some remarks of conclusion.

§ 2. Preliminary formulation

We develop the discussion by presenting some formal relations given by Brueckner and others.²⁾ We suppose the Schrödinger equation to have the form

$$(H_0 + H'_m + H_r)\psi = E\psi, \quad (1)$$

where H_0 is the deuteron Hamiltonian including the nuclear potential V , and

$$H'_m = H_m - V; H_m = (g/2M) \sigma_i \cdot \nabla \tau_i \phi(\mathbf{r}_i).^*$$

The first term is the nonrelativistic form of pion-nucleon interaction which is suitable for the theoretical interpretation of p -wave scattering. The s -wave interaction may be neglected here, because we are interested in the p -wave resonance scattering.

$$H_r = H_{RN} + H_{RM}, \quad (2)$$

$$H_{RN} = \sum_{i=1}^n \left\{ \frac{e}{M} \frac{1 + \tau_i^z}{2} \mathbf{P}_i \cdot \mathbf{A}(\mathbf{r}_i) + \frac{1}{2} [(\mu_P + \mu_N) + (\mu_P - \mu_N) \tau_i^z] \sigma_i \cdot \mathbf{H}(\mathbf{r}_i) \right\}, \quad (3a)$$

$$H_{RM} = ie \mathbf{A}(\mathbf{r}) (\phi^* \cdot \nabla \phi - \phi \cdot \nabla \phi^*). \quad (3b)$$

E is the energy of the system, H_m the interaction energy between meson and nucleon, and H_r the sum of the interaction energies of nucleon and meson with the electromagnetic field, respectively. We proceed with our discussion by neglecting such an interaction energy as H_{RM} , since it involves s -wave pion and has nothing to do with resonance scattering.

Now, if ϕ_i is an eigenstate vector of H_0 , the true state vector Ψ_i is defined by the integral equation

$$\Psi_i = \phi_i + (1/d') (H'_m + H_r) \Psi_i, \quad (4)$$

where

$$d' \equiv E - H_0 + i\eta.$$

The above equation is rewritten down by introducing the Möller wave matrix Ω as follows

$$\Psi_i = \Omega \phi_i, \quad (5)$$

where Ω satisfies the Lippmann-Schwinger integral equation³⁾

$$\Omega = 1 + (1/d') (H'_m + H_r) \Omega. \quad (6)$$

In solving the above integral equation, H_m and H_r are not dealt with as small perturbations simultaneously; that is to say, H_m is stronger than H_r in the coupling strength. Remembering this fact let us solve eq. (6) instead of eq. (1). Thus we obtain the formal solution to eq. (6) with respect to H_m .

$$\Omega = 1 + \frac{1}{d' - H'_m} H'_m + \frac{1}{d' - H'_m} H_r \Omega. \quad (7)$$

In order to clarify the features of eq. (7), it is useful to rewrite it in the form of

$$\Omega = \omega (1 + \frac{1}{d'} H_r \Omega), \quad (7a)$$

where

$$\omega = 1 + \frac{1}{d' - H'_m} H'_m. \quad (8)$$

* We use the natural unit $\hbar = c = 1$ here and in any place.

The probability of transition to another free two nucleon state, which is specified ϕ_f , is proportional to the absolute square of

$$T_{fi} \equiv \langle \phi_f | T | \phi_i \rangle = \langle \phi_f | H'_m + H_r | \psi_i \rangle, \quad (9)$$

where T_{fi} is the transition matrix defined by Lippmann-Schwinger.

If we confine ourselves to the first order in electric charge e , then the transition matrix element is obtained from

$$T_{fi} = \langle \phi_f | H'_m + H_r | \omega \phi_i \rangle + \langle \phi_f | (H'_m + H_r) \omega \frac{1}{d'} H_r | \omega \phi_i \rangle, \quad (10)$$

where the first term of the transition operator is

$$H_r + H_r \frac{1}{d' - H'_m} H'_m,$$

and, if we neglect the multiple scattering of virtual mesons in the initial state, the second term becomes

$$H'_m \frac{1}{d' - H'_m} H_r + H'_m \frac{1}{d' - H'_m} H_r \frac{1}{d'} H'_m.$$

Hence, we have

$$T = H_{rN} + H_r \frac{1}{d' - H'_m} H'_m + H'_m \frac{1}{d' - H'_m} H_r + H'_m \frac{1}{d' - H'_m} H_r \frac{1}{d'} H'_m. \quad (11)$$

The first term represents the multipole transition neglecting the meson corrections.¹⁾ The second and third terms do not include the $(3/2, 3/2)$ state, and may be estimated by the perturbation theory. Then

$$-\frac{1}{d' - H'_m} H'_m = \frac{1}{a - L} H_m + \frac{1}{a - L} L - \frac{1}{a - L} V - \frac{1}{a - L} H_m \frac{1}{a} V \quad (12)$$

by making use of the identity relation

$$-\frac{1}{a - H_m} H_m = \frac{1}{a - L} H_m + \frac{1}{a - L} L$$

and

$$L \equiv H_m \frac{1}{a} H_m; \quad d' - H'_m = a - H_m.$$

If one assumes that the nuclear force is obtained by second order meson process, the second and third terms are approximately cancelled out each other. On the other hand, since the first term in eq. (11) has been already studied by many authors, we do not consider it here, but the last term in eq. (11) will give rise to the meson correction which seems to be effective at high energies.

Now, the analysis of recent pion-nucleon scattering experiments suggests that for the

scattering of p -wave pion its theoretical treatment is in fair agreement with observation and the pion-nucleon interaction is very strong in the $(3/2, 3/2)$ state⁵⁾. In our calculation of the deuteron photodisintegration cross section it is, therefore, hopeful that the p -wave pion interaction is taken into account to explain the recent experimental results.⁵⁾ Our purpose will be attained by calculating the potential which can describe the emission of a pion by the first operator $(g/2M)\sigma \cdot \nabla \tau \phi$, its multiple scattering between the two nucleons in the deuteron, and finally its reabsorption by the second one. However, in this case, we omit the charge renormalization, the vertex part, and so on that appear from the last term in eq. (11). Here the terms which we wish to calculate are the graphs (a), (b) and (c) in Fig. 1.

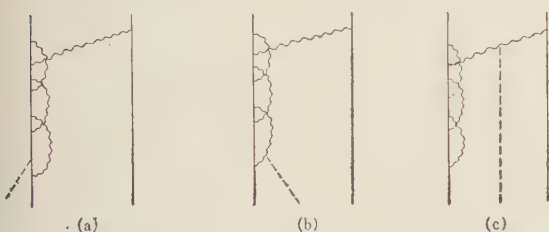


Fig. 1. —: Nucleon line,
 ~: Meson line,
 ---: Photon line,

The graph (b) gives the contribution that results from the charged pion, which is considered to be not so effective in the present process, and further the lowest order process of the graph (c) is one that has been already calculated by Bruno-Dempken.⁷⁾ Therefore we consider only the contribution from graph (a).

Thus the transition operator, which is appropriate to the photodisintegration of deuteron in the present case, has the

form of

$$T = H_m^{(-)j} \frac{1}{a-L} H_{RN} \frac{1}{a} H_m^{(+i)}, \quad (13)$$

where the operators with $(+)$ emit pions, the ones with $(-)$ absorb pions, and super-indices i, j run from 1 to 2.

§ 3. Cross section

The transition operator given in the preceding section has much complex structure and consequently the calculation of matrix elements will become complicated and cumbersome. Therefore, we first investigate the scattering matrix and its properties. As is well known, the term $1/(a-L)$ occurring in eq. (13) may be written as $W \cdot 1/a$, where W satisfies the equation

$$W = 1 + (1/a)LW \quad (14)$$

which describes the multiple scattering of the virtual mesons by the two nucleons and has the solution

$$W = 1 + \frac{1}{a-L} L. \quad (15)$$

Following the method developed by Brueckner-Watson,¹⁾ and Aitken and others,⁸⁾ the term

$1/(a-L) \cdot L$ can be written as $1/a \cdot t$, which satisfies the integral equation for scattering matrix :

$$t = L + L(1/a)t, \quad (16)$$

and, if we neglect the multiple meson exchange in the present case, we can write

$$t = t^{(1)} + t^{(2)}. \quad (17)$$

Eq. (16) is resolved into equations for the four substates of spin and isotopic spin for the scattering of a meson from either nucleon. Supposing that the meson-nucleon interaction is strongest in the $(3/2, 3/2)$ state, the integral equation for the scattering of the meson from nucleon "1" is

$$\begin{aligned} \langle q' | t^{(1)}(3/2, 3/2) | q \rangle = & \lambda \frac{q'q}{(W_{q'}W_q)^{1/2}} \frac{1}{E - W_{q'} - W_q} \\ & + \lambda \int \frac{q'q''}{(W_{q'}W_{q''})^{1/2}} \frac{dq''}{(E - W_{q'} - W_{q''})} \frac{1}{(E + i\epsilon - W_{q''})} \langle q'' | t^{(1)}(3/2, 3/2) | q \rangle, \end{aligned} \quad (18)$$

if the nucleon recoil is neglected, with a similar equation for nucleon "2". To solve this equation we introduce trial function as follows :

$$\begin{aligned} \langle q' | t^{(1)}(3/2, 3/2) | q \rangle = & -\lambda R \frac{q'q}{(W_{q'}W_q)^{1/2}} \times \begin{cases} 1/W_{q'}; & W_{q'} > W_q, \\ 1/W_q; & W_{q'} < W_q, \end{cases} \\ \lambda = & (g^2/4\pi)(1/3\pi M^2), \quad W_q = (q^2 + t^2)^{1/2}, \quad \text{etc.} \end{aligned} \quad (19)$$

Then we have

$$R = 1 + \lambda R \int \frac{q''^4 dq''}{W_{q'}^2 (W_{q'} + W_{q''})} = 1 + \lambda R K_{q'}. \quad (20)$$

Eq. (18) is so much singular that it has no sensible solution. However, if the integral in eq. (18) is assumed to be cut off at the nucleon rest mass, then a solution exists.

Now, to clarify the occurrence of resonance in the deuteron photodisintegration, it is convenient to replace the factor R by the following equation in order to fit the $(3/2, 3/2)$ state to the one level resonance formula along the lines of the resonance theory studied by Brueckner. Then we have

$$R = \frac{1}{1 - (\epsilon + \frac{i}{2}\Gamma')/W_{q'}} \quad (21)$$

where ϵ is the resonance energy and Γ' represents the width of resonance level. The value of resonance energy is to be about 2μ and the width about $\mu/2$ so as to be accordant with the meson-nucleon scattering experiments; μ is the meson rest mass.

Since we have obtained the expressions for the scattering matrix, we insert them into eq. (13) and the transition matrix element consequently is written in the form of

$$\begin{aligned}
T_{fi} = & \langle o, o | H_m^{(-)} | o, q \rangle \langle o, q | 1/a | o, q \rangle \langle o, q | H_{RN} | \kappa, q \rangle \langle \kappa, q | 1/a | \kappa, q \rangle \langle \kappa, q | H_m^{(+)} | \kappa, o \rangle \\
& + \langle o, o | H_m^{(-)} | o, q \rangle \langle o, q | 1/a | o, q \rangle \lambda R_q K_q \langle o, q | H_{RN} | \kappa, q \rangle \langle \kappa, q | 1/a | \kappa, q \rangle \langle \kappa, q | H_m^{(+)} | \kappa, o \rangle.
\end{aligned} \quad (22)$$

This matrix element represents the contribution from $(3/2, 3/2)$ state. The contributions from other states also are to be taken into account at the same time, but, here the former contribution to deuteron photodisintegration mainly must be clarified on our assumption. Hence we have

$$T_{fi} = \langle o, o | H_m^{(-)} | o, q \rangle \frac{1}{\kappa - W_q + i\eta} \langle o, q | H_{RN} | \kappa, q \rangle \frac{1}{W_q} \langle \kappa, q | H_m^{(+)} | \kappa, o \rangle \cdot R_q E_{3/2} F_{3/2}, \quad (23)$$

where $E_{3/2}$ and $F_{3/2}$ are projection operators for the substate of isotopic spin $3/2$ and angular momentum $3/2$.

Our next work is the calculation of the differential cross section with eq. (23), which is

$$d\sigma = 2\pi |T_{fi}|^2 \rho_f \quad (24)$$

where ρ_f is the final state density, which is

$$\rho_f = (2\pi)^{-3} p^2 \frac{dp}{dE} d\Omega = \frac{1}{(16\pi^3)} M p d\Omega.$$

$d\Omega$ is the solid angle into which proton is directed.

In order to obtain the angle dependence of the cross section we have to select suitable ones, which satisfy the exclusion principle, out of various final wave functions appearing as result of application of various operators on the initial deuteron function and the intermediate wave functions. First, the deuteron wave function is written

$$\psi_D = (\alpha/2\pi)^{1/2} \frac{e^{-\alpha r}}{r} {}^3\chi_i(\sigma) {}^1\eta_0(\tau); \alpha = (M \cdot B)^{1/2}. \quad (25)$$

Here B is the deuteron binding energy. On the other hand, the final wave functions are secured by introducing the following projection operators for the spin and isotopic spin states:

$$O_1 = \frac{1}{4} [3 + (\sigma_1 \cdot \sigma_2)], \quad (26)$$

$$O_0 = \frac{1}{4} [1 - (\sigma_1 \cdot \sigma_2)],$$

and

$$T^1 = \frac{1}{4} [3 + (\tau_1 \cdot \tau_2)], \quad (27)$$

$$T^0 = \frac{1}{4} [1 - (\tau_1 \cdot \tau_2)].$$

That is to say, by combining these operators, i. e., $O_1 T^1$, $O_1 T^0$, $O_0 T^1$, $O_0 T^0$, we can select the appropriate ones out of the final wave function

$$\psi = A(j_i(pr)/r) Y_{l''}(\theta, \varphi) {}^o\chi_f(\sigma) {}^\tau\eta_f(\tau), \quad (28)$$

where Y_l^m is spherical harmonics of order l and m , and they are ${}^A\zeta_l(\mathbf{r})^3\chi_l^3\chi_l$, ${}^S\zeta_l(\mathbf{r})^3\chi_l^1\chi_{00}$, ${}^A\zeta_l(\mathbf{r})^1\chi_{00}^1\chi_{00}$, and ${}^S\zeta_l(\mathbf{r})^1\chi_{00}^3\chi_{lf}$; the supersuffices A, S denote symmetry properties of wave functions, and ${}^a\chi_l(\sigma)$ and ${}^s\chi_l(\tau)$ are the spin and isotopic spin functions, respectively.

Thus, the differential cross section is given:

$$\frac{d\sigma}{d\Omega} = \frac{1}{108} e^2 \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{M^2} \frac{\pi\alpha}{12} \frac{p^3}{M^3\kappa} \sum_l \left\{ \frac{31}{2} I_1 |Y_{2l}^m|^2 + 5 I_2 |Y_{2l+1}^m|^2 \right\} \cdot |R_q|^2 \quad (29a)$$

$$+ \frac{1}{128} (\mu_p - \mu_N)^2 \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{M^2} \frac{\pi\alpha}{18} \frac{p^3}{M^3\kappa} \sum_l \{ I_1' |Y_{2l+1}^m|^2 + I_2' |Y_{2l}^m|^2 \} \cdot |R_q|^2, \quad (29b)$$

where

$$\begin{aligned} I_1 &= \sum_l \frac{\Gamma(l+2)}{\Gamma(l+1)} \int \frac{[(\mathbf{q} \cdot \mathbf{q}')^2 + \frac{1}{3} q^2 q'^2] (\frac{1}{2} p)^{2l}}{\{ [p^2 + (\mathbf{q} - \boldsymbol{\kappa}/2)^2 + \alpha^2] [p^2 + (\mathbf{q}' - \boldsymbol{\kappa}/2)^2 + \alpha^2] \}^{(l+2)/2}} \\ &\quad \times F(l; q) F(l; q') \frac{d\mathbf{q} d\mathbf{q}'}{W_q^2 W_{q'}^2}, \\ I_2 &= \sum_l \frac{\Gamma(l+2)}{\Gamma(l+2)} \int \frac{[(\mathbf{q} \cdot \mathbf{q}')^2 - q^2 q'^2] (\frac{1}{2} p)^{2l}}{\{ [p^2 + (\mathbf{q} - \boldsymbol{\kappa}/2)^2 + \alpha^2] [p^2 + (\mathbf{q}' - \boldsymbol{\kappa}/2)^2 + \alpha^2] \}^{(l+2)/2}} \\ &\quad \times F(l; q) F(l; q') \frac{d\mathbf{q} d\mathbf{q}'}{W_q^2 W_{q'}^2}, \\ I_1' &= \sum_l \frac{\Gamma(l+2)}{\Gamma(l+1)} \int \frac{[7(\mathbf{q} \cdot \mathbf{q}')^2 - \frac{1}{3} q^2 q'^2] (\frac{1}{2} p)^{2l}}{\{ [p^2 + (\mathbf{q} - \boldsymbol{\kappa}/2)^2 + \alpha^2] [p^2 + (\mathbf{q}' - \boldsymbol{\kappa}/2)^2 + \alpha^2] \}^{(l+2)/2}} \\ &\quad \times F(l; q) F(l; q') \frac{d\mathbf{q} d\mathbf{q}'}{W_q^2 W_{q'}^2}, \\ I_2' &= \sum_l \frac{\Gamma(l+2)}{\Gamma(l+1)} \int \frac{[(\mathbf{q} \cdot \mathbf{q}')^2 + \frac{1}{3} q^2 q'^2] (\frac{1}{2} p)^{2l}}{\{ [p^2 + (\mathbf{q} - \boldsymbol{\kappa}/2)^2 + \alpha^2] [p^2 + (\mathbf{q}' - \boldsymbol{\kappa}/2)^2 + \alpha^2] \}^{(l+2)/2}} \\ &\quad \times F(l; q) F(l; q') \frac{d\mathbf{q} d\mathbf{q}'}{W_q^2 W_{q'}^2}, \end{aligned} \quad (30)$$

and F is the hypergeometric function

$$F(l; q) = F\left(\frac{l+2}{2}, \frac{l-1}{2}; l+1; \frac{p^2}{[\alpha - i(\mathbf{q} - \boldsymbol{\kappa}/2)]^2 + p^2}\right) \quad (31)$$

and

$$\begin{aligned} \int_0^\infty e^{-\alpha r} j_l(pr) r dr &= \frac{(\frac{1}{2} p)^l}{(\alpha^2 + p^2)^{(l+2)/2}} \frac{\Gamma'(l+2)}{\Gamma(l+1)} \\ &\quad \times F\left(\frac{l+2}{2}, \frac{l-1}{2}; l+1; -\frac{p^2}{\alpha^2 + p^2}\right). \end{aligned} \quad (32)$$

Here p, q and κ are final relative momentum of proton, meson momentum, and incident photon momentum in the center of mass system, respectively. Of course, the meson

momentum appearing in the intermediate states will be taken as the relative momentum with respect to the center of mass of the two nucleons in the approximation where terms of relative order $\mu/2M$ will be dropped.

However, since the above calculations are very complicated, we are satisfied with explaining only the features of the total cross section and we adopt the free wave functions as the initial and final state functions for the first term of eq. (29). Then the total cross section is

$$\sigma_d = \frac{1}{48} e^2 \left(\frac{g^2}{4\pi} \right)^2 \frac{1}{M^2} \frac{\pi \alpha}{3} \frac{p^3}{M^3 \kappa} \cdot S \cdot |R_{\eta \max}|^2 \quad (33)$$

with

$$S = \pi (J^2 - 4J \cdot J' + 6J'^2), \quad (34)$$

where, if J and J' are calculated in the approximation that the exchanged mesons are regarded as real, then they are

$$J = \frac{(\kappa^2 - \mu^2)}{2p\kappa} \ln \frac{\alpha^2 + (p + \sqrt{\kappa^2 - \mu^2})^2}{\alpha^2 + (p - \sqrt{\kappa^2 - \mu^2})^2},$$

$$J' = \frac{(\alpha^2 + p^2 + \kappa^2 - \mu^2)}{8p^3\kappa} \left\{ (\alpha^2 + p^2 + \kappa^2 - \mu^2) \ln \frac{\alpha^2 + (p + \sqrt{\kappa^2 - \mu^2})^2}{\alpha^2 + (p - \sqrt{\kappa^2 - \mu^2})^2} - 4p(\kappa^2 - \mu^2)^{1/2} \right\}. \quad (35)$$

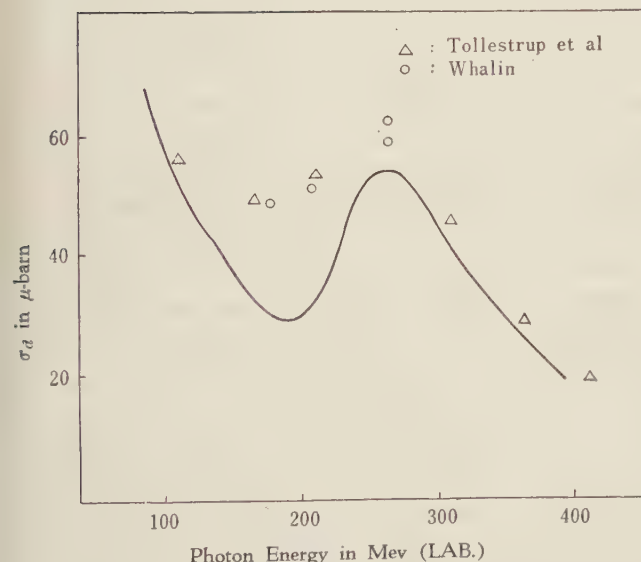


Fig. 2 Total cross section for deuteron photodisintegration.

The calculated total cross section including the resonant effects, is indicated in Fig. 2, where the coupling constant is taken as $g^2/4\pi = 22$, to be possible to explain the experimental results. This result is still unsatisfactory because the meson in the intermediate states is regarded as free and, further, the contributions from the meson current and the second terms of eq. (29) are entirely neglected.¹⁰⁾ However, it is concluded at least that the Born approximation is not qualitatively bad if one is satisfied only with explaining the occurrence of resonant peak in

the total cross section by assuming that the meson-nucleon interaction is the strongest in the $(3/2, 3/2)$ state.

§ 4. Conclusions

We have seen that if we take account of, in addition to the interaction of deuteron with electromagnetic field, the contributions from the meson scattering in the intermediate states, it is possible to explain the observed cross section. This has, as repeatedly stated in the preceding sections, a connection with the process which can be interpreted in terms of the idea of an isobaric state ($3/2, 3/2$). Therefore it may be said that our recipe is fairly suitable, although it is pretty satisfactory but not too rigorous. However, we can say little of the angular distribution even though the meson interaction is taken into account in the intermediate states.

In our procedure the following approximations are used: neglecting nucleon recoil, dropping terms of order $\mu/2M$, and reducing the factor R tentatively to the one level formula. Though it seems that these approximations do not lose the validity for clarifying the observed cross section, it still retains ambiguities occurring from being impossible to integrate the terms which involve the hypergeometric function. Hence it may be concluded that the Born approximation result is qualitatively not a bad approximation for the deuteron photodisintegration at high energies, except with the angular distribution.

The author wishes to thank professors S. Tomonaga and T. Miyazima for their interest in this work. He is also much indebted to Dr. N. Fukuda, Mr. D. Ito and Mr. Y. Takahashi for useful discussion for this problem.

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Elastic Scattering of Alpha-Particles by Heavy Elements

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The elastic scattering of 13–42 Mev alpha-particles by heavy elements has been analysed, using the boundary condition model of Feshbach and Weisskopf and the optical model with a complex potential of exponential shape in addition to the Coulomb potential. Results for Ag and Au have been compared with experiments for the energy dependence of the elastic scattering and its angular distribution at 22-Mev. As for the energy dependence the experimental data have been fitted fairly well with the results derived from both models, but for the angular distribution the agreement between experiments and theories is less satisfactory, especially for Ag data. The best-fit procedure compels us to assume a considerably larger value for the radius of the alpha particle than the usually accepted value, $R_\alpha \sim 1.2 \times 10^{-13}$ cm, using $R = r_0 A^{1/3}$ with $r_0 = 1.5 \times 10^{-13}$ cm for the nuclear radius.

§ 1. Introduction

Recently, Farwell and Wegner¹⁾ measured the energy dependence of the elastic scattering of 13 — 42 Mev alpha-particles by heavy elements and investigated the character of deviation from the Rutherford scattering. The results show the general tendency that the cross section for each element represents the usual Rutherford scattering up to the critical energy E_0 which increases with Z , but when the energies increase over E_0 the cross sections suddenly drop. More recently, Wall, Rees and Ford²⁾ have measured the angular distributions of the elastic scattering of 22 Mev alpha-particles by Ag, Au, and Pb. The differential cross sections were found to follow the Rutherford's formula at the forward angle for all elements mentioned above, and, after a slight rise with the increase in angle, they are observed to decrease rather monotonously. We shall analyse these experimental data in the light of the continuum theory which is shown to be successful for the scattering and capture of neutron, proton and the other charged particles^{3)–9)} in the intermediate energy region.

The usual continuum-theoretic treatments of scattering by nuclei are conveniently classified into the following two groups:

(a) Strong absorption model: The main characteristic of this model is based on the assumption that the impinging particle is absorbed by the struck nucleus to form the compound nucleus with probability unity once the particle entered into the nucleus. The black body model and the boundary condition model belong to such strong absorption model; the former is a simple model used by Blair¹⁰⁾ in accounting for the experiment of Farwell and Wegner¹⁾, and the latter is characterized by the simple boundary condition that there is only incoming wave in the entrance channel just inside the nuclear boundary.³⁾

(b) Optical potential model: The main characteristic is based on the assumption that the interaction between the impinging particle and the struck nucleus is represented by a complex potential, allowing for both absorption and refraction effects of the nucleus (i. e., showing the partial transparency of nucleus).

Therefore the model (b) seems to be a more general description for the nuclear scattering and reaction than the model (a). In fact, recently, the model (b) has given fairly reasonable results for the total cross sections and the angular dependence of the neutron scattering in the energy region between 50 Kev and 3 Mev,⁹⁾ and, further, the angular dependence of elastic scattering of protons in the intermediate energy region was examined by this model.⁸⁾

The black body model used by Blair¹⁰⁾ is based on two assumptions. First, if the potential barrier for the l -th partial wave allows the nucleus and the impinging particle to overlap with each other when considered classically, the outgoing l -th partial wave is assumed to vanish on account of its complete absorption. Second, if the barrier is such that the nucleus and the particle do not classically overlap, the outgoing l -th wave has the phase characteristic of the pure coulomb scattering. Hence we may define the critical angular momentum $l'b$ by the following relation:

$$E = 2Ze^2/R + \hbar^2 l'(l' + 1)/2M_\alpha R^2 \quad (1 \cdot 1)$$

where E is the channel energy, R the channel radius, and M_α the reduced mass of the alpha particle. In the present case, l' runs to about 16 the at maximum energy, so that the classical path might be defined qualitatively. The first assumption neglects the reflection due to the sudden change of the potential at the surface of the nucleus, as well as the reflection due to the potential barrier (coulomb and centrifugal) outside the nucleus. On the other hand, the second assumption neglects the absorption due to the penetration of the incoming particle through the potential barrier. According to the counter balance of the effects due to the above two neglects, the black body model might accidentally give a rather good fit to the experimental curves. In fact, the result of calculation by Blair seems to give a fairly good agreement with the experimental curves of energy dependence of the cross sections. However, a closer examination shows that it gives a rather larger cross section than the experimental one in the higher energy region (about 30 Mev and higher), and moreover, the nuclear radius which gives the best fit to the experimental curve must take a rather larger value than the usual one obtained from the neutron and proton scattering, particularly, for Au the best fit radius at 60 differing from that at 95°. Although the latter might be able to be explained by that classically the apsidal distance of the path which gives the larger angle scattering is smaller, it is unsatisfactory from the wave-mechanical point of view.

The analysis of the angular distribution with same model has been done by Wall, Rees and Ford.²⁾ The results do not agree with the experiments in the points that their calculations give large diffraction pattern and, especially, very large scattering in backward direction. To obtain the better fit with the experiments they have adopted a fuzzy model in which for $l < l'$ and $l > l'$ the partial waves are still the same as the Blair's, while for

$l=l'$ they have the pure Coulomb phases but half the usual amplitudes. This modification gives the reasonable results for Pb and Au but it is still unsatisfactory for Ag. Since their critical angular momentum l' is arbitrarily chosen, the original meaning defined by (1.1) is blurred out in this model. However, the success of this fuzzy model apparently seems to suggest that the nucleus would have the diffuse boundary rather than the sharp one.

In order to analyze the experimental observations mentioned above, we shall here calculate the cross sections of elastic scattering of alpha-particles by Au and Ag, using the boundary condition model in section 2, and using the diffuse optical potential model in section 3, with the comparison of the obtained results with the experimental ones. Using these models mentioned above, we can take into account the reflections due to the variation of the potential at the nuclear surface and outside the nucleus, and the barrier penetration that are neglected in Blair's model. However, in our actual calculation, the neglect of the barrier penetration (corresponding to the "sharp angular momentum cut-off approximation") was taken over in most cases. In section 4, the discussion of the results will be added with the concluding remarks. Since our calculation are made by means of the partial wave method, the actual procedure becomes rather laborious owing to a great number of phase shifts for high energy scattering.*)

§ 2. Boundary condition model

In the following, we shall adopt the boundary condition model of Feshbach and Weisskopf,³⁾ in which it is assumed that, in the entrance channel, the wave function just inside the nucleus is of the form of an incoming wave only:

$$u_l = r\psi_l \sim \exp(-iKr), \quad (2.1)$$

that is independent of the angular momentum l/\hbar of the incident particle. The logarithmic derivative of the wave function at the boundary, therefore, is

$$f_l = -iKR. \quad (2.2)$$

K is the average wave number of the incident particle inside the nucleus

$$K = (K_0^2 + k^2)^{1/2}, \quad (2.3)$$

where k is the wave number of the incident particle outside the nucleus and K_0 the adjustable parameter which may depend on each nucleus. Therefore we have two adjustable parameters K_0 and R in this model. Since for the alpha particle there is no plausible criterion for the choice of K_0 we shall here tentatively assume the following value according to B.W.,¹²⁾

* Calculation using the other way than the partial wave analysis in the model (b) was recently done by Izumo,¹¹⁾ which is directly based on the variational method by Montroll and Greenberg¹¹⁾ Although he does not say anything about the validity of this method in his calculation, it seems to us that this method is not so much justified in his case, i. e., when the potential does not more rapidly damp than $1/r$ as $r \rightarrow \infty$ and the nucleus is not a soft scatterer.

$$K_0 \sim 1 \times 10^{13} \text{ cm}^{-1}. \quad (2.4)$$

Indeed, our calculation has shown the dependence of the theoretical cross section on K_0 to be negligible in our energy region. Another adjustable parameter R is the channel radius,

$$R = r_0 A^{1/3} + R_\alpha, \quad (2.5)$$

where R_α is the radius of the alpha-particle. The differential elastic scattering cross section for a charged particle is evaluated by using the following well-known formula¹²⁾:

$$\begin{aligned} d\sigma = & |(Z Z' e^2 / 2 M v^2) \operatorname{cosec}^2 (\theta/2) \exp[-2i\gamma \ln \sin (\theta/2)] \\ & - i\lambda \sqrt{\pi} \sum_{l=0}^{\infty} \sqrt{2l+1} \exp(2i\sigma_l - 2i\sigma_0) \{1 - \exp(2i\xi_l - 2i\sigma_l) [1 + A_{\text{res}}^l]\} \\ & \times Y_{l,0}(\theta)|^2 d\Omega. \end{aligned} \quad (2.6)$$

The notations are all the same as ones of *B-W*. Since the Coulomb wave functions at the nuclear surface, required for our case, are not tabulated for values of Z and energy under consideration, we have approximately evaluated them, i. e., $U_l^{(+)}$ and $U_l^{(-)}$ by the use of the *WKB* method.¹²⁾ The summation over l in (2.6) is extended up to the highest angular momentum \bar{l} which substantially contributes to the nuclear scattering. In our actual calculation, \bar{l} is assumed to take the critical angular momentum l' defined by expression (1.1) which depends on energy and runs from 0 to 16. This sharp cut-off approximation represents the disregard of the penetrability of potential barrier for the $l(>l')$ th partial wave. In order to test this approximation we have calculated the Au cross section at 90° for $\bar{l} = l' + 2$. The results of our calculation are shown in Figs. 1-5, together with experimental curves for Au and Ag. We have chosen the nuclear radius $R_n = 1.5 \times A^{1/3} (10^{-13})$ cm, in order to estimate the radius of the alpha-particle from the channel radius R entered in our calculation. From the other experiments, several kinds of values have been derived for the radius of the alpha-particle, R_α ; for example, $\sim 1.2 (10^{-13})$ cm from the alpha decay and $\sim 2.3 (10^{-13})$ cm from the scattering of neutrons on alpha particles.¹³⁾ For comparison with the channel radii R used in our calculation, the ones derived from the above two values for R_α are listed in Table I. In our actual calculation

Table I. Nuclear radii. All radii in units 10^{-13} cm.

R = channel radius $= R_n + R_\alpha$

R_α = radius of alpha-particle

R_n = nuclear radius $= r_0 \times A^{1/3} = 1.50 \times A^{1/3}$

Element	R_n	R_α	R
Au	8.73	1.2	9.93
		2.3	11.03
Ag	7.14	1.2	8.34
		2.3	9.44

tion we shall adopt the similar values for the channel radii to Blair's one in order to compare our results with his. In Fig. 1 the experimental cross section for Au at the

90° part is compared with the theoretical ones computed for $R=10.85$ (10^{-13}) cm and 10.0 (10^{-13}) cm and $\theta=90^\circ$. (see Table I) For comparison the best fit curve obtained by Blair for $R=10.95$ (10^{-13}) cm is drawn. The theoretical curve is sensitive to the choice of R ; for $R=10.0$ (10^{-13}) cm corresponding to $R_\alpha \sim 1.2$ (10^{-13}) cm in (2.5), the curve has the shape nearly parallel to the experimental but is considerably displaced to the right hand side, whereas, for a little larger value of $R=10.85$ (10^{-13}) cm corresponding to $R_\alpha \sim 2.1$ (10^{-13}) cm, the comparison is improved. The general tendency of the large drop of the experimental curves in the high energy region is well reproduced in our calculated ones and the role of the variation of R is to displace the curves after the sudden drop nearly in parallel with each other. Therefore, it might be possible, with some appropriate choice of R , to obtain a better agreement with the experimental curve. Furthermore a curve with $R=10.85$ (10^{-13}) cm and $\bar{l}=l'+2$ is drawn, showing that, due

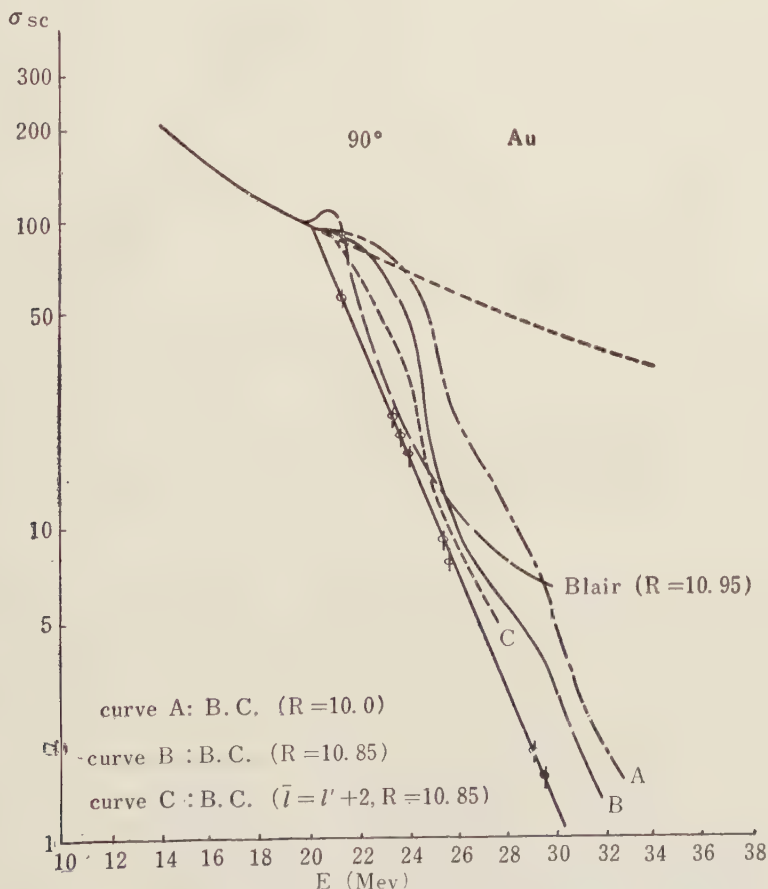


Fig. 1 Differential scattering cross section for Au at $\theta=90^\circ$ as a function of the alpha-particle energy. Three theoretical curves computed from the boundary condition model are shown. Points are experimental.

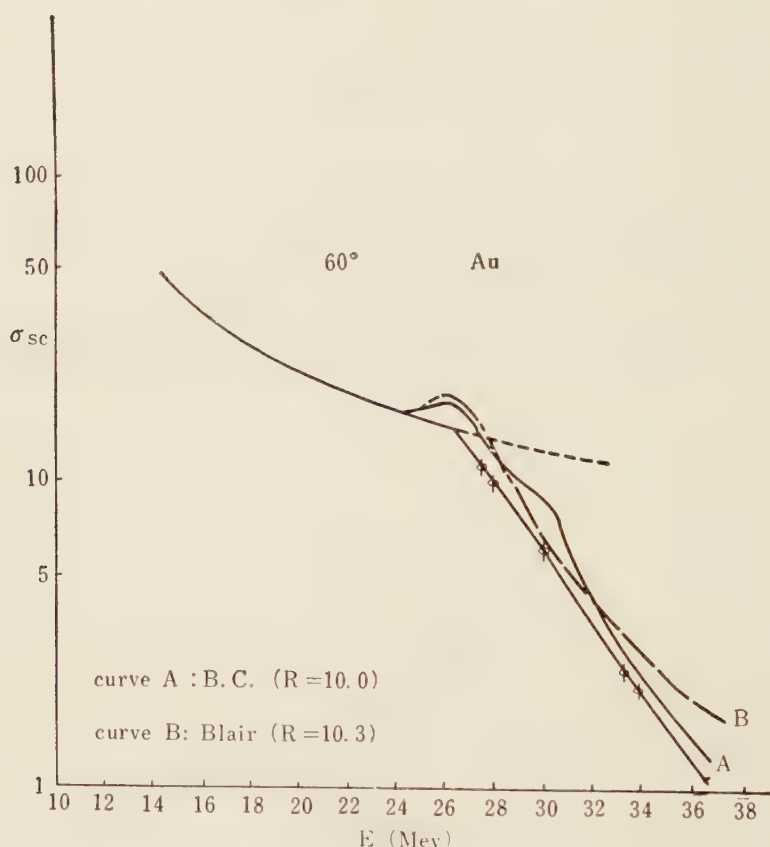


Fig. 2 Differential scattering cross section for Au at $\theta=60^\circ$ as a function of the alpha-particle energy. The theoretical curve is computed from the boundary condition model.

to the inclusion of the penetrability of the potential barrier (Coulomb and centrifugal) for the partial waves of $l \neq l'$, the cross section is reduced, especially in the region near the critical energy E_0 at a sudden drop. Since the penetrability of the partial waves with the larger angular momentum than $l' + 2$ has shown to become quite small, the inclusion of them was not worked here.

The comparison of the experimental cross section with theoretical one for Au at the 60° port is shown in Fig. 2, where the theoretical curve is drawn for $R=10.0$ (10^{-13}) cm and $\bar{l}=l'$. For comparison the best fit curve obtained by Blair for $R=10.3$ (10^{-13}) cm is shown. Although in this case the theoretical curve with $R=10.85$ (10^{-13}) cm was not drawn, it seems highly possible that we shall be able to obtain the best fit to the experiment with the use of a slightly larger value of R than $R=10.0$ (10^{-13}) cm together with $\bar{l}=l'+2$. The comparison for the case of Ag at 60° scattering is given in Fig. 3,

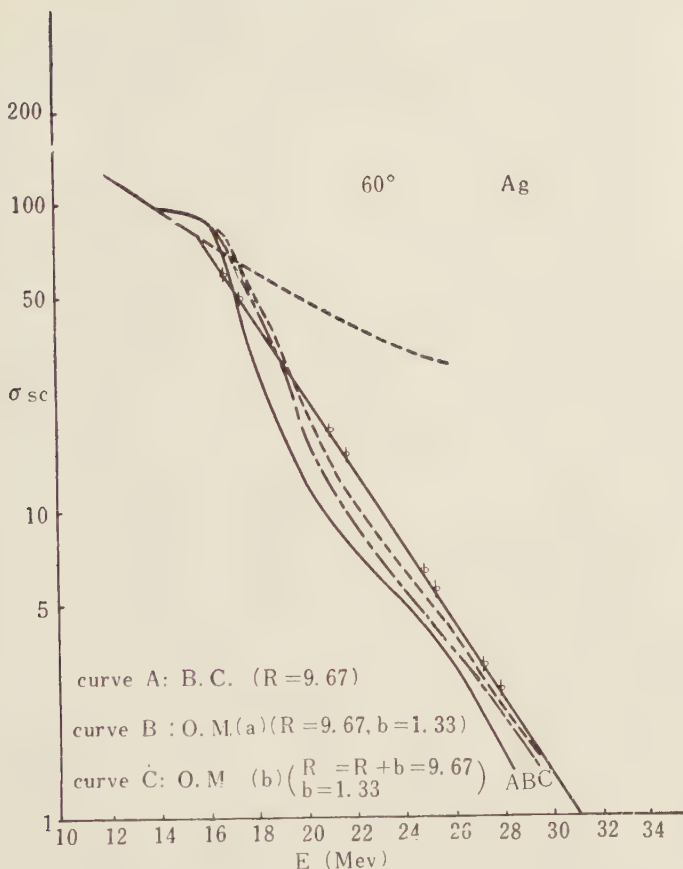


Fig. 3. Differential scattering cross section for Ag at $\theta=60^\circ$ as a function of the alpha-particle energy. Three theoretical curves are computed by A: boundary condition model, B: optical model (a) $R=R'$ and C: and optical model (b) $R'=R+b$.

where the theoretical curve is for $R=9.67 (10^{-13})$ cm and $\bar{l}=l'$. The reason why the theoretical values become too small perhaps comes from the choice of too large a value of R (the value of R using $R_\alpha=2.0 (10^{-13})$ cm in (2.5) is $9.14 (10^{-13})$ cm), so that a little smaller value of R and $\bar{l}=l'+2$ will give rise to a more reasonable curve. Finally, in Figs. 4 and 5 we show the angular distribution of the elastic scattering of 22 Mev alpha particles by Au and Ag. For Au, the theoretical curve for $R=10.85 (10^{-13})$ cm and $\bar{l}=l'=4$ (i. e., the sharp angular momentum cut-off) is represented, reproducing the qualitative features of a rise followed by a sharp fall of the relative cross section at the large angles. The theoretical curve, however, shows a diffraction pattern with a large amplitude which is not observed in the experimental data. For Ag, the theoretical curve for $R=9.67 (10^{-13})$ cm and $\bar{l}=l'=11$ was drawn, showing the excellent agreement with the experiment in the general trend, especially in larger angles. In order to see whether

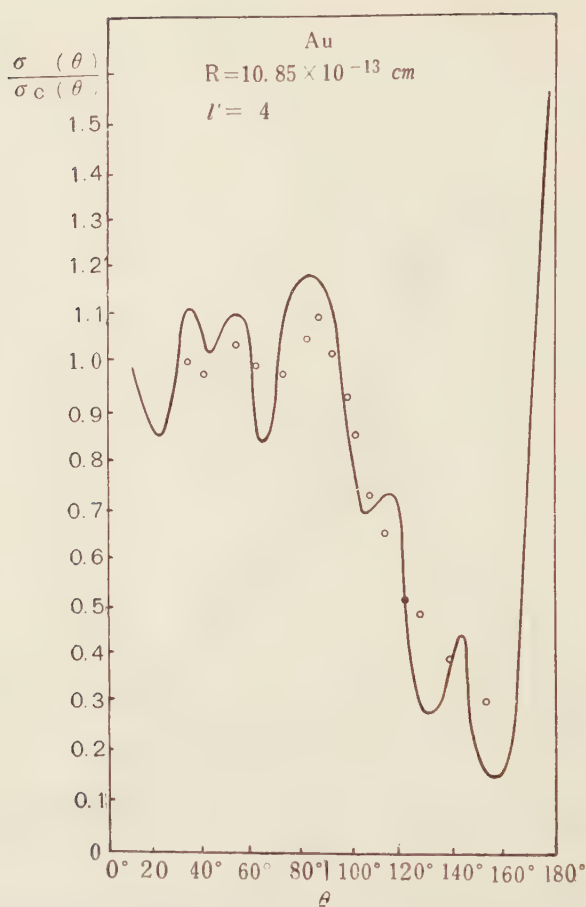


Fig. 4. Comparison of boundary-condition model with σ/σ_c for Au.

this success with the sharp angular momentum cut-off is incidental or not, we have performed a calculation which includes the effect of the transparency of the potential barrier (i. e. $\bar{l} = l' + 2 = 13$), with the result that the inclusion of two more partial waves destroys the agreement between theory and experiment above 80° and gives rise to a very marked diffraction pattern of which only the minima come close to the experimental points.

§ 3. Optical potential model with diffuseness

For the angular distribution of the proton scattering in the 20-Mev region, Chase and Rohrlich⁸⁾ have worked in some detail the optical model with sharp boundary such as the square well in comparison with the experiments and pointed out the disagreement between theoretical and experimental results. However, Woods and Saxon¹⁴⁾ have shown

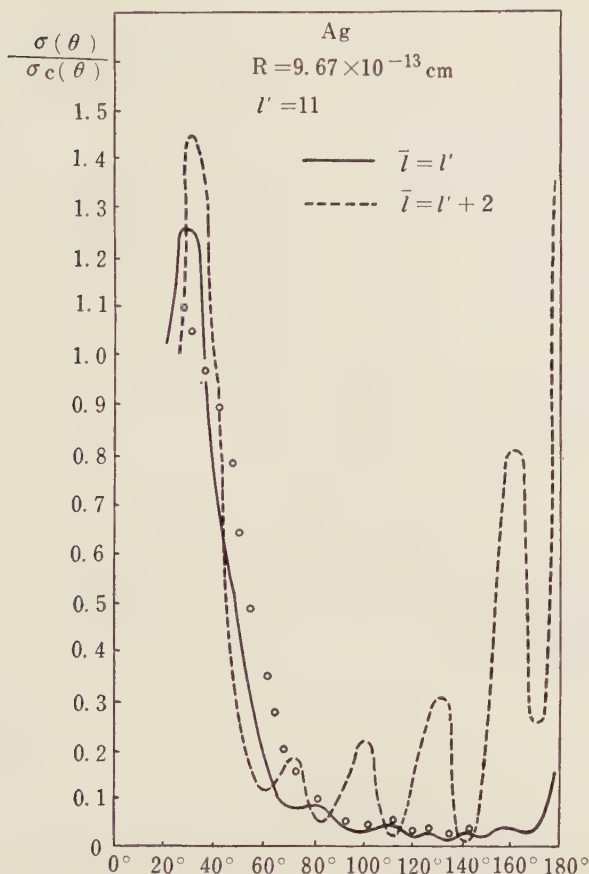


Fig. 5. Comparison of boundary-condition model with σ/σ_c for Ag.

recently that a smoothing-out of the sharp boundary even within the small interval of $0.5 \times (10^{-13})$ cm changes the results considerably and brings them into much better agreement with the experiments. In the present problem, therefore, we also use the optical potential with diffuseness. Since the exact calculation is much complicated owing to the lack of the table of Coulomb wave functions and, further, to a large number of the partial waves required (up to 16), we shall here take provisionally the potential form used by Bethe,¹⁵⁾ in order to know the qualitative effect of the diffuse boundary upon the alpha-particle scattering. Hence, the radial Schrödinger wave equation to solve is

$$\frac{d^2 u}{dr^2} + \frac{2M_\alpha}{\hbar^2} (E - V_{\text{ext}} - V_n + iW) u = 0, \quad (3.1)$$

where V_n and W are the real and imaginary part of the nuclear potential respectively, and V_{ext} is the Coulomb plus the centrifugal potential. We shall here assume that V_n is proportional to W , viz.

$$V_n = \alpha W, \quad (3.2)$$

where V_n is attractive for $\alpha < 0$ and repulsive for $\alpha > 0$, and W is the following from

$$W = (b^2/2M_\alpha b^2) \exp \{-(r-R)/b\}. \quad (3.3)$$

b is the "diffuseness of the nuclear boundary."* From (3.3) R the value of r for which W has the value of $\hbar^2/2M_\alpha b^2$. There is no definite criterion in choice of b , but $b=0.86$ (10^{-13}) cm, i.e., the range of nuclear force, will result in $\hbar^2/2M_\alpha b^2 \sim 7$ Mev. Though W becomes very large inside the nucleus ($r < R$), this does not perceptibly affect the results because the behavior of the wave function in the boundary region depends only on the value of W in that region.** Since the diffuseness of the nuclear boundary is small, we may consider the slow varying part V_{ext} as constant over the region just outside the nucleus in which V_n and W are still appreciable.*** Therefore, we shall introduce

$$k^2 = (2M_\alpha/\hbar^2) (E - V_{ext}(R')) \quad (3.4a)$$

$$= (2M_\alpha/\hbar^2) (E - 2Ze^2/R') - l(l+1)/R'^2, \quad (3.4b)$$

i.e., the wave number of the incident particle near the nucleus, where $l\hbar$ is the orbital angular momentum of the incident particle, Ze the nuclear charge and R' the radius of the nucleus which may be chosen independent of R introduced in (3.3). Now, introducing the independent variable

$$x = e^{-(r-R)/2b}, \quad (3.5)$$

equation (3.1) becomes

$$\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + 4 \left(\frac{k^2 b^2}{x^2} + (i - \alpha) \right) u = 0. \quad (3.6)$$

This has the solution which tends to zero for large x , i.e., inside the nucleus,

$$u = c H_{2i/\hbar b}^{(1)} (2(i - \alpha)^{1/2} x), \quad (3.7)$$

where c is a constant, $H^{(1)}$ the Hankel function. The asymptotic form of (3.7) far outside the nucleus ($r \gg R$), choosing c appropriately, becomes

$$u = \exp \{-ik(r-R)\} - \exp \{ik(r-R) - (\pi + 2 \arctan \alpha) kb + 2i\eta - ikb \log(1 + \alpha^2)\}, \quad (3.8)$$

where η is the complex phase of $(2ikb)!$, i.e.,

$$e^{i\eta} = (2ikb)! / |(2ikb)!|.$$

Now the amplitude of the elastically scattered wave can be obtained from the behavior

* The model used by Bethe treats complex nuclear potential with diffuseness, which is however very large inside the nucleus. So, it might have to be called a *strong absorption optical model* rather than an optical model which is usually used for the neutron and proton scatterings.⁸⁾¹⁴⁾ Since, however, the absorption in this model is not so strong as in the strong absorption model used in § 2, we have used in this section the name "*optical potential model with diffuseness*" for this model.

** For this argument see reference¹⁵⁾, p. 1130, §4.

*** Concerning the validity of this approximation see reference 15), pp. 1131 and § 8.

of the wave function for large r . Since the incident particle is an alpha-particle in our case, it must be noticed that the Coulomb potential produces the scattering by itself. Let σ_l be the Coulomb phase shift, then the actual scattered amplitude $F(\theta)^{(16)}$ becomes

$$F(\theta) = (1/2ik) \sum_l [\exp 2i\sigma_l \cdot \exp \{ -(\pi + 2 \tan^{-1} \alpha) kb + 2i\eta - ikb \log(1 + \alpha^2) - 2ikR \} - 1] \\ \times (2l+1) P_l(\cos \theta) \\ = f_c(\theta) + \frac{1}{2ik} \sum_l e^{2i\sigma_l} (e^{2i\delta_l} - 1) (2l+1) P_l(\cos \theta), \quad (3.9)$$

where $f_c(\theta)$ is the Coulomb amplitude and

$$2i\delta_l = -(\pi + 2 \tan^{-1} \alpha) kb + 2i\eta - ikb \log(1 + \alpha^2) - 2ikR. \quad (3.10)$$

For the summation over l in (3.9), we shall use the sharp cut-off approximation as in the section 2, that is, l runs up to the critical angular momentum l' defined in (1.1). The theoretical results did not depend sensitively on $\alpha = V_n/W$ in the range of its plausible values, so that for simplicity most of the calculations were performed approximately with $V_n = 0$. The cross sections calculated for non-zero α are represented for the case of Au (90° , 32 Mev) (See Fig. 6). The mentioned approximation is probably not so bad, because the one-particle potential V_n for the alpha-particle inside the nucleus must not be identified with the deep one-particle potential for the nucleons constituting the alpha-particle and might be close to zero due to the large binding energy of the alpha-particle (if we adopt the alpha-particle model for the nucleus this circumstance shall be more definite).

Now we have two parameters R and R' as the nuclear radii in our case; R is the one determining the extent of nuclear potential, while, R' the one which appears in the expression of V_{el} (see Eq. (3.4a, b)) and corresponds to a point where the potential barrier sharply drops provided $k(R')$ is chosen to be at the top of the potential barrier. Since it may be thought that the sudden drop at R' of the potential acting on an alpha-particle is due to the effect of the attractive nuclear potential, there is naturally an intimate relation between R and R' . In case of the diffuse boundary in the nuclear potential we have no definite information about the relation mentioned above, so that the following two cases are here chosen for trial: (a) $R = R'$ and (b) $R' = R + b$. R' in the case (b) is taken in a similar way to the case of the S-wave elastic scattering of slow neutrons in which the scattering behaves as the potential scattering by a hard sphere of an effective radius $R' \sim R + 1.15b$.⁽¹⁵⁾

The comparison of the calculated results for the optical model with the experimental data for Au and Ag are represented in Figs. 3 and 6—9.

For Au, at $\theta = 90^\circ$, the case (a) with $R = R' = 10.85 (10^{-13})$ cm and $b = 0.8 (10^{-13})$ cm gives the nearly similar curve as the one of the boundary-condition model with $R = 10.85 (10^{-13})$ cm (Fig. 6). The case (b) with $R' = R + b = 10.85 (10^{-13})$ cm and $b = 0.8 \times (10^{-13})$ cm gives a larger cross section than the case (a) over the whole energy region, because the absorption potential enters further inwards than in the case (a) so that the absorption is reduced at the nuclear surface. Since it was found later that the average

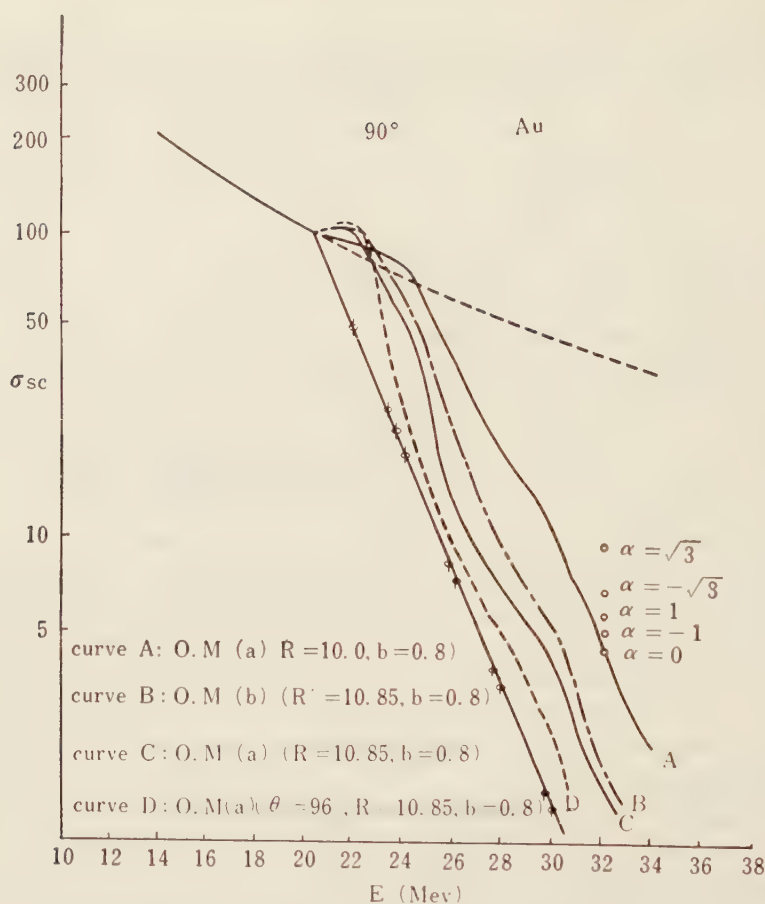


Fig. 6 Differential scattering cross section for Au at $\theta = 90^\circ$ as a function of the alpha-particle energy. Theoretical curves are computed from optical model (a) and optical model (b). Curve D is computed at $\theta = 96^\circ$. Circles represent the α -dependences of cross sections at 32 Mev.

experimental angle θ corresponding to the 90° port was 97° in the center-of-mass system, we have also calculated the case (a) for $\theta = 96^\circ$. The result shows that the cross section is fairly reduced over the whole energy region, compared with the one for $\theta = 90^\circ$, so that it becomes more close to the experimental curve. For Au, at $\theta = 60^\circ$, we adopt the same values for various parameters as at $\theta = 90^\circ$ (Fig. 7). In this case, also, the case (b) gives a little larger cross section than the case (a) over the whole energy region. In Fig. 3 the theoretical curves are compared with the experimental data for Ag at 60° port. The case (a) is with $R' = R = 9.67(10^{-13})$ cm and $b = 1.33(10^{-13})$ cm, and the case (b) with $R' = R + b = 9.67(10^{-13})$ cm and $b = 1.33(10^{-13})$ cm. In a similar fashion as in the case of Au, the case (b) gives a slightly larger cross section than the case (a) over the whole energy region. In the same way as in the boundary-condition model, we shall be able to get the reasonable fit with the experiment, using a slightly smaller value for

R' than $9.67 (10^{-13})$ cm and probably including higher l -th partial waves than l' -th.

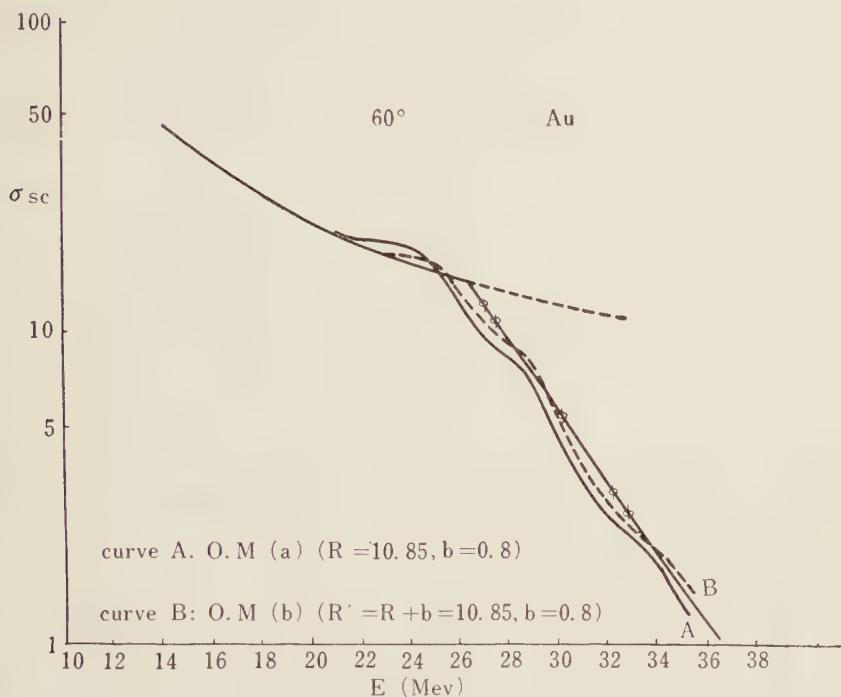


Fig. 7 Differential scattering cross section for Au at $\theta=60^\circ$ as a function of the alpha-particle energy. Curves are computed from the optical model (a) and optical model (b).

Thus regarding the general trend of energy dependence of the cross sections, the agreement between theories and experiments is fairly well except when the energy is close to the critical one E_0 . Our approximations, i.e., the sharp angular momentum cut-off and $V_{\text{ext}}=V_{\text{ext}}(R')$ over the range where the nuclear potential is still appreciable (Eqs. (3, 4a), (3, 4b)), would not be so much justified in the region near the E_0 where the number of partial waves required is very few and $E-V_{\text{ext}}$ changes by its own amount and more in very short distance. In Fig. 8, the theoretical curves are compared with the experimental one for Au, for the case (a) with $R'=R=10.85(10^{-13})$ cm, $b=0.85(10^{-13})$ cm, and $l'=4$, and for the case (b) with $R'=R+b=10.85(10^{-13})$ cm, $b=0.85(10^{-13})$ cm, and $l'=4$. Both cases give the nearly similar curves, reproducing the qualitative features of the experiments. The theory, however, gives a diffraction pattern which is not seen in the experimental data in the angles larger than 90° . This defect of the theory may readily be attributed to the sharp angular momentum cut-off approximation, in spite of the use of a diffuse absorption potential. The diffuseness of the absorption potential will naturally require the inclusion of the more partial waves than the ones limited by l' . But we can see that owing to the diffuseness of the potential this theory is superior to the boundary-condition model, for Au, in showing that in the angles smaller than 90° the diffraction pattern given by the former theory is damped out compared with

the one by the latter theory and more closely approaches to the experimental curve (refer Fig. 4). In Fig. 9, the theoretical curves are compared with the experimental one for Ag. The case (a) is with $R'=R=9.67(10^{-13})\text{cm}$, $b=1.33(10^{-13})\text{cm}$ and $l'=11$, while the case (b) with $R'=R+b=9.67(10^{-13})\text{cm}$, $b=1.33(10^{-13})\text{cm}$ and $l'=11$. Since the above remarks concerning Au can be applied equally well to Ag, the detailed discussion about the theoretical curves is not presented here.

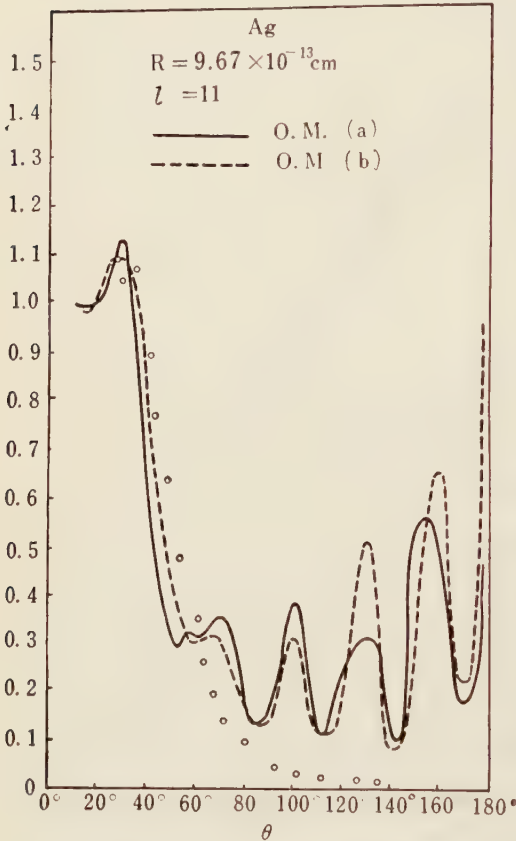


Fig. 8 Comparison of optical model with σ/σ_0 for Au.

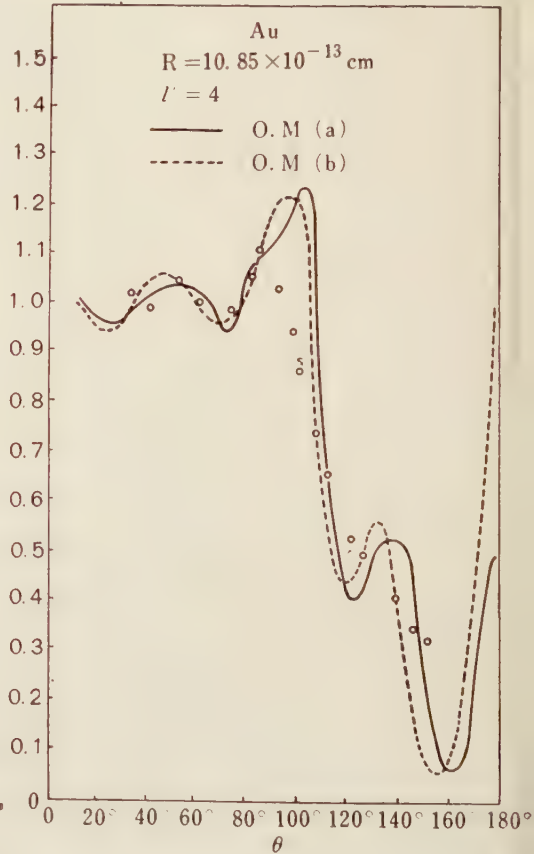


Fig. 9 Comparison of optical model with σ/σ_0 for Ag.

§ 4. Concluding remarks

The energy dependence of the cross sections by both the boundary-condition model and the diffuse optical model is fairly well fitted to the experimental data by choosing appropriate values for the parameters. With the same value for the nuclear radius (though in the diffuse optical model the definition of the nuclear radius might be rather obscure), both theories give the nearly similar curves. This may be understood by the fact that in the diffuse optical model we made a real part of the nuclear potential zero

mainly for the reason of convenience for the computations and the imaginary part of the nuclear potential used here became considerably deep inside the nucleus owing to the exponential shape. The sharp angular momentum cut-off approximation used in both theories will not be so much justified in the region near the critical energy E_{0c} , where the calculated results of the energy dependence of cross sections by both theories give too large values compared with the experimental ones. The reason for this may be readily understood from the fact that, in the region in question, owing to the relatively small number of the partial waves limited by the critical angular momentum l' the contribution of the larger l -th waves than the l' -th to the cross section through the penetration through the V_{ext} will be considerably appreciable (this feature was verified in the boundary-condition model, Fig. 1). Furthermore, since in the diffuse optical model the nuclear potential has a tail perceptibly diffused out beyond the nuclear radius R' (3, 4a), the l -th waves with l larger than l' might possibly be absorbed by the tail of the diffuse absorption potential even if these waves could not appreciably penetrate through the V_{ext} .

Although, concerning the angular distribution, both theories have reproduced the qualitative features of the experimental curve, they show a diffraction pattern with large amplitude not observed in the experiment, which is in particular striking for Ag. Even the use of the diffuse potential could not remove this diffraction pattern except the part below about 80° for Au. This defect of the theory might readily be attributed to the sharp angular momentum cut-off as mentioned above. Finally, the radius of the alpha-particle R_α depends on the model employed to fit the data, but it seems better for R_α to take a larger value than the value $1.2 \times (10^{-13})\text{cm}$ derived from the alpha decay.¹³⁾

Recently, after this paper was completed, the experiments of the angular distributions for 40-Mev alpha-particles by Wegner, Eisberg, and Igo¹⁷⁾, and for 48.2 Mev by Ellis and Schecter¹⁸⁾ has been reported. Since at these energies the partial wave analysis given in this paper seems not so convenient and from the survey of the experimental data it seems rather important to take into account the detailed form of the distribution of protons inside the nucleus, we are now engaged in the calculation by the model other than the ones used in this paper.

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The Theory of Radiative K -Capture, I

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The bremsstrahlung spectrum accompanying K -electron capture are calculated for allowed transitions as well as first and second forbidden transitions, including the effect of nuclear Coulomb field. The calculations are done with the second order perturbation method. Capture interactions are adopted of the scalar, vector, tensor, axial vector and pseudoscalar forms. The plane wave approximation used in the previous calculations is improved by correct Coulomb wave function.

§ 1. Introduction

The study of β -disintegration process has the dual purpose of determining the intrinsic properties of interaction between nucleons and leptons, and providing information on the nuclear structure.

The recent progress of experimental clarification has greatly reduced the initial ambiguities in the theory of β interaction and the classification of transitions in degrees of forbiddenness has provided fruitful evidence on the spins and parities of nuclear states. Moreover, the theoretical estimation of nuclear matrix elements has given more detailed information on the nuclear coupling scheme.

On the other hand, orbital electron capture has not been so useful in narrowing the alternatives of nucleon-lepton coupling, and in providing above information of nuclear structure, because only an emerging particle is a neutrino unobserved and such characteristic continuous spectrum as in usual β -decay can not be available. But orbital electron capture is always by emission of γ -radiation of weak intensity, in addition to characteristic X -ray. Recently, owing to the development of experimental technique the experiments¹⁾ on this γ -ray have become to be actively performed.

The origin of this radiation is due to the sudden change of charge from atom to nucleus. This emission is analogous to the emission of the internal bremsstrahlung due to usual β -decay. The latter process has been studied by many authors both theoretically²⁾ and experimentally,³⁾ and for allowed as well as for certain forbidden transitions. The spectra and angular correlations of this γ -ray for all cases agree quite well with predictions of the semiclassical theory of Knipp and Uhlenbeck.²⁾

Concerning radiative K -capture which are treated in the present article, the theoretical calculation on its bremsstrahlung spectrum has been made, first by Morrison and Schiff⁴⁾ with very simplified assumptions, and is in agreement with the measured spectra. Their theory has been given only for allowed transitions and may generally disagree with forbidden spectra.

Moreover, owing to the simplicity of their assumptions, its discrepancies are expected with the detailed aspect of experimental spectra. Recently, the continuous spectra of this γ -ray have been measured for many elements. Measured intensities of low energy γ -ray were higher than predicted by Morrison and Schiff.

Glauber and Martin have explained its experimental anomaly for allowed transitions, taking account of the effect of nuclear Coulomb field and including the contribution from L -electron in addition to K -electron. Their skilful method consists of calculating the intermediate electron propagation function in the Coulomb field, and treating nonrelativistically, but almost completely the effect of nuclear Coulomb field. Glauber and Martin, as well as Morrison and Schiff, however restricted their discussions only to allowed transitions, so their results are not always useful for analysis of all electron-capturing nuclei.

The purpose of the present article is to obtain the theoretical energy spectra of internal bremsstrahlung accompanying by K -electron capture for allowed transitions as well as first and second forbidden transitions, and to show that one can obtain as much information about electron-capturing nuclei and nucleon-lepton interaction by studying the γ -rays as one can obtain about beta-emitting nuclei and its interaction by studying their β -radiation. We treat here five forms of interaction between nucleons and leptons separately but not their mixture which we shall discuss in the future. Moreover, we calculate relativistically the propagation function of electron in the intermediate state, but introduce certain approximation to Coulomb potential, which is verified to be appropriate in the space region of our problem.

§ 2. Condition to available energy

If the mass difference W of two nuclei having same mass number, and charge numbers Z and $Z-1$ respectively, satisfies the condition

$$W \geq -mc^2, \quad (1)$$

the nucleus Z is unstable, and transforms the nucleus $Z-1$ by capturing an orbital electron and emitting a neutrino, m is an electron rest mass and c is light velocity. Strictly speaking, to the right side of formula (1), the binding energy of orbital electron must be added, but it is neglected owing to its smallness as compared with mc^2 . The binding energy of K -electron is $(1-\gamma)mc^2 = (1-\sqrt{1-\alpha^2 Z^2})mc^2$, where α is the fine structure constant.

In the case that the bremsstrahlung γ -ray is accompanied by K -electron capture, available energy W shared between γ -ray and a neutrino may be expressed by

$$W = W + mc^2. \quad (2)$$

Since the condition of β^+ activity is

$$W \geq mc^2, \quad (3)$$

the condition that only the orbital electron capture can proceed is, from (1) and (3), given by

$$mc^2 > \Delta W \geq -mc^2. \quad (4)$$

Using eq. (2), this condition is rewritten as

$$2mc^2 > W \geq 0. \quad (5)$$

When this inequality (5) is fulfilled, only orbital electron capture is energetically possible, a neutrino and a γ -quantum are emitted from final nucleus, sharing available energy W , and the γ -ray spectrum should be observed without background of any external radiation except for very low energy X-ray.

In other words, the best condition for investigating γ -ray spectrum we are interested in, is as follows: (a) the case in which no positron activity exists and only electron capture happens, that is, available energy W satisfies the inequality (5), and (b) the case in which nuclear charge Z is not so large and X-rays are restricted to low energy region.

§ 3. Method of calculation

The process which we wish to investigate is as follows: either of two electrons circulating on the K -orbit of the atom with atomic number Z , emits a photon, is excited virtually and is subsequently captured by the nucleus, which transforms into a nucleus with charge number $Z-1$ and emits a neutrino.

In this process, the expression for the total probability $w(k)dk$ for the emission of γ -quantum with energy between k and $k+dk$ is*

$$w(k)dk = \frac{1}{(2\pi)^5} \frac{1}{2} \int d\omega_\nu \int d\omega_k \left| \sum_j \frac{(H_c)_{fj}(H_R)_{ji}}{E_j - E_i + k} \right|^2 (W-k)^2 k^2 dk, \quad (6)$$

where E_i , E_j , k are the total energies of K -electron, intermediate electron and emitted γ -quantum respectively. The integrals over ω_ν and ω_k represent integrations over the directions of the neutrino and photon momenta and summations over the directions of the neutrino spin and photon polarization. W is the available energy characteristic to this process. Moreover, $(H_c)_{fj}$ and $(H_R)_{ji}$ represent matrix elements for capture and radiative interactions respectively and have the following forms

$$(H_c)_{fj} = G_\lambda \int \Psi_{\text{final}} O_\lambda \Phi_{\text{initial}} \varphi_f^*(\mathbf{r}_n) O_\lambda \psi_j(\mathbf{r}_n) d\tau_n, \quad (7)$$

and

$$(H_R)_{ji} = -e \sqrt{\frac{2\pi}{k}} \int \psi_j^*(\mathbf{r}) \alpha_\mu e_\mu e^{-ik \cdot \mathbf{r}} \psi_i(\mathbf{r}) d\tau, \quad (8)$$

where G_λ 's are strengths of capture interaction, O_λ 's, interaction operators and λ is a suffix referring scalar, vector, tensor, axial vector and pseudoscalar interactions. Φ_{initial} , Ψ_{final} , ψ_i , ψ_j and φ_f are wave functions of initial nucleus, final nucleus, K -electron intermediate electron and final neutrino respectively. α_μ and e_μ are components of Dirac operator in

* In the following, the usual relativistic units will be used throughout.

the direction of polarization and unit vectors of its direction. \mathbf{r}_n and \mathbf{r} are position vectors of capture and radiative interactions respectively. \mathbf{k} is a momentum of photon. The integral in eq. (7) is taken over the coordinates of all the nucleons.

The summation over j in eq. (6) can be written, using eqs. (7) and (8), as follows,

$$\begin{aligned} M &= \sum_j \frac{(H_c)_{fj}(H_R)_{ji}}{E_j - E_i + k} = -e \sqrt{\frac{2\pi}{k}} G_\lambda \langle O_\lambda \rangle \\ &\quad \times \sum_j \frac{(\varphi_f^*(\mathbf{r}_n) O_\lambda \psi_j(\mathbf{r}_n)) \left(\int \psi_j^*(\mathbf{r}) \alpha_\mu \mathbf{e}_\mu e^{-i\mathbf{k} \cdot \mathbf{r}} \psi_i(\mathbf{r}) d\tau \right)}{E_j - E_i + k} \\ &= -e \sqrt{\frac{2\pi}{k}} G_\lambda \langle O_\lambda \rangle \int [\varphi_f^*(\mathbf{r}_n) O_\lambda \mathcal{G}_{E_i - k}(\mathbf{r}_n, \mathbf{r}) \alpha_\mu \mathbf{e}_\mu e^{-i\mathbf{k} \cdot \mathbf{r}} \psi_i(\mathbf{r})] d\tau \quad (9) \end{aligned}$$

where

$$\langle O_\lambda \rangle = \int \Psi_{\text{final}}^* O_\lambda \Psi_{\text{initial}} d\tau_n$$

is nuclear matrix element and

$$\mathcal{G}_{E_i - k}(\mathbf{r}_n, \mathbf{r}) = \sum_j \psi_j(\mathbf{r}_n) \psi_j^*(\mathbf{r}) / (E_j - E_i + k) \quad (10)$$

is a propagation function of electron with energy $E_i - k$, satisfying Dirac equation in the nuclear Coulomb field and the summation is taken over all intermediate states. Gauder and Martin adopted, as the explicit form of $\mathcal{G}_{E_i - k}$, the solution of non-relativistic Schrödinger equation, while we calculate its relativistic but approximate form with a method explained in next section.

The neutrino wave function can be taken as plane wave having four components, and K -electron wave function is given as

$$\left. \begin{aligned} \phi_1 &= -i \sqrt{\frac{1}{4\pi}} \cos \theta \sqrt{1 - \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \\ \phi_2 &= -i \sqrt{\frac{1}{4\pi}} \sin \theta e^{i\varphi} \sqrt{1 - \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \\ \phi_3 &= \sqrt{\frac{1}{4\pi}} \sqrt{1 + \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \\ \phi_4 &= 0, \end{aligned} \right\} j=1/2, m=1/2, \quad (11a)$$

$$\left. \begin{aligned} \phi_1 &= i \sqrt{\frac{1}{4\pi}} \sin \theta e^{-i\varphi} \sqrt{1 - \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \\ \phi_2 &= -i \sqrt{\frac{1}{4\pi}} \cos \theta \sqrt{1 - \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \\ \phi_3 &= 0, \\ \phi_4 &= -\sqrt{\frac{1}{4\pi}} \sqrt{1 + \gamma} D r^{\gamma-1} e^{-\alpha Z r}, \end{aligned} \right\} j=1/2, m=-1/2, \quad (11b)$$

and

$$D = (2\alpha Z)^{\gamma+1/2} [2\Gamma(2\gamma+1)]^{-1}. \quad (11c)$$

Substituting eqs. (11a), (11b) and (11c) into eq. (9), and using the explicit form of $\mathcal{G}_{E_i-k}(\mathbf{r}_n, \mathbf{r})$ in next section, we can calculate eq. (9) straightforward. In this way, we shall obtain final results from eq. (6).

§ 4. Calculation of electronic propagation function

As is mentioned in § 3, Glauber and Martin obtained non-relativistically eq. (10) as solution for Schrödinger equation.

Only for allowed transitions, its solution may be valid, but for higher forbidden transitions, a more correct relativistic solution may be favourable. At present, the explicit form of electron propagation function satisfying Dirac equation in the Coulomb field is not available, so here we calculate this with a kind of perturbation method, considering Coulomb field as a perturbation.

Then we introduce an approximation, $kr \gg 1$. This is a reasonable approximation for our consideration.

Now, $\psi_i(\mathbf{r})$ being solutions of the following Dirac equation for the nuclear Coulomb field,

$$H\psi_j(\mathbf{r}) = E_j\psi_j(\mathbf{r}) \quad (12)$$

where

$$H = (1/i)(\boldsymbol{\alpha} \cdot \mathbf{V}) + \beta - eV, \quad (12a)$$

$$V = Ze/r, \quad (12b)$$

then the eigensolutions $\psi_{j\rho}$ satisfy the completeness relation

$$\sum_j \psi_{j\rho}(\mathbf{r}_n) \psi_{j\sigma}^*(\mathbf{r}) = \delta_{\rho\sigma} \delta(\mathbf{r} - \mathbf{r}_n) \quad (13)$$

where a summation over j expresses that over all states, ρ and σ are spinor indices, and \mathbf{r}_n and \mathbf{r} are coordinate vectors mentioned in § 3.

On both sides of eq. (13), operate $1/\{H - (E_i - k)\}$ on the left, and we obtain

$$\text{the left hand side} = \frac{1}{H - (E_i - k)} \sum_j \psi_{j\rho}(\mathbf{r}_n) \psi_{j\sigma}^*(\mathbf{r}) = \sum_j \frac{\psi_{j\rho}(\mathbf{r}_n) \psi_{j\sigma}^*(\mathbf{r})}{E_j - E_i + k} = \mathcal{G}_{E_i-k}(\mathbf{r}_n, \mathbf{r}),$$

$$\text{the right hand side} = \frac{1}{H - (E_i - k)} \delta_{\rho\sigma} \delta(\mathbf{r} - \mathbf{r}_n).$$

Rewritten $\mathbf{r} - \mathbf{r}_n$ by \mathbf{r} ,

$$\mathcal{G}_{E_i-k}(\mathbf{r}) = \frac{1}{H - (E_i - k)} \delta_{\rho\sigma} \delta(\mathbf{r}).$$

Substituting $\delta(\mathbf{r}) = 1/(2\pi)^3 \cdot \int e^{-i\mathbf{p} \cdot \mathbf{r}} d\mathbf{p}$ in the right hand side,

$$\mathcal{G}_{E_i-k}(\mathbf{r}) = \frac{1}{(1/i)(\boldsymbol{\alpha} \cdot \nabla) + \beta - eV - E_i + k} \frac{1}{(2\pi)^3} \int e^{-i\mathbf{p} \cdot \mathbf{r}} d\mathbf{p} \quad (14)$$

are obtained.

Then, we wish to perform a differentiation in the denominator of eq. (14) but it is generally not easy, for ∇ and V are not commutable with each other.

To the volume integral of eq. (9), the values of integrand at greater r contribute predominantly compared with those at smaller r and in fact its value around $r \sim 1/\alpha Z$ is important, where $1/\alpha Z(b/mc)$ is dimension of K -electron wave function. Thus, in the following, we make use of the fact that the space region where $r \sim 1(b/mc)$ is important to our problem. If we accept such a fact, it turns out that we may be possible to estimate as if ∇ and V are commutable.

To prove it explicitly, let us introduce the abbreviations

$$P \equiv (1/i)(\boldsymbol{\alpha} \cdot \nabla) + \beta - E_i + k, \quad Q \equiv -eV \quad (15)$$

then the first factor of the right side in eq. (14) is written in expanded form

$$\frac{1}{P+Q} = \frac{1}{P} - \frac{1}{P} Q \frac{1}{P} - \frac{1}{P} Q \frac{1}{P} Q \frac{1}{P} - \dots \quad (16)$$

Now

$$Q \frac{1}{P} = \frac{1}{P} ([P, Q] \frac{1}{P} + Q).$$

Therefore, if

$$|[P, Q](1/P)| \ll |Q|, \quad (17)$$

we have an approximate relation

$$Q(1/P) \approx (1/P)Q$$

and so in eq. (16), all Q 's in each term may be removed to left ends.

Now

$$\begin{aligned} [P, Q] &= [(1/i)(\boldsymbol{\alpha} \cdot \nabla) + \beta - (E_i - k), -eV] \\ &= (1/i)[(\boldsymbol{\alpha} \cdot \nabla) - eV] = -(e/i)(\boldsymbol{\alpha} \cdot \nabla)V \end{aligned}$$

therefore

$$[P, Q] \frac{1}{P} = \frac{-(e/i)(\boldsymbol{\alpha} \cdot \nabla)V}{\boldsymbol{\alpha} \cdot \mathbf{p} + \beta - (E_i - k)} \sim -\frac{e}{ik}(\boldsymbol{\alpha} \cdot \nabla)V \sim -e \text{ grad } V/k.$$

Hence, the inequality (17) may be written in the form

$$|\text{grad } V|/k \ll |V|$$

or

$$kr \gg 1.$$

Since $r_{\text{eff}} \sim 1/\alpha Z$ where r_{eff} is the effective value of r important for our problem, results

of our following discussion will be quite reliable for the photon energy more than few hundreds kev.

Now, on the above consideration, eq. (14) can be written as follows:

$$\mathcal{G}_{E_i-k}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int \frac{1}{(\boldsymbol{\alpha} \cdot \mathbf{p}) + \beta - eV - E + k} e^{-i\mathbf{p} \cdot \mathbf{r}} d\mathbf{p}. \quad (18)$$

After elementary but tedious evaluation, this integral is expressed in the form

$$\begin{aligned} \mathcal{G}_{E_i-k}(\mathbf{r}) = & \frac{1}{4\pi} \left[(\beta + eV + E_i - k) - i \frac{(\boldsymbol{\alpha} \cdot \mathbf{r})}{r^2} \right. \\ & \left. \times \left(\sqrt{1 - (eV + E_i - k)^2} + \frac{1}{r} \right) \right] e^{-r\sqrt{1 - (eV + E_i - k)^2}} \end{aligned} \quad (19)$$

where k is restricted to the following region

$$1 + eV + E_i \geq k \geq eV + E_i - 1. \quad (19')$$

The inequality (19') is a condition by which the result of integration (19) is meaningful, and is compatible with (5).

§ 5. The allowed spectra

In the last section, the propagation function $\mathcal{G}(\mathbf{r})$ of electron in the intermediate state could be evaluated, and therefore substituting it in eq. (9), matrix element M can be estimated.

Neutrino wave function $\varphi_f^*(\mathbf{r}_n)$ is a solution of free Dirac equation, which is expressed, denoting momentum of emitted neutrino by \mathbf{q} , as

$$\varphi_f(\mathbf{r}_n) = u_f(\mathbf{q}) e^{i\mathbf{q} \cdot \mathbf{r}_n}$$

where $u_f(\mathbf{q})$ is four-component spinor part which is a function of momentum \mathbf{q} only.

Substituting it, eq. (9) is given by

$$M = -e \sqrt{\frac{2\pi}{k}} G_\lambda \langle O_\lambda \rangle \int [u_f^* e^{-i\mathbf{q} \cdot \mathbf{r}_n} O_\lambda \mathcal{G}(\mathbf{r} - \mathbf{r}_n) \alpha_\mu \mathbf{e}_\mu e^{-i\mathbf{k} \cdot \mathbf{r}} \psi_i(\mathbf{r})] d\tau$$

or

$$M = -e \sqrt{\frac{2\pi}{k}} G_\lambda \langle O_\lambda \rangle e^{-i(\mathbf{k} + \mathbf{q}) \cdot \mathbf{r}_n} \int [u_f^* O_\lambda \mathcal{G}(\mathbf{r} - \mathbf{r}_n) \alpha_\mu \mathbf{e}_\mu e^{-i\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}_n)} \psi_i(\mathbf{r} - \mathbf{r}_n + \mathbf{r})] d\tau$$

Rewriting $\mathbf{r} - \mathbf{r}_n$ by \mathbf{r} , and neglecting \mathbf{r}_n compared with \mathbf{r} in $\psi_i(\mathbf{r} - \mathbf{r}_n + \mathbf{r}) = \psi_i(\mathbf{r} + \mathbf{r}_n)$, M can be expressed as

$$M = -e \sqrt{\frac{2\pi}{k}} G_\lambda \langle O_\lambda \rangle e^{-i(\mathbf{k} + \mathbf{q}) \cdot \mathbf{r}_n} \int (u_f^* O_\lambda \mathcal{G}(\mathbf{r}) \alpha_\mu \mathbf{e}_\mu \psi_i(\mathbf{r})) e^{-i\mathbf{k} \cdot \mathbf{r}} d\tau. \quad (20)$$

The energy release usual in radiative K-capture limits $(\mathbf{k} + \mathbf{q})$ to values of only a few mc units. As \mathbf{r}_n is of nuclear dimensions, and therefore of at most the order of

$(1/40)(\hbar/mc)$, the successive terms of eq. (20) using the expanded form of $e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_n}$

$$e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_n} = 1 - i(\mathbf{k} + \mathbf{q} \cdot \mathbf{r}_n) - \frac{1}{2}(\mathbf{k} + \mathbf{q} \cdot \mathbf{r}_n)^2 - \dots$$

are smaller by factors of at least $1/10$ (only $1/100$ on squaring). For allowed transitions, all terms except the first term can be neglected

As an example, let us show the calculation of the case when λ is scalar, that is, $O_\lambda = \beta$. The momentum direction of emitted photon being z axis and O_λ being β , for allowed transitions eq. (20) is expressed as

$$M = -e \sqrt{\frac{2\pi}{k}} G_s \langle \beta \rangle \int (u_f^* \beta \mathcal{G}(\mathbf{r}) \alpha_2 \psi_i(\mathbf{r})) e^{-i k r \cos \theta} r^2 dr \sin \theta d\theta d\varphi \quad (21)$$

where α_1 and α_2 are x and y components of Dirac operators.

Substituting eq. (19) in eq. (21), the term in parentheses is written as

$$\begin{aligned} (u_f^* \beta \mathcal{G}(\mathbf{r}) \alpha_2 \psi_i) &= \frac{1}{4\pi} \left[(u_f^* \beta (\beta + eV + E_i - k) \alpha_2 \psi_i) \frac{1}{r} e^{-r\sqrt{1 - (eV + E_i - k)^2}} \right. \\ &\quad - i(u_f^* \beta (\mathbf{a} \cdot \mathbf{r}) \alpha_2 \psi_i) \frac{1}{r} \left\{ \sqrt{1 - (eV + E_i - k)^2} \frac{1}{r} e^{-r\sqrt{1 - (eV + E_i - k)^2}} \right. \\ &\quad \left. \left. + \frac{1}{r^2} e^{-r\sqrt{1 - (eV + E_i - k)^2}} \right\} \right]. \end{aligned} \quad (22)$$

Using K -electron wave functions (11a) and (11b), matrix elements in parentheses in the right side of eq. (22) are expressed as

$$\begin{aligned} &(u_f^* \beta (\beta + eV + E_i - k) \alpha_1 \psi_i^\dagger) \\ &= (1 + eV + E_i - k) u_2^* \sqrt{\frac{1}{4\pi}} \sqrt{1 + \gamma} Dr^{\gamma-1} e^{-\alpha Z r} \\ &\quad - i(1 - eV - E_i + k) u_4^* \sqrt{\frac{1}{4\pi}} \cos \theta \sqrt{1 - \gamma} Dr^{\gamma-1} e^{-\alpha Z r}, \end{aligned} \quad (23a)$$

$$\begin{aligned} &(u_f^* \beta (\mathbf{a} \cdot \mathbf{r}) \alpha_1 \psi_i^\dagger) \\ &= i u_2^* \sqrt{\frac{1}{4\pi}} \sqrt{1 - \gamma} Dr^{\gamma-1} e^{-\alpha Z r} r \cos^2 \theta + u_4^* \sqrt{\frac{1}{4\pi}} \sqrt{1 + \gamma} Dr^{\gamma-1} e^{-\alpha Z r} r \cos \theta \end{aligned} \quad (23b)$$

for K electron spin up and direction 1 of γ polarization,

$$\begin{aligned} &(u_f^* \beta (\beta + eV + E_i - k) \alpha_1 \psi_i^\dagger) \\ &= (1 + eV + E_i - k) u_1^* \left(-\sqrt{\frac{1}{4\pi}} \right) \sqrt{1 + \gamma} Dr^{\gamma-1} e^{-\alpha Z r} \\ &\quad - i(1 - eV - E_i + k) u_3^* \sqrt{\frac{1}{4\pi}} \cos \theta \sqrt{1 - \gamma} Dr^{\gamma-1} e^{-\alpha Z r}, \end{aligned} \quad (24a)$$

$$(u_f^* \beta (\mathbf{a} \cdot \mathbf{r}) \alpha_1 \psi_i^\dagger)$$

$$= -iu_1^* \sqrt{\frac{1}{4\pi}} \sqrt{1-\gamma} Dr^{\gamma-1} e^{-\alpha Zr} r \cos^2 \theta + u_2^* \sqrt{\frac{1}{4\pi}} \sqrt{1+\gamma} Dr^{\gamma-1} e^{-\alpha Zr} r \cos \theta \quad (24b)$$

for K -electron spin down and direction 1 of γ polarization,

$$(u_f^* \beta (\beta + eV + E_i - k) \alpha_2 \phi_i^\dagger) = i(u_f^* \beta (\beta + eV + E_i - k) \alpha_1 \phi_i^\dagger), \quad (25a)$$

$$(u_f^* \beta (\boldsymbol{\alpha} \cdot \mathbf{r}) \alpha_2 \phi_i^\dagger) = i(u_f^* \beta (\boldsymbol{\alpha} \cdot \mathbf{r}) \alpha_1 \phi_i^\dagger) \quad (25b)$$

for K electron spin up and direction 2 of γ polarization

$$(u_f^* \beta (\beta + eV + E_i - k) \alpha_2 \phi_i^\dagger) = -i(u_f^* \beta (\beta + eV + E_i - k) \alpha_1 \phi_i^\dagger), \quad (26a)$$

$$(u_f^* \beta (\boldsymbol{\alpha} \cdot \mathbf{r}) \alpha_2 \phi_i^\dagger) = -i(u_f^* \beta (\boldsymbol{\alpha} \cdot \mathbf{r}) \alpha_1 \phi_i^\dagger) \quad (26b)$$

for K -electron spin down and direction 2 of γ polarization,

where vanishing terms in the integration of (21) are omitted. u_1, \dots are four components of $u_f(\mathbf{q})$, explicit forms of which are expressed for positive energy as follows:

$$u_1 = -\frac{1}{\sqrt{2}} \frac{q_z}{q}, \quad u_2 = -\frac{1}{\sqrt{2}} \frac{q_x + iq_y}{q}, \quad u_3 = \frac{1}{\sqrt{2}}, \quad u_4 = 0 \quad \text{for spin up,} \quad (27a)$$

$$u_1 = -\frac{1}{\sqrt{2}} \frac{q_x - iq_y}{q}, \quad u_2 = \frac{1}{\sqrt{2}} \frac{q_z}{q}, \quad u_3 = 0, \quad u_4 = \frac{1}{\sqrt{2}} \quad \text{for spin down.} \quad (27b)$$

Now, we substitute eqs. (23), (24), (25), (26) and (27) in eq. (21), square these matrix elements after integrations individually, perform summations over directions of γ polarization and neutrino spin directions, and average over K -electron spin directions.

Finally, after integration over directions of momenta of neutrino and photon, we obtain energy spectrum of γ -ray:

$$w(k) dk = \frac{G_s^2 e^2}{2\pi^3} \frac{(2\alpha Z)^{2\gamma+1}}{4\Gamma(2\gamma+1)} \frac{\exp\left(2\alpha Z \frac{\gamma-k}{\sqrt{1-(\gamma-k)^2}}\right)}{[\alpha Z + \sqrt{1-(\gamma-k)^2}]^{2\gamma}} (\mathcal{A}^2 + \mathcal{B}^2) |\langle \beta \rangle|^2 (W-k)^2 k dk \quad (28)$$

where

$$\begin{aligned} \mathcal{A} &= \sqrt{1+\gamma} \Phi_1 - \sqrt{1-\gamma} \Phi_3, \quad \mathcal{B} = \sqrt{1-\gamma} \Phi_2 - \sqrt{1+\gamma} \Phi_4, \\ \Phi_1 &= \alpha Z \Gamma(\gamma) F\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{3}{2}; -\frac{k^2}{a^2}\right) \\ &\quad + (1+\gamma-k) \frac{\Gamma(\gamma+1)}{a} F\left(\frac{\gamma+1}{2}, \frac{\gamma+2}{2}; \frac{3}{2}; -\frac{k^2}{a^2}\right), \\ \Phi_2 &= \frac{k}{3} \left\{ \frac{\alpha Z}{a} \Gamma(\gamma+1) F\left(\frac{\gamma+1}{2}, \frac{\gamma+2}{2}; \frac{5}{2}; -\frac{k^2}{a^2}\right) \right. \\ &\quad \left. + (\gamma-k-1) \frac{\Gamma(\gamma+2)}{a^2} F\left(\frac{\gamma+2}{2}, \frac{\gamma+3}{2}; \frac{7}{2}; -\frac{k^2}{a^2}\right) \right\}, \end{aligned}$$

$$\begin{aligned}
\phi_3 = & \frac{k^2}{15} \left\{ \frac{\sqrt{1-(\gamma-k)^2}}{a^3} \Gamma(\gamma+3) F\left(\frac{\gamma+3}{2}, \frac{\gamma+4}{2}; \frac{7}{2}; -\frac{k^2}{a^2}\right) \right. \\
& + \left(1 - \alpha Z \frac{\gamma-k}{\sqrt{1-(\gamma-k)^2}}\right) \frac{\Gamma(\gamma+2)}{a^2} F\left(\frac{\gamma+2}{2}, \frac{\gamma+3}{2}; \frac{7}{2}; -\frac{k^2}{a^2}\right) \Big\} \\
& - \frac{1}{3} \left\{ \frac{\sqrt{1-(\gamma-k)^2}}{a} \Gamma(\gamma+1) F\left(\frac{\gamma+1}{2}, \frac{\gamma+2}{2}; \frac{5}{2}; -\frac{k^2}{a^2}\right) \right. \\
& + \left(1 - \alpha Z \frac{\gamma-k}{\sqrt{1-(\gamma-k)^2}}\right) \Gamma(\gamma) F\left(\frac{\gamma}{2}, \frac{\gamma+1}{2}; \frac{5}{2}; -\frac{k^2}{a^2}\right) \Big\}, \\
\phi_4 = & \frac{k}{3} \left\{ \frac{\sqrt{1-(\gamma-k)^2}}{a^2} \Gamma(\gamma+2) F\left(\frac{\gamma+2}{2}, \frac{\gamma+3}{2}; \frac{5}{2}; -\frac{k^2}{a^2}\right) \right. \\
& + \left(1 - \alpha Z \frac{\gamma-k}{\sqrt{1-(\gamma-k)^2}}\right) \frac{\Gamma(\gamma+1)}{a} F\left(\frac{\gamma+1}{2}, \frac{\gamma+2}{2}; \frac{5}{2}; -\frac{k^2}{a^2}\right) \Big\}
\end{aligned}$$

and

$$a = \alpha Z + \sqrt{1 - (\gamma - k)^2}, \quad \gamma = \sqrt{1 - \alpha^2 Z^2}. \quad (29)$$

The functions F 's appearing in the above expressions are solutions for the following type of radial integration in eq. (21)

$$\begin{aligned}
& \int_0^\infty e^{-(\alpha Z + \sqrt{1-(\gamma-k)^2})r} J_\nu(kr) r^{\mu-1} dr \\
& = \frac{k^\nu \Gamma'(\mu+\nu)}{2^\nu (\alpha Z + \sqrt{1-(\gamma-k)^2})^{\mu+\nu} \Gamma(\nu+1)} F\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{k^2}{a^2}\right).
\end{aligned}$$

In the case of allowed transitions, the same form of energy spectrum is obtained for all types of capture interaction.

§ 6. Forbidden spectra

For forbidden transitions, we must take account of the second and third terms of expanded form of $e^{-i(\mathbf{k}+\mathbf{q})\cdot\mathbf{r}_n}$ in eq. (20) or the terms including velocity terms of capture interaction, i.e., α or γ . In an analogous fashion to the calculations in usual β disintegration, we obtain the results

$$w(k) dk = \frac{G_\lambda^2 \epsilon^2}{2\pi^3} \frac{(2\alpha Z)^{2\gamma+1}}{4\Gamma(2\gamma+1)} \frac{\exp\left(2\alpha Z \frac{\gamma-k}{\sqrt{1-(\gamma-k)^2}}\right)}{[\alpha Z + \sqrt{1-(\gamma-k)^2}]^{2\gamma}} C_\lambda (W-k)^2 k dk \quad (30)$$

where C_λ 's are represented, for five forms of interaction and for first and second forbidden transitions, in the following expressions:

(a) First forbidden transitions

$$\begin{aligned}
C_{S1} &= |\langle \beta \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/3) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \}, \\
C_{T1} &= |\langle \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/3) [k^2 + (W-k)^2] - \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad + |\langle \mathbf{a} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) - i \{ \langle \mathbf{a} \rangle \cdot \langle \mathbf{r}^* \rangle - \text{c.c.} \} (1/3) \\
&\quad \times \{ 2k\mathcal{A}\mathcal{B} - (W-k) (\mathcal{A}^2 + \mathcal{B}^2) \}, \\
C_{T1} &= |\langle \beta (\boldsymbol{\sigma} \cdot \mathbf{r}) \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/9) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad + |\langle \beta (\boldsymbol{\sigma} \times \mathbf{r}) \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/6) [k^2 + (W-k)^2] - \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad + \sum_{mn} |\langle \beta L_{mn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/12) [k^2 + (W-k)^2] + |\langle \beta \mathbf{a} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) \\
&\quad - i \{ \beta (\boldsymbol{\sigma} \times \mathbf{r}) \rangle \langle \beta \mathbf{a}^* \rangle - \text{c.c.} \} \{ (1/3) (W-k) (\mathcal{A}^2 + \mathcal{B}^2) - (2/3) k\mathcal{A}\mathcal{B} \}, \quad (30a) \\
C_{A1} &= |\langle (\boldsymbol{\sigma} \cdot \mathbf{r}) \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/9) [k^2 + (W-k)^2] - \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad + |\langle (\boldsymbol{\sigma} \times \mathbf{r}) \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/6) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad + \sum_{mn} |\langle L_{mn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/12) [k^2 + (W-k)^2] + |\langle \gamma_5 \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) \\
&\quad - \{ \langle \boldsymbol{\sigma} \cdot \mathbf{r} \rangle \langle \gamma_5^* \rangle + \text{c.c.} \} \cdot \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/3) (W-k) - \mathcal{A}\mathcal{B} (2/3) k \}, \\
C_{P1} &= |\langle \beta \gamma_5 \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2).
\end{aligned}$$

(b) Second forbidden transitions

$$\begin{aligned}
C_{S2} &= \sum_{mn} |\langle \beta R_{mn} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) [(1/30) k^4 + (1/9) k^2 (W-k)^2 + (1/30) (W-k)^4] \\
&\quad + \mathcal{A}\mathcal{B} (4/45) [k^2 + (W-k)^2] (W-k) k \}, \\
C_{T2} &= \sum_{mn} |\langle R_{mn} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) [(1/30) k^4 + (1/9) k^2 (W-k)^2 + (1/30) (W-k)^4] \\
&\quad - \mathcal{A}\mathcal{B} (4/45) [k^2 + (W-k)^2] \} + \sum_{mn} |\langle L_{mn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/12) [k^2 + (W-k)^2] \\
&\quad + |\langle \mathbf{a} \times \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/6) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \} \\
&\quad - \{ \sum_{mn} \langle L_{mn} \rangle \langle R_{mn}^* \rangle + \text{c.c.} \} \{ (\mathcal{A}^2 + \mathcal{B}^2) [(1/30) (W-k)^3 + (1/18) k^2 (W-k)] \\
&\quad - \mathcal{A}\mathcal{B} [(1/30) k^3 + (1/18) k(W-k)^2] \}, \\
C_{T2} &= \sum_{lmn} |\langle \beta S_{lmn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/1080) [k^4 + (10/3) (W-k)^2 k^2 + (W-k)^4] \\
&\quad + \sum_{mn} |\langle \beta T_{mn} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/180) [k^4 + (10/3) (W-k)^2 k^2 + (W-k)^4] \\
&\quad - \mathcal{A}\mathcal{B} (1/45) [k^2 + (W-k)^2] k(W-k) \} \\
&\quad + \sum_{mn} |\langle \beta A_{mn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/12) [k^2 + (W-k)^2] \\
&\quad - \{ \sum_{mn} \langle \beta T_{mn} \rangle \langle \beta A_{mn}^* \rangle + \text{c.c.} \} \cdot \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/60) [(W-k)^3 + (5/3) k^2 (W-k)] \\
&\quad - \mathcal{A}\mathcal{B} (1/30) [k^3 + (5/3) k(W-k)^2] \} \\
&\quad + |\langle \mathbf{a} \cdot \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/9) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \}, \quad (30b)
\end{aligned}$$

$$\begin{aligned}
C_{A2} = & \sum_{lmn} |\langle S_{lmn} \rangle|^2 (\mathcal{A}^2 + \mathcal{B}^2) (1/1080) [k^4 + (10/3)k^2(W-k)^2 + (W-k)^4] \\
& + \sum_{mn} |\langle T_{mn} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/180) [k^4 + (10/3)k^2(W-k)^2 + (W-k)^4] \\
& + \mathcal{A}\mathcal{B} (1/45) [k^2 + (W-k)^2] k(W-k) \} + |\langle \gamma_5 \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/3) [k^2 \\
& + (W-k)^2] - \mathcal{A}\mathcal{B} (4/9) k(W-k) \}, \\
C_{I2} = & |\langle \gamma_5 \mathbf{r} \rangle|^2 \{ (\mathcal{A}^2 + \mathcal{B}^2) (1/3) [k^2 + (W-k)^2] + \mathcal{A}\mathcal{B} (4/9) k(W-k) \},
\end{aligned}$$

where

$$\begin{aligned}
L_{mn} &= \int [\sigma_i x_j + \sigma_j x_i - (2/3) (\boldsymbol{\sigma} \cdot \mathbf{r}) \delta_{ij}], \quad R_{mn} = \int (x_i x_j - (1/3) \delta_{ij} r^2), \\
S_{lmn} &= \int \{ \sigma_{(i} x_j x_{k)} - (1/5) \delta_{(ij} [\sigma_{k)} r^2 + 2x_{k)} (\boldsymbol{\sigma} \cdot \mathbf{r}) \}, \\
T_{mn} &= \int \{ [\boldsymbol{\sigma} \times \mathbf{r}]_i x_j + [\boldsymbol{\sigma} \times \mathbf{r}]_j x_i \}, \\
A_{mn} &= \int \{ \alpha_i x_j + \alpha_j x_i - (2/3) \delta_{ij} (\boldsymbol{\alpha} \cdot \mathbf{r}) \},
\end{aligned} \tag{30c}$$

and enclosing indices in parenthesis signifies a sum of terms for all permutations of the indices.

§ 7. Concluding remarks

We have evaluated the bremsstrahlung energy spectra originated due to the sudden change of charge in the moment of K -electron capture. We introduced an approximation in the course of calculation of propagation function expressing the behaviour of electron excited virtually by emitting a photon. This approximation becomes worse, the position of K -electron being closer to nucleus as in the heaviest nuclei. Therefore our results must be improved in the case of heaviest nuclei. It is however very difficult to perform straightforward the second order perturbation calculation without previous evaluation of electronic propagation function in the intermediate state, because one must take summation over all intermediate states, using bound and continuous state solutions of Dirac equation for the nuclear Coulomb field. Moreover, the electron propagation function, satisfying Dirac equation for a Coulomb field, is not available, so it was natural to introduce any approximations in order to take account for the effect of nuclear Coulomb field. Glauber and Martin adopted a non-relativistic approximation and Cutkosky⁽⁴⁾ used the approximate solutions which asymptotically have the form of plane waves (with ingoing spherical waves). Our approximate procedure as well as their techniques improved certainly Morrison-Schiff's previous calculation. Moreover, we proceeded the calculation to allowed transition as well as first and second forbidden transitions for several types of capture interaction. However, it shall be desirable to improve our results further by devising more skilful method.

Now, to compare our results with experimental ones, it is convenient to make an analogous graph to "Kurie plot" in usual β -decay. The numbers of photon with the

energy interval between k and $k+dk$ being $N(k)dk$,

$$N(k)dk = C(k) \frac{G_\lambda e^2}{2\pi^3} \frac{(2\alpha Z)^{2\gamma+1}}{4\Gamma(2\gamma+1)} |\langle O \rangle|^2 (W-k)^{2k} dk \quad (31)$$

is obtained, where $C(k)$ includes all k -dependent factors except for the last factor $(W-k)^{2k}$ and $|\langle O \rangle|^2$ is the square of any nuclear matrix element.

If $C(k)$ is able to be considered as a constant,

$$[N(k)/Ck]^{1/2} = K(W-k) \quad (32)$$

is obtained, where K is a constant. This $[N(k)/Ck]^{1/2}$ vs. k -plot is analogous to "Kurie plot" in usual β -decay.

However, as seen in eqs. (28), (29) and (30), even for allowed transitions, $C(k)$ is a complicated function of k and is difficult to estimate, for lack of mathematical tables of hypergeometrical functions appearing in eq. (29). These tabulations are being prepared at present and numerical results and graphs will be published in another paper.

For forbidden transitions, as easily seen from eqs. (30a), (30b) and (30c), there appear appreciably the differences of energy spectra among several types of capture interaction and linearity of Kurie plot will be lost if the C value for allowed one is taken, and its disparity from linearity is characteristic of the types of interaction as in usual β transitions.

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Interpretation on Nitrogen- and Proton-Induced Nuclear Reactions

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Excitation functions for α -, $2p$ - and 2α -emissions are calculated on the basis of the compound nucleus formalism for both of nitrogen-induced reactions on ^{12}C and proton-induced reactions on ^{25}Mg . The results are compared with the observed excitation curves in respective cases. A good agreement between the theoretical and the experimental results in $(p; 2p)$, $(p; \alpha)$ and $(p; 2\alpha)$ reactions indicates that the evaporation theory is valid for explaining the intermediate energy reactions even in those light nuclei. The overall behaviour of observed $(N; \alpha)$ and $(N; 2p)$ excitation curves is in agreement with those obtained by the calculation and suggests that some sort of compound nucleus is formed in the nitrogen induced reaction. However the large magnitude of $(N; 2\alpha)$ cross section cannot be explained by the successive emission of two alpha particles, if the first alpha particle emerges from the conglomerate nucleus proceeding toward the complete compound state. Hence the large $(N; 2\alpha)$ cross section is regarded as due to direct processes such as alpha-capture or stripping. The occurrence of direct processes may explain a small discrepancy in the shape of $(N; \alpha)$ excitation curve at high incident energies.

§ 1. Introduction

Although in the last two decades an enormous amount of studies has been made on the nuclear reactions produced by artificially accelerated particles, almost all of this has been concerned with the bombardment of nucleons, deuterons, or alpha particles. Works have scarcely been done on nuclear reactions produced by bombarding particles heavier than alpha particles, partly on account of the complexity of the reactions which might be expected in bombardments by the heavier nuclei and partly of the difficulty confronted in producing useful beams of the heavier nuclei. However a remarkable progress has been made in the experimental technique in accelerating ions of light elements such as carbon, nitrogen or oxygen, and further Breit et al.¹⁾ have predicted that the bombardment of nuclei by multiply charged ions of medium atomic weight might be expected to give useful informations on the nuclear structure. On the other hand, the recent survey of nuclear reactions has indicated that the process of compound nucleus formation is not the whole picture of nuclear reactions and various processes should be considered to explain a wide range of experimental data. Hence experimental results²⁻⁵⁾ obtained by the bombardment with heavier nuclei are also available to clear up to what extent the compound nucleus theory can do with describing the nuclear reactions. In the production of radioactive isotopes by bombardments with high energy nitrogen and oxygen ions, Chackett et al.²⁾ have pointed out that the experimental observations are not easily explained if the collid-

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ing nuclei simply fuse into a compound nucleus which then undergoes evaporation or fission, but it is rather possible that only a part of one nucleus sticks to the other. This conclusion seems to be reasonable in view of the high incident energies which might be expected to induce various non-compound processes. To the contrary, Cohen et al.⁽⁶⁾ have concluded that the compound nucleus model is of good validity in explaining the $2p$ -, 2α - and α -excitation curves when carbon nuclei are bombarded with nitrogen ions of moderate energies and that deviations from the theoretical predictions may be due to the lowering down of exit coulomb barriers, in which discussions the direct processes such as "stripping" or "break up" are excluded. In this paper we concentrate ourselves to investigate some discrepancies between the conventional theory and the experimental results for nitrogen induced reactions on carbon.⁽⁴⁾

Bohr's assumption of the compound nucleus requires that the interaction between an incident particle and a target nucleus is so strong that the complete "mix-up" condition or the thermodynamic equilibrium may be formed in the compound state and then it decays by emitting some particles and gamma rays. However, in the reaction of nitrogen ion on carbon, one may expect that some of particles are emitted from ^{12}C or ^{14}N , or from a dumbbell-shaped structure consisting of these two nuclei in contact before the compound nucleus is formed, or that the complete equilibrium cannot be reached due to the "memory" of the alpha particle structure in the initial carbon nuclei. These reactions induced by a nitrogen beam are compared with those induced by a proton beam bombarding ^{25}Mg nuclei in which the same compound nucleus ^{26}Al is formed. From the very feature of the theory, the latter is likely to be better explained on the basis of the compound nucleus model than the former. This is the case, as will be shown below, in the absolute magnitude of 2α -excitation cross section. The theoretical prediction agrees with the observed yield in the $(p; 2\alpha)$ excitation, while it falls significantly below the magnitude of the experimental $(N; 2\alpha)$ cross section. On the other hand, the calculated cross sections for α - and $2p$ -excitations are found to be in good agreement with the observed results in both proton- and nitrogen-induced reactions. And the close correspondence in respective excitation curves for both cases suggests us that the compound nucleus is formed also in nitrogen-induced reactions, and the large $(N; 2\alpha)$ cross section observed seems to be due to the contribution from another reaction mechanism, which has been excluded by Cohen et al.⁽⁶⁾

§ 2. The procedure of theoretical calculations

The method to calculate excitation functions predicted on the compound nucleus formation is a straightforward application of the formulas from Blatt and Weisskopf⁽⁷⁾. According to the compound nucleus formalism, the cross section $\sigma(i; j, k)$ of the $(i; j, k)$ reaction is given by the product of $\sigma_c^{(i)}$, the cross section for the formation of the compound nucleus by the incident particle i , and p_{jk} , the decay probability for the successive emission of particles j and k from the compound nucleus, as

$$\sigma(i; j, k) = \sigma_c^{(i)} p_{jk} \quad (1)$$

The probability p_{jk} is expressed as

$$p_{jk} = J_{jk} / \sum_{l,m} J_{lm} = J_{jk} / \sum_l F_l, \quad (2)$$

where J and F have been defined in the previous papers^(8,9), cited as I and II, and of usual meaning. The summation in the denominator must be extended over all particles and gammas as long as their emission is energetically possible. In the evaluation of integrals F and J , the density of energy levels (or nuclear temperature) is taken to be $w(E) = c \exp 2\sqrt{aE}$, in a usual manner. Where c and a are energy independent constants; and E is the excitation energy. Both c and a can be adjusted so that the formula reproduces our rather scant knowledge on level densities. Since this statistical formula may be less valid for such light nuclei as concerned here, the results obtained with such $w(E)$ are inevitably less reliable. In our calculation they are chosen as

$$\left. \begin{aligned} a &= 0.1 + 0.01 A \text{ Mev}^{-1}, \\ c_{\text{even-odd}} &= 0.77 - 0.01 A \text{ Mev}^{-1} \end{aligned} \right\} \quad (3)$$

where A is the mass number of the nucleus concerned and for c the even-odd rule is taken into account. The choice (3) is very hypothetical and merely represents a gross picture of excitation levels. Data of the resonance levels observed by the bombardment of ^{25}Mg and ^{23}Na nuclei with low energy protons are available to check how our choice

of the constants can reproduce them. Average level spacings obtained from the above two cases are 0.09 and 0.06 Mev, for which the theory gives us the right orders of magnitudes, 0.05 and 0.04 Mev respectively. Moreover the low excitation levels of lighter nuclei, ^{18}F or ^{21}Ne , seem to be reproduced by the constants, except the levels nearest to the ground state. The agreement, better than expected, encourages us in the risk of employing the Fermi-gas level density formula.

Table 1.

S_n	13.1* Mev	$S_{\alpha n}$	11.0* Mev
S_p	5.4	$S_{\alpha p}$	6.9
S_α	8.4	$S_{\alpha\alpha}$	8.6
S_{pn}	7.3	$S_{\text{intermediate}}$	14.1
S_{pp}	11.6		
$S_{p\alpha}$	9.9		

* These are calculated by the semi-empirical mass formula.

The integrals F and J are closely dependent upon the energy available for the emission of any particle. This energy corresponds to the maximum kinetic energy and equals to the excitation energy minus its separation energy from the compound nucleus or from the intermediate residual nucleus. The values of these separation energies in respective cases are easily calculated from the atomic masses obtained by various measurements. Since no measurement has been done for the masses of ^{25}Al and ^{21}Na nuclei, we are obliged to employ the semi-empirical mass formula. In Table I are shown the separation energies used in our calculation; notations are the same as in I and II, S_j is the separation energy of the particle j from the compound nucleus and S_{jk} that of k from the intermediate residual nucleus left after the emission of the particle j .

The cross section for the formation of the compound nucleus by neutrons has been

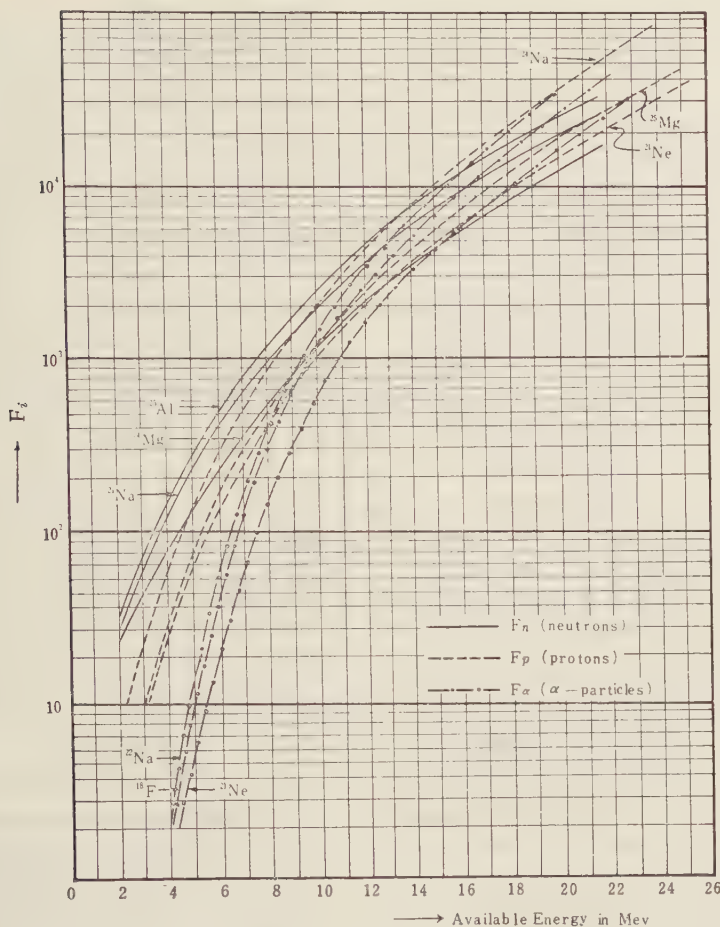


Fig. 1. The function F_i , which is proportional to the width of i -particle emission vs. the available energy.

calculated by the formula given in Feshbach and Weisskopf's paper¹⁰⁾ and those for protons and alpha particles have been interpolated from the table given by Blatt and Weisskopf⁷⁾, in all cases assuming a sticking probability of unity. The compound nucleus formation cross section for incident nitrogen ions can be estimated by the same method as those for protons and alpha particles, assuming the reasonable value of the reaction channel radius. In our work, the channel radius is taken as 6×10^{-13} cm, which is somewhat smaller than the sum of the nuclear radii of ^{12}C and ^{14}N nuclei, and larger than the radius of the compound nucleus ^{26}Al .

In the calculation of integrals F and J , a special attention has been paid to the spin weight of the particles evaporated from excited nuclei and the factor 2 has been multiplied on F 's for a nucleon emission.

The integrals F and decay probabilities p_{jk} thus calculated are shown in Fig. 1 and Fig. 2 for the respective processes.

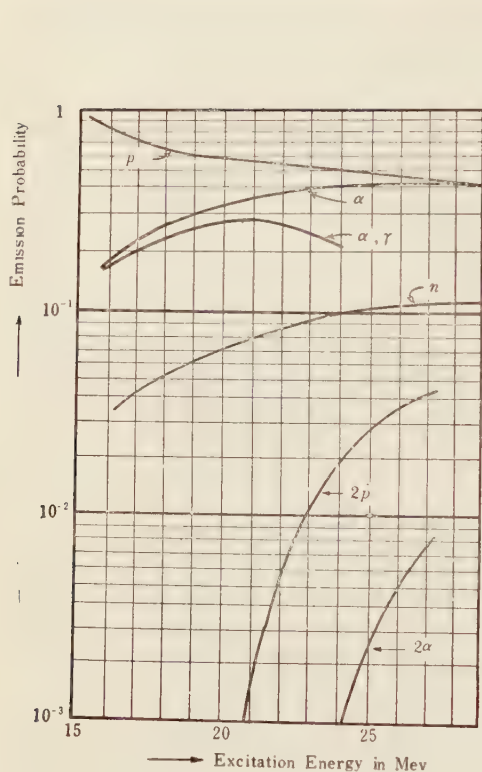


Fig. 2. The calculated probability $\sum_k p_{jk}$ and p_{jk} for the emission of j and the successive emission of j and k respectively vs. the excitation energy of the compound nucleus ^{26}Al . The figures attached to respective curves indicate j or j and k .

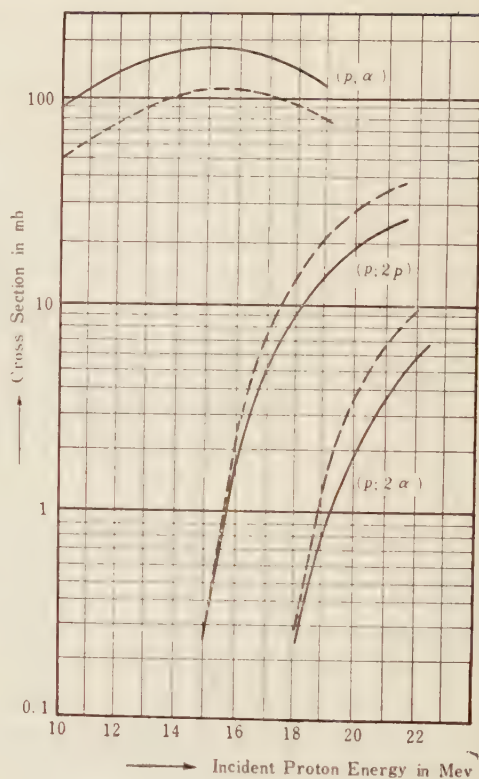


Fig. 3. Comparison between theoretical calculations and experimental results for the proton induced reactions on ^{25}Mg . The solid curves refer to the calculated cross sections and the dashed curves are those for the experimental cross sections.

§ 3. Results and discussions

Firstly the comparison between theoretical and experimental results will be made in the case of the proton-induced reactions in ^{25}Mg nuclei. It is apparent from Fig. 3 that the compound nucleus theory is able to explain the gross behaviour of the excitation curves in a satisfactory manner. Some discrepancies in the absolute magnitude are perhaps due to the application of the statistical treatment to light nuclei or to the rather frivolous choice of the level density constants. The comparatively copious evaporation of alpha particles is understandable in view of the following several points; since the excitation energy of the compound state is high enough, the alpha particles can easily penetrate the coulomb barrier. Further the disadvantage due to the spin weight is compensated by the odd-odd nature of the residual nucleus ^{22}Na and the integral F is proportional to the reduced mass of the emerging particle. The theoretical curve for the $(p; \alpha)$ cross section indicates maximum at about 15 Mev incident energy, which is in good agreement with

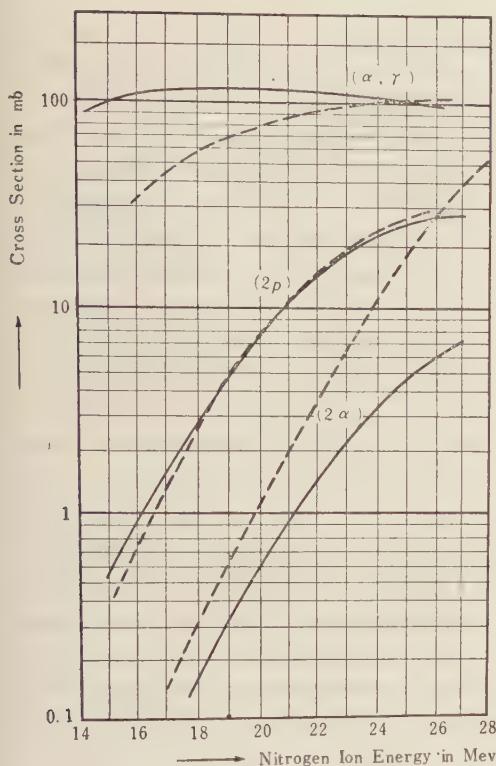


Fig. 4. Comparison between theoretical calculations and experimental results for the nitrogen induced reactions on carbon. The solid curves refer to the calculated cross sections and the dashed curves are those for the experimental cross sections.

the experimental data, and the subsequent drop-off is explained by the occurrence of the $(p; \alpha p)$ reaction. So far as the proton-induced reaction is concerned, the compound process seems to be the main event occurring and the non-compound process plays a small role.

Now we analyse the reactions of the same types induced by incident nitrogen ions. In this case the discrepancies are remarkable in the energy dependence of the $(N; \alpha)$ excitation curve and the absolute magnitude of the $(N; 2\alpha)$ cross section.

The probability $p_{\alpha T}$ shows the maximum at the excitation energy around 21 Mev, as shown in Fig. 2, which corresponds to the incident nitrogen energy of about 14 Mev. Near this incident energy, the observed $(N; \alpha)$ curve indicates such a rapid increase with increasing incident energy that it can not be explained by the energy dependence of σ_c for the incident nitrogen, since the coulomb barrier height is much lower than the incident energies concerned. It may be interpreted if it is assumed that

the ground state of ^{22}Na nucleus has the isotopic spin $T=1$ and consequently many energy levels up to several Mev excitation energy have the same isotopic spin. If it is the case, these levels do not contribute to the $(N; \alpha)$ cross section, since the compound states formed by incident nitrogen nucleus bombarding on carbon nucleus are of zero isotopic spin and the emission of alpha particles does not change the isotopic spin of residual nucleus, provided that the mixing of different isotopic spin states is insignificant.

As to the yield of alpha particles, it has been pointed out by Cohen et al.⁽⁶⁾ that alpha particles can be emitted from ^{12}C or ^{14}N nucleus, or from a dumbbell-shaped structure consisting of these two nuclei in contact. An alpha particle emitted from such nuclear states carries away the kinetic energy at least comparable to the coulomb barrier, the remnants being driven by its recoil toward the intermediate residual nucleus which perhaps satisfies the condition of the compound nucleus. The excitation energy of ^{22}Na nuclei thus left is at best 10~13 Mev when the incident energy increases from 17 Mev to 25 Mev. Around this excitation energy, the emission of protons predominates over those of other particles and the emission probability of second alpha particles, if no "memory" remains, ranges from 10^{-1} to 10^{-2} . If it is assumed that the absolute magnitude of the $(N; 2\alpha)$

cross section can be explained by the successive evaporation of two alpha particles, this probability facilitates us to estimate the cross section for the emission of alpha particles from the separate two nuclei or from the dumbbell-shaped structure consisting of these two nuclei. In order to yield the observed $(N; 2\alpha)$ cross section, the estimated cross section for the above process rises up over the compound nucleus formation cross section for the incident nitrogen ions. When the first alpha particles are emitted from a conglomerate nucleus of spherical shape which is proceeding toward the complete compound state, the coulomb barrier becomes higher than that of each nucleus, so that the emission of the second alpha particle from the residual nucleus would be more difficult. Further several processes considered above, if they took place, hardly make the $(N; 2\alpha)$ cross section larger, but they would produce more yield of the $(N; \alpha, p)$ excitation and consequent drop-off in the $(N; \alpha)$ excitation curve in contradiction to the observed curve. In view of the above considerations, it rather seems that the large yield of 2α -emission is hardly attributed to the lowering down of the coulomb barrier in the conglomerate or the dumbbell-shaped nucleus, and the large contribution must arise from processes such as the simultaneous emission of two alpha particles due to "three body break-up" or "stripping". And the occurrence of these direct processes may explain the energy dependence of the $(N; \alpha)$ curve, which does not show the drop-off occurring in the $(p; \alpha)$ excitation curve. It may be reasonable that the binding among alpha particles in the target carbon nuclei is set loose by the interaction with the approaching nitrogen ions¹⁾ and one of these alphas is captured into by the nitrogen nucleus or, on the contrary, it is stripped off, the other two being caught by the nitrogen ions.

The cross section for an alpha particle process is especially simple to estimate in a classical picture, if we adopt the assumption that the carbon nucleus is constructed in a regular triangle shape of 3α -particles in glazing contact and the constituents rotate rapidly with their internal kinetic energy around the centre of gravity of the regular triangle. For simplicity, we consider the inverse case of the actual experiment, that is, nitrogen nuclei bombarded with incident carbon beam. The cross section $\sigma_{\alpha\text{-cap}}$ for one alpha particle captured and the other missed by the target nitrogen nucleus is, in a crude approximation,

$$\sigma_{\alpha\text{-cap}} \simeq 2\pi R_N^2 (1 - B/E_c) \hat{\xi}, \quad (4)$$

where R_N and ρ are radii of the nitrogen nucleus and the alpha particle respectively, E_c is the energy of the incident carbon beam and B is the coulomb barrier height between carbon and nitrogen nuclei. The quantity $\hat{\xi}$ is the probability that one constituent in the 3α -structure of carbon nucleus comes into contact with the nuclear surface and the other does not. In the simple model here adopted $\hat{\xi}$ is estimated as $(2 - \sqrt{3})/4$ from the geometrical consideration. For $E_c \sim 21$ Mev, $B/E_c \sim 0.5$ and the equation (4) gives us about 10 mb in the right order of the observed $(N; 2\alpha)$ cross section.

The authors wish to express their hearty thanks to Professor K. Husimi for his interest in this work and for the careful reading of the manuscript before publication. They also wish to express their sincere gratitude to Professors S. Hayakawa and S. Takagi for their valuable comments on this work.

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) Thanks are due to Professor B. L. Cohen for his valuable communication about his experiment

Letters to the Editor

Surface Rigidity and Shell Structure of Nuclei

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The nuclear collective model^{(1),(2)} has been successful in accounting for the qualitative features in the nuclear structure. Some problems resulting from this model may, however, require a more profound analysis of this model. One of these problems is to make clear the relation between the surface rigidity of the core of the nucleus and its shell structure. For example, there is a remarkable discrepancy between the surface rigidity suggested from the empirical quadrupole moments⁽³⁾ and that from the hydrodynamical model.⁽¹⁾ In this letter, we shall discuss the relation between the surface rigidity of the core and the proper shell structure of it.

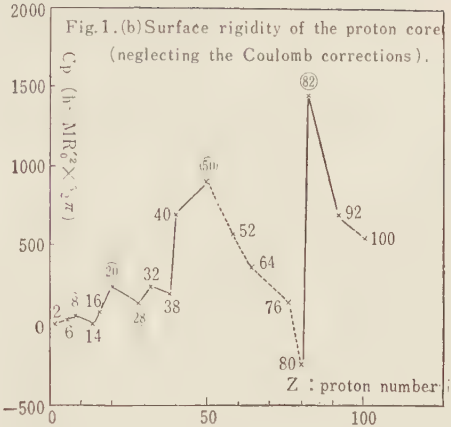
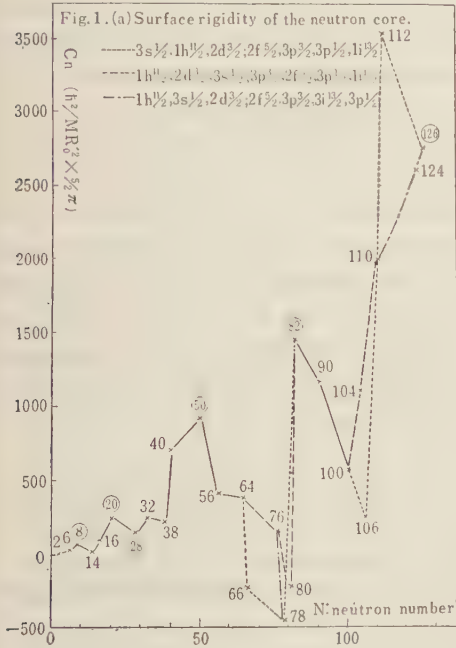
By using the method of the quantum mechanical description of the collective motion which has been formulated by one of the present authors⁽¹⁾ and others,^{(5),(6)} we can reduce the Schrödinger equation for the nuclear system to such an equation for the coupled system of the collective motion of the core and the individual particle motion

of extra-particles, that was proposed by Bohr and Mottelson.⁽¹⁾ Here the degrees of freedom of extra-particles represent the low frequency modes of excitation of the particle structure, associated with the particles in the last unfilled level. And the motion of the particles forming the core manifests itself only through the collective motion of the core.

In this case we assume that the motion of the particles forming a core can be treated adiabatically against the deformation of the core. Assuming the picture of the shell model, therefore, the motion of such particles is reduced to one particle problem in a deformed average potential, and the energy eigen-value of such a particle can be solved for a given value of the deformation described by the collective coordinates α . The sum of such energy eigen-values, $E(\alpha)$, gives the rigidity of the core, C , through $C = \partial^2 E(\alpha) / \partial \alpha^2 |_{\alpha=0}$. The rigidity thus derived may be different from that based on the hydrodynamical model, and the dependence on the shell structure can be implied in $E(\alpha)$.

To make clear the essential point of the problem we calculated the surface rigidity of nuclei which consist only of the core. In obtaining $E(\alpha)$, we assumed the deformation of the average potential to be of the spheroidal type, and used the wave functions of individual particles in a spherical potential as zeroth-order functions.⁽⁷⁾ The features of this potential were chosen so that the level scheme for the zeroth-order approximation reproduced approximately Klinkenberg's one.⁽⁵⁾ We can separate the surface rigidity into two parts: One depends strongly on

the level spacing (strongly shell dependent part) and another depends rather weakly on it. In the calculation of the former, we used the values of the level spacing which were deduced from the separation



energy. The result of the calculation is shown in Fig. 1 (a) and (b).*, **

According to these figures, we can find the following qualitative conclusions: i) The values of the surface rigidity of the closed shells corresponding to the magic numbers are very large except for 28 shell. ii) The values of the surface rigidity of the subshells between the neutron or proton numbers 50 and 82 become very small, irrespectively of the various possible choices

of the level ordering. iii) The values of the surface rigidity of the subshells between the neutron or proton numbers 82 and 126 are also small except for a particular choice of the level ordering.

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* Here $R_0' = R_0 + \lambda_0$, $\lambda_0 = [\hbar^2/2MV_0]^{1/2}$, where R_0 is the nuclear radius and V_0 is the depth of the potential. The dotted lines are used for the region in which the assignment of subshells is uncertain because of the pairing effect. The lines joining the points for the closed shells have no meaning. For the other nuclei than those in the proximity of the closed shells, the surface rigidity may be complicated because of the crossing of particle levels due to the large deformation of the core⁹⁾ which is caused by the coupling of the core and extra-particles.

** A printing mistake is found in proof. "52" in Fig. 1 (b) should be read correctly "58".

Nuclear Quadrupole Moment and Rigidity of Nuclear Core

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The magnitude of the electric quadrupole moments reveals directly their collective origin. The empirical quadrupole

moments give the direct information on the rigidity of the nucleus. In the hydrodynamical model,^{(1),(2),(3),(4),(5)} we can not explain the rapid variation of the rigidity, which must be accepted as an empirical fact from the nuclear quadrupole moments. Still more critical for this hydrodynamical model is the remarkable variation of the surface rigidity in different isotopes of the same element, which can be seen from the quadrupole moment ratio of isotopes with the same spin. We think that these discrepancies come from the neglect of the shell structure of the nuclear core (closed shell). Therefore, we analyse the electric quadrupole moments of "core \pm one extra-particle" type nuclei using the results of the preceding letter.⁽⁶⁾

Table I. The comparison of the surface rigidity C and the quadrupole moment Q of the nucleus of the type of core \pm one extra-particle. (We omit the nuclei in the region that the assignment of the subshell is uncertain because of the pairing effect.)

Nucleus	Configuration		I	C (Mev)		Q ($\times 10^{-24}$ cm ²)			
	proton	neutron		Cal.	Hydro.	Obs.	Cal.	Hydro.	S. P.
$^5_6\text{B}^{11}$	(1p _{3/2}) ⁻¹	—	3/2	117	24.1	+0.036	+0.044	+0.093	+0.025
$^{13}_{14}\text{Al}^{27}$	(1d _{5/2}) ⁻¹	—	5/2	26	38.8	+0.16	+0.37	+0.24	+0.062
$^{17}_{20}\text{Cl}^{37}$	1d _{3/2}	—	3/2	348	45.6	-0.062	-0.087	-0.25	-0.051
$^{19}_{20}\text{K}^{39}$	(1d _{3/2}) ⁻¹	—	3/2	505	45.6	+0.14	+0.089	+0.33	+0.055
$^{27}_{32}\text{Co}^{59}$	(1f _{7/2}) ⁻¹	—	7/2	299	54.8	+0.5	+0.30	+0.90	+0.12
$^{31}_{38}\text{Ga}^{69}$	(2p _{3/2}) ⁻¹	—	3/2	323	58.6	+0.23	+0.20	+0.52	+0.08
$^{31}_{40}\text{Ga}^{71}$	(2p _{3/2}) ⁻¹	—	3/2	691	60.2	+0.14	+0.14	+0.52	+0.08
$^{37}_{50}\text{Rb}^{87}$	(2p _{3/2}) ⁻¹	—	3/2	736	64.5	+0.14	+0.17	+0.68	+0.09
$^{59}_{82}\text{Pr}^{141}$	2d _{5/2}	—	5/2	973	67.0	-0.05	-0.35	-1.7	-0.18
$^{83}_{126}\text{Bi}^{209}$	1h _{9/2}	—	9/2	1580	58.0	-0.4	-0.55	-5.2	-0.30
$^8_9\text{O}^{17}$	—	1d _{5/2}	5/2	174	28.6	-0.005	-0.031	-0.014	-0.0013
$^{16}_{17}\text{S}^{33}$	—	1d _{3/2}	3/2	188	41.2	-0.055	-0.059	-0.20	0.
$^{16}_{19}\text{S}^{35}$	—	(1d _{3/2}) ⁻¹	3/2	350	45.6	+0.038	+0.036	+0.20	0.
$^{32}_{41}\text{Ge}^{73}$	—	1g _{9/2}	9/2	677	60.2	-0.2	-0.11	-1.1	0.
$^{38}_{49}\text{Sr}^{87}$	—	(1g _{9/2}) ⁻¹	9/2	733	64.5	—	+0.14	+1.4	0.
$^{40}_{51}\text{Zr}^{91}$	—	2d _{5/2}	5/2	1030	62.2	—	-0.083	-0.90	0.

According to the collective model,¹⁾ the quadrupole moment of the nucleus of this type is expressed by the formula $Q = Q_{sp} + Q_s$, of which the first part is due to the extra-particle outside the core. The second part is due to the surface deformation and is given by $Q_s = P_Q(x) Q_0$, where Q_0 is the intrinsic quadrupole moment of the core relative to the axes fixed in the core. x in the projection factor $P_Q(x)$ is a dimensionless parameter which is related to the strength of the particle-surface coupling. (cf. reference 1, p. 56) Using the values of the surface rigidity C given by Fig. 1 in the preceding letter,⁶⁾ we calculated the quadrupole moments of "core \pm one extra-particle" type nuclei. Table I shows the comparison of the quadrupole moments obtained by our calculation with observed ones and hydrodynamical ones.

The results of our calculation agree qualitatively with the observed values and can explain a remarkable variation of the quadrupole moments of isotopes with the same nuclear spin. This tendency caused by the rapid variation of the rigidity of the core can never be explained in terms of the hydrodynamical model or the method of the configuration mixing.⁷⁾ For example, the ratios, $|Q_{ob}(S^{33})|/|Q_{ob}(S^{35})| \cong 1.45$ and $|Q_{ob}(Ga^{69})|/|Q_{ob}(Ga^{71})| \cong 1.6$, are caused by the rapid variations of the rigidity of the cores composed of 16, 20 neutron subshells and 38, 40 neutron subshells respectively.

The values of the rigidity of 20, 82 and 126 major closed shells which are suggested by the quadrupole moments are remarkably large. The large value of $Q_{ob}(Co^{59})$ indicates the relatively small surface rigidity of the 28 proton major closed shell. These tendencies agree with the results in

the preceding letter.⁶⁾

In the regions between the neutron or proton number, 50 and 82 and between 82 and 126, the assignment of the closed subshells is uncertain because of the pairing effect. In these regions, the surface rigidity of other nuclei than those in the proximity of the closed shells may be reduced very much⁸⁾ because of the crossing of particle levels due to the large deformation of the core which is caused by the coupling of the core and extra-particles. This will be the reason why the quadrupole moments of nuclei are very large in these regions.

The facts that $Q_{ob}(Bi^{209})$ is nearly equal to Q_{sp} and $Q_{ob}(Au^{197}) (=0.56)$ is much larger than $Q_{sp}(Au^{197}) (=0.18)$ mean that there is an abrupt change of the rigidity between 80 proton core and 82 proton core. This fact can be explained, if one sees Fig. 1 of the preceding letter.⁶⁾

More detailed analysis of the nuclear quadrupole moments will be published together with the full account of the preceding letter in this journal.

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Rotational Excitation with Neutron Inelastic Scattering

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The recent experiments have shown that the excitation function of the inelastic scattering of neutrons depends on the nature of the excited states of the nucleus. For instance, as Weisskopf pointed out about the results of the experiments by Kiehn-Goodmann^(1,2), the nucleus which shows the Bohr-Mottelson spectrum would not have the excitation function of \sqrt{E} -type expected from the statistical theory of the nuclear reaction but the one which is like linear type (Lead, etc.). The nuclei having the excitation function of \sqrt{E} -type (Iron, etc.) have been analysed mostly by means of the statistical theory of the nuclear reaction. Recently S. Hayakawa-S. Yoshida⁽³⁾ and D. M. Brink⁽⁴⁾ dealt with this process by taking into account the direct interaction, but they concluded that the main contribution comes from the reaction process through the compound nucleus. On the other hand the collective model of the nucleus proposed by A. Bohr succeeded to describe well the properties of the low excited states in the heavy nucleus. And we think such low excited states play an important role in the inelastic scattering. Therefore we adopt, here, the collective model to describe the compound nucleus and the target nucleus, and calculate the excitation function according to the compound theory of Teichmann-Wigner. We are mainly interested in the rotational excitation of the strongly deformed nuclear surface, so we use the strong coupl-

ing approximation in this calculation.

When an incident neutron with the spin σ and the orbital angular momentum l collides with a target nucleus having the spin I , forming the compound nucleus with the spin J and the other quantum number λ , the reduced width $\gamma_{\lambda sl}$ for the channel s is given by⁽⁵⁾

$$\sum_s \gamma_{\lambda sl}^2 = (\hbar^2/2M) \cdot \sum_{jj'} |\alpha_{jj'} \cdot u_{\lambda j'}(a)|^2 \cdot \theta_{\lambda j}^2, \quad (1)$$

where

$$\theta_{\lambda j} = \int_S \Psi_\lambda^* \psi_j dS. \quad (2)$$

In the above equations, $u_{\lambda j'}(r)$ is the wave function of the single particle state in the compound state which is connected at the nuclear boundary S to the external wave function. This was adopted by A. M. Lane⁽⁶⁾ to obtain the good results. $u_{\lambda j'}(a)$ is the value of this wave function on the nuclear surface S . And $\alpha_{jj'}$ is the coefficient of the configuration mixing of the extra particle induced by the coupling to the nuclear surface. Ψ_λ is the wave function of the compound nucleus and ψ_j is the spin-angle wave function in the external region in j - j coupling scheme. Using B-M's wave functions as those of the target and the compound nuclei, the overlapping integral is given by

$$I_{\lambda j}^2 = 2(2I+1)/(2J+1) \cdot P_\varphi \cdot |(Ij0K|JK)|^2. \quad (3)$$

For simplicity, we regarded the target nucleus as the core. Of course this can be generalized easily. P_φ is the overlapping integral in reference to the vibrational states and it can be put equal to unity in a good approximation. The resonance levels are obtained by solving the characteristic equation in the internal region with Teichmann-

Wigner's boundary condition. The potential contains the spin-orbit interaction, its strength being determined by Inglis' formula. As K , the z -component of J , may not be determined for excited level, we assume the appropriate value here. Among the set of ortho-normal functions for compound nucleus, those which contribute mainly to this process may be the first excited state in the rotational mode of the core. And the higher excitation may be neglected.

Then, the spin J of the compound state is given by $l + \sigma \geq J \geq |l - \sigma|$, because the spin of the target nucleus is zero. When the incident neutron is s -wave, then only $J = (1/2)^+$ occurs. According to the above assumption, the rotational excitation of the core which is taken into consideration has the angular momentum 2 and so an extra particle of the nucleus has the mixed configuration of the states $(3/2)^+$ and $(5/2)^+$. In this case, however, the overlapping integral Eq. (2) vanishes because of the orthogonality of the spherical harmonics. This fact shows that the s -wave of the impinging neutron does not contribute to this reaction process. In the case of p -wave, the resonance level

is very high so that p -wave also has no contribution near threshold. The d -wave mainly contributes to this reaction. In this case $J = (3/2)^+$ and $(5/2)^+$. In the state of the $J = (3/2)^+$ the state of the extra particle in the compound nucleus is the mixture of the four states $(1/2)^+$, $(3/2)^+$, $(5/2)^+$ and $(7/2)^+$. The coefficients of these configuration mixing are determined in appropriate approximation. The figure shows the result of the numerical calculation for Pb^{206} which has the closed shell. The wave function of the collective model makes the reduced width small and this trend is increased by the configuration mixing of the extra particle in the compound state, so that our calculation agrees with the experimental data better than the case of Hauser-Feshbach. Thus we may conclude that if the rotational excitation levels are the only excited states of the above mentioned nuclei in the low energy region, we have the good result. And from this calculation, we will find the reason why the mass number dependence of the strength function is so small at such nuclei as Lead, etc. The depression between the resonance

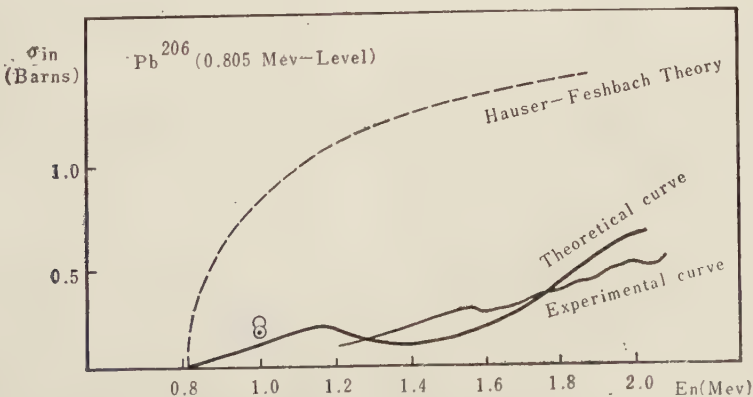


Fig. 1. Inelastic scattering of neutrons by Pb^{206} . Solid line is the data by Kiehn-Goodmann.¹⁾ \circ is the Beyster et al's and \bullet is the Walt et al's²⁾.

levels may be expected to be raised by the small contribution from the excited states of the collective mode and the single particle mode of excitation. In this treatment, however, any discussions about the region far from the resonance level is not so significant. Furthermore, the approximation of putting the overlapping of the vibrational state nearly equal to unity in the present calculation will cause a small depression of the curve as a whole.

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On the Hydrodynamical Description of the Behaviours of Elementary Particles

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In 1928, L. de Broglie¹⁾ rewrote the Schrödinger equation into the hydrodynamical

form, but his formulation did not attract due attention at that time. D. Bohm²⁾ developed this formulation in a different way. Recently, T. Takabayasi³⁾ proposed to rewrite the Klein-Gordon equation into the expression with a picture of hydrodynamics.

In order to investigate the behaviour of interacting elementary particles, we attempt to develop the formulation of the meson field equation with higher derivative interactions, using the above Takabayasi's method. As the results, it is proposed in this article that a non-local interaction contains the behaviour analogous to the ordinary viscous effects in relativistic hydrodynamics and also its effect plays an important role in the small domain of space-time.

At first, we consider the charged boson field with following Lagrangian:

$$\begin{aligned} \mathcal{L}_0 = & \partial_\nu \phi^* \cdot \partial_\nu \phi + \kappa^2 \phi^* \phi + i/2 \cdot \\ & (\square \partial_\nu \phi^* \cdot \partial_\nu \phi + \square \phi^* \cdot \square \phi + \square \partial_\nu \phi^* \cdot \\ & \square \partial_\nu \phi + \square \phi^* \cdot \square^2 \phi + \partial_\nu \phi^* \cdot \square^2 \partial_\nu \phi). \end{aligned} \quad (1)$$

We then obtain the field equation

$$(\square - \kappa^2)\phi = -1/2 \cdot g \square^3 \phi, \quad (g \sim l^4) \quad (2)$$

and the symmetrized energy momentum tensor⁴⁾

$$\begin{aligned} T_{\mu\nu} = & \partial_\mu \phi^* \cdot \partial_\nu \phi + 1/2 \cdot g (\square^2 \partial_\mu \phi^* \cdot \partial_\nu \phi \\ & + \square \partial_\mu \phi^* \cdot \square \partial_\nu \phi + \partial_\mu \phi^* \cdot \square^2 \partial_\nu \phi) \\ & + \text{conj.} - \delta_{\mu\nu} \cdot \mathcal{L}_0. \end{aligned} \quad (3)$$

To rewrite the eqs. (2) and (3) into the form with hydrodynamical expression, we assume that $\phi = R \exp[iS/\hbar]$, $\phi^* = R \cdot \exp[-iS/\hbar]$, (R, S ; real) in which S differs from S_0 satisfying the free field equation. For the energy momentum tensor, we have then

$$\begin{aligned}
T_{\mu\nu} = & 2 \cdot \partial_\mu R \cdot \partial_\nu R - 2 (i/\hbar) R^2 \cdot \partial_\mu S \cdot \partial_\nu S \\
& + g/2 \cdot \{ 2 \cdot \square^2 \partial_\mu R \cdot \partial_\nu R + \dots \\
& - 4 (i/\hbar)^2 \cdot \square^2 R \cdot R \cdot \partial_\mu S \cdot \partial_\nu S - \dots \\
& - 8 (i/\hbar)^4 \cdot R \cdot \partial_\mu R \cdot \partial_\nu S \cdot \partial_\mu S \cdot \partial_\nu S \cdot \partial_\mu S \cdot \partial_\nu S \\
& - (\mu \leftrightarrow \nu) \\
& - 8 (i/\hbar)^4 \cdot R \cdot \partial_\mu R \cdot \partial_\nu S \cdot \partial_\mu S \cdot \partial_\nu S \cdot \square S + \dots \} \\
& - \partial_{\mu\nu} \{ (\partial_\alpha R)^2 + \kappa^2 R^2 - (i/\hbar)^2 \cdot (\partial_\alpha S)^2 \\
& + g/2 \cdot [2 \square^2 \partial_\alpha R \cdot \partial_\alpha R \dots - 4 (i/\hbar)^4 \cdot R \cdot \\
& \partial_\mu R \cdot \partial_\mu S \cdot (\partial_\alpha S)^2 \cdot \square S + \dots] \}, \quad (4)
\end{aligned}$$

where... in (4) are composed of many terms with various characters, but, for the present, these terms are unnecessary for our following discussion. And we obtain the following two equations for the hydrodynamical expression of eq. (2),

$$\left\{ (\partial_\mu S)^2 - \frac{\hbar^2 (\square - \kappa^2) R}{R} \right\} = \frac{\hbar^2}{R} \cdot \rho_R, \quad (5)$$

$$\partial_\mu (R^2 \cdot \partial_\mu S) = (\hbar^2/i) \cdot R \cdot \rho_I,$$

where ρ_R , ρ_I are respectively the real and imaginary parts of the interaction given by the relation $-g/2 \cdot \square^3 \phi = \rho \cdot e^{iS/\hbar} = (\rho_R + i\rho_I) \cdot e^{iS/\hbar}$.

We can solve eq. (5) under the assumption that R does not contain the higher derivatives than the second with respect to space and time. In this case, our hydrodynamical description has about the same applicability as that of free Klein-Gordon equation. Therefore, we will not meet with any kind of difficulty in our following discussions in which the viscous effects are discussed.

We now analyse the hydrodynamical behaviour of $T_{\mu\nu}$ under the assumption that the velocity of charge density of boson field corresponds to the velocity of fluid in relativistic hydrodynamics. Though this

assumption cannot be used generally, it can be used so far as we consider the charged boson field.

In this connection, the energy momentum tensor⁽⁶⁾ with the viscosity in relativistic hydrodynamics is given by

$$\begin{aligned}
\theta_{\mu\nu} = & p \delta_{\mu\nu} + (\mathcal{E} + p) v_\mu v_\nu \\
& - \eta \{ \partial_\nu v_\mu + \partial_\mu v_\nu + v_\nu v_r \partial_r v_\mu \\
& + v_\mu v_r \partial_r v_\nu \} - \xi \{ \partial_{\mu\nu} + v_\mu v_\nu \} \cdot \partial_r v_r. \quad (6)
\end{aligned}$$

η , ξ are respectively the transversal and longitudinal viscosities ($c=1$).

Comparing the terms marked by underlines in eq. (6) with those in eq. (4) under a correspondence $1/\kappa \cdot (\partial S/\partial x_\mu) \sim v_\mu$, we obtain the same expression for the viscosities η and ξ ,

$$\eta \sim 4g (i/\hbar)^4 \kappa^3 R \partial_r R \partial_r S. \quad (7)$$

Integrating $T_{\mu\nu}$ over a volume, the viscous effects may give no explicit contribution to the integrated energy momentum tensor. However, the viscous effects will contribute implicitly to the energy momentum tensor since these effects are contained explicitly in the density of energy momentum tensor.

We now investigate the contributions of such effects to the interaction in small domain of space-time. The contribution of viscous effects in the expression with hydrodynamical picture is known by comparing with that of the effect of inertia. In this case, we cannot discuss the contribution of such effects to the interaction by using the coefficient of viscosity or coupling constant, but by the quantity analogous to Reynolds number in hydrodynamics. Therefore, we introduce the Reynolds number from the eq. (6)

$$\Re = \varepsilon v \lambda / \eta \equiv \lambda v / \nu^*,$$

(ν : kinematic coefficient of viscosity) (8)

where ε , λ are the energy density of the expression (6) and a length of path of fluid motion.

Assuming the correspondence between the relations (7) and (8), we have $1/\Re \cdot \varepsilon v \lambda \sim 2/\hbar^4 \cdot g \cdot \kappa^3 \cdot \partial R^2 / \partial x \cdot \partial S / \partial x$. If we remember that the dimension of ϕ is given by the inverse ratio of a length $[\phi] = [R] = [\lambda^{-1}]$, using a distance λ describing the spread of charge density, and that ε , κ have respectively the dimensions $[l^{-4}]$ and $[l^{-1}]$, in the end we obtain

$$1/\Re \sim g \cdot \lambda^{-4} \equiv (l/\lambda)^4. \quad (9)$$

In the above equation we used the natural unit ($\hbar = c = 1$). l is the universal length. As is well known, the viscous effects cannot be neglected for the effect of inertia in hydrodynamics when \Re has relatively a small value.

From relation (9), it will be concluded that the behaviour analogous to the viscous motion of the interacting boson field has relatively a remarkable effect in the small distance satisfying $\lambda < l$.

* The conception of the viscosity in the usual gas theory is different from that of our hydrodynamical treatments. But, we may think that our proposal is allowed in the circumstances treated here. The detailed account of this problem will be given elsewhere.

This conclusion obtained by us does not contradict with the fact that we consider the higher derivative interaction.

Now, this work has been done in order to modify the non-local interaction between nucleon and meson, $\bar{\psi} O \psi \cdot F[\square] \phi$, in which the property analogous to the viscosity contained in $F[\square]$ will be related to the internal structure of elementary particles and to the multiplicity of created particles in multiple production of mesons. The development of the above attempt will be done in future.

I wish to thank Professor T. Inoue for his kind interest and continual encouragement to my work, and to Dr. H. Kita for his helpful discussion.

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AUTHORS INDEX TO VOLUME 15

- Araki-G & Huzinaga-S**, Absorption Spectra of Unsymmetrical Cyanines 307
- Fukuda-N**, Some Relations between the Bound State Problem and Scattering Theory .. 414 (L)
- Goto-K**, On a Regular Formulation of Quantum Field Theory, I — *Non-Relativistic Theory* —.. 167
- Hamaguchi-M**, On the Hydrodynamical Description of the Behaviours of Elementary Particles 588 (L)
- Harada-K** (see Oda-N) 545
- Hasegawa-H** (see Nogami-Y) 137
- Hashitsume-N**, A Statistical Theory of Linear Dissipative Systems, II..... 369
- Hatanaka-T** (see Taketani-M) 89
- Hatano-S & Kaneno-T**, On the Exchange Magnetic Moment of the Two Nucleon System 63
- Hatano-S** (see Iwadare-J) 185 (L)
- Hayakawa-S**, Supernova Origin of Cosmic Rays 111
- Hayakawa-S & Hokkyo-N**, Electromagnetic Radiation from Electron Plasma..... 193
- Hiida-K, Iwadare-J & Machida-S**, On the One Pion Exchange Potential..... 189
- Hokkyo-N** (see Hayakawa-S) 193
- Hori-S** (see Ōneda-S) .. 300 (L), 302 (L), 304 (L)
- Hsieh-S.H & Nakagawa-M**, Nature of Nuclear Force indicated by the Photodisintegration of the Deuteron, I..... 79 (L)
- Huzinaga-S** (see Araki-G) 307
- Huzinaga-S**, Interaction between Electrons in Two-Dimensional Free-Electron Model for Conjugated Systems 495
- Huzinaga-S**, One-Center Expansion of Molecular Wave Function 501
- Ichimura-H**, Quantum Statistical Mechanics of Electron-Phonon System 151
- Ikeda-M**, On Boundary Conditions in the Non-Symmetric Unified Field Theory 1
- Inoue-K, Machida-S, Taketani-M & Toyoda-T**, Pion Theory of Nuclear Forces 122
- Ishiguro-E** (see Sakamoto-M) 37
- Iso-C & Sato-M**, Hydrodynamical Theory of the Multiple Production of Particles in High Energy Nucleon-Nucleus Collision..... 510 (L)
- Ito-D, Ono-M, Kato-T & Takahashi-Y**, The Photodisintegration of Deuteron at High Energy 74 (L)
- Iwadare-J, Otsuki-S, Tamagaki-R & Watari-W**, Determination of the Pion Coupling Constant in Nuclear Forces 86 (L)
- Iwadare-J & Hatano-S**, Interaction of Antinucleons in Matter, I..... 185 (L)
- Iwadare-J** (see Hiida-K) 189
- Iwata-G**, Orbits of an Electron in Static Electromagnetic Fields, I 513
- Kamefuchi-S & Umezawa-H**, On the Renormalization Theory of Quantum Electrodynamics 298 (L)
- Kamefuchi-S** (see Umezawa-H) 417 (L)
- Kanazawa-H**, Coulomb Interactions and the Diamagnetism of Free Electrons..... 273
- Kaneno-T** (see Hatano-S) 63
- Kato-T** (see Ito-D) 74 (L)
- Kawaguchi-M & Nishijima-K**, Note on the Decay Interactions of Hyperons and Heavy Mesons 180 (L)
- Kawaguchi-M & Nishijima-K**, Note on the Decays of Σ Particles 182 (L)
- Kikuchi-K** (see Nakasima-R) 574
- Kikuta-T**, Extensions of Variational Methods, II — *Two Parameter Eigenvalue Problem for the Deuteron State* — 50
- Kikuta-T, Morita-M & Yamada-M**, Effect of Hard Core on the Binding Energies of H^3 and He^3 , I..... 222
- Kita-H**, Remarks on Heisenberg's Non-Linear Field Theory 83 (L)
- Koba-Z**, Interaction of π -Mesons with π -Mesons 294 (L)
- Koba-Z**, Remarks on the Pion-Pion Interaction 461
- Konuma-M** (see Umezawa-H) 417 (L)
- Machida-S** (see Inoue-K) 122
- Machida-S** (see Hiida-K) 189
- Maki-Z**, On the Use of Feynman Amplitudes in the Quantum Field Theory 237
- Marumori-T, Suekane-S & Yamamoto-A**, Surface Rigidity and Shell Structure of Nuclei 582 (L)
- Marumori-T, Suekane-S & Yamamoto-A**, Nuclear Quadrupole Moment and Rigidity of Nuclear Core 584 (L)
- Minami-S**, Meson Reactions in Two-Nucleon System 12

- Miyazima-T & Tamura-T**, A Collective Description of the Surface Oscillation of Atomic Nuclei — *Extension of Tomonaga's Method to Three Dimensional Nucleus* — 255
- Morita-M** (see Kikuta-T) 222
- Morita-M**, Proposal for Experiments for Determination of Beta-Decay Interaction and Theory of Triple Cascade Transition 445
- Murakami-K** (see Murayama-T) 421
- Murayama-T, Murakami-K, Tanaka-R & Ogawa-S**, The Atmospheric Effects on the Intensity of High Energy μ -Mesons 421
- Murty-G. S.**, On the Relativistic Thomas-Fermi Atom 473
- Nakagawa-M** (see Hsieh-S.H) 79(L)
- Nakano-H**, A Method of Calculation of Electrical Conductivity 77(L)
- Nakano-T**, A Relativistic Field Theory of an Extended Particle, I 333
- Nakasima-R, Tanaka-Y & Kikuchi-K**, Interpretation on Nitrogen- and Proton-Induced Nuclear Reactions 574
- Nanda-V.S** (see Trikha-S.K) 178(L)
- Nishijima-K** (see Kawaguchi-M) ..180(L), 182(L)
- Nogami-Y & Hasegawa-H**, Intermediate Coupling Meson Theory of Nuclear Forces, II .. 137
- Obi-S** (see Taketani-M) 89
- Oda-N & Harada-K**, Elastic Scattering of Alpha-Particle by Heavy Elements 545
- Ogawa-S** (see Murayama-T) 421
- Ogawa-S**, On the Universality of the Weak Interaction 487
- Okada-K** (see Yamamoto-T) 184(L)
- Okai-S & Sano-M**, Deuteron Stripping Reactions and Nuclear Shell Structure 203
- Okamoto-K**, Relation between the Quadrupole Moments and the Widths of the Giant Resonance of Photonuclear Reaction 75(L)
- Ôneda-S, Hori-S & Wakasa-A**, On the Elementarity of the Weak Eoson-Fermion Interaction, I, II 300(L), 302(L)
- Oneda-S, Hori-S & WaKasa-A**, On K-Meson Decays and the Universal Interactions .. 304(L)
- Ôno-M** (see Ito-D) 74(L)
- Otsuki-S** (see Iwadare-J) 86(L)
- Ouchi-T, Senba-K & Yonezawa-M**, Theory of Mass Reversal in the Quantized Field Theory 431
- Rustgi-O.P** (see Trikha-S.K) 296(L)
- Sakai-M**, Note on the Nuclear Level of Spin 2⁺ 416(L)
- Sakamoto-M & Ishiguro-E**, He-He Repulsive Potential, I. 37
- Sano-M** (see Okai-S) 203
- Sato-M** (see Iso-C) 510(L)
- Senba-K** (see Ouchi-T) 431
- Suekane-S** (see Marumori-T)582(L),584(L)
- Suzuki-R**, Deuteron Photodisintegration at High Energies 536
- Takahashi-Y** (see Ito-D) 74(L)
- Takeno-H & Ueno-Y**, On the Wave Theory of Light in General Relativity, III — *Electromagnetic Four Potential* — 322
- Taketani-M, Hatanaka-T & Obi-S**, Populations and Evolution of Stars 89
- Taketani-M** (see Inoue-K) 122
- Tamagaki-R** (see Iwadare-J) 86(L)
- Tamura-T** (see Miyazima-T) 255
- Tanaka-R** (see Murayama-T) 421
- Tanaka-Y** (see Nakasima-R) 574
- Tani-K** (see Yamamoto-T) 184(L)
- Taniuti-T**, On the Theories of Higher Derivatives and Non-Local Couplings, II. 19
- Teramoto-E**, The Statistical Mechanical Aspect of the H-Theorem, II 480
- Tomozawa-Y** (see Umezawa-H) 417(L)
- Toyoda-T** (see Inoue-K) 122
- Trikha-S.K & Nanda-V.S**, On Solutions of He³ and He⁴ 178(L)
- Trikha-S.K & Rustgi-O.P**, Surface Tension of Liquid He³ and Liquid He³ 296(L)
- Ueno-Y** (see Takeno-H) 322
- Umezawa-H** (see Kamefuchi-S) 298(L)
- Umezawa-H, Kamefuchi-S, Tomozawa-Y & Konuma-M**, On the Renormalization Cut-off 417(L)
- Wakasa-A** (see Ôneda-S) 300(L), 302(L), 304(L)
- Watanabe-S**, Mass Reversal and Space-Time Inversions 81(L)
- Watanabe-S**, Symmetry in Time and Tanikawa's Method of Superquantization in Regard to Negative Energy Fields 523
- Watari-W** (see Iwadare-J) 86(L)
- Yamada-M** (see Kikuta-T) 222
- Yamamoto-A** (see Marumori-T) . 582(I), 584(L)
- Yamamoto-T, Tani-K & Okada-K**, On the Energy Dissipation of Conduction Electrons undergoing Elastic Scattering by Impurities 184(L)

Yamashita-J , Theory of Electron Multiplication in Silicon	95
Yasuno-M , Rotational Excitation with Neutron Inelastic Scattering.....	586(L)
Yamazaki-K , On the Feynman's Theory of Polarons	508(L)
Yonezawa-M (see Ouchi-T)	431
Yukawa-J , The Theory of Radiative K Capture, I	561
Errata	187, 306, 419

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN

Volume 11, Number 6, June 1956

CONTENTS

	Page
Ikuro SHIMOSE; Lattice Theory of Liquid State	615
Sigetosi TANISAKI; Growth Spirals on Crystals of WO_3	620
Terutaro NAKAMURA; On the Domain Structure of Rochelle Salt Crystal	624
Toshinosuke MUTO and Hiroshi OKUNO; Electronic Structure of the Exciton. I. Localized Exciton in KCl	633
Sumitada ASANO and Yasuo TOMISHIMA; Calculation of the Cohesive Energy of Zincblende	644
Kazuko MOTIZUKI and Takeo NAGAMIYA; Effect of Self-diffusion on the Ortho- para Conversion in Oxygen Contaminated Solid Hydrogen	654
Gunji SHINODA, Tatsuro SUZUKI and Susumu KATO; The Soft X-ray Spectroscopy of the Solid State by the Electronic Differentiating Method; Aluminium L_3	657
Ichiro TSUBOKAWA; The Magnetic Properties of Chromium-Tellurium-Selenium System	662
Tokutaro HIRONE and Shu CHIBA; The Magnetic Properties of $FeSe_x$ with the NiAs Structure	666
Yoshika MASUDA; Nuclear Magnetic Resonance in Molecular Chlorine Compounds	670
Yoshiyuki TAKEISHI; Auger Ejection of Electrons from Barium Oxide by Inert Gas Ions and the Cathode Fall in the Normal Glow Discharges	676
Hiroomi FUJIKAWA; The Forces acting on Two Circular Cylinders of arbitrary Radii placed in a Uniform Stream at Low Values of Reynolds Number	690
Hiroshi SATO; Experimental Investigation on the Transition of Laminar Separated Layer	702
Takeo SAKURAI; The Flow past a Flat Plate accompanied with an Unsymmetric Dead Air at Mach Number 1	710

SHORT NOTES

Takeo YOKOBORI; On the Frequency Dependence of the Fatigue of Metals	715
Kimio HASHIMOTO and Kazuyosi HIRAKAWA; Electrical Properties of Stannous Telluride $SnTe$	716
Tadanobu KOJIMA; Mechanism for Self-Diffusion in Metallic Lithium	717
Jiro YAMAGUCHI; On the Inductive Reactance and Negative Resistance in the Transistor	717
Sōshin CHIKAZUMI; On the Interpretation of Magnetic Torque Reversal	718
Bogdan SUJAK; A Remark on the Excitement of the Electrodes when a Positive Joshi Effect Inversion is Observed under Continuous Discharge	719

Errata

Hidegori HASIMOTO; Note on Rayleigh's Problem for a Circular Cylinder with Uniform Suction and Related Unsteady Flow Problem	721
---	-----